Chapter 4

WELFARE PROPERTIES OF "JUNGLE" EXCHANGE

1. Power as a Basis for Exchange

We will now describe a model of exchange that is quite different from barter exchange or market exchange. In both barter and market transactions, exchanges are voluntary. Person i does not swap a bundle of commodities to person j unless he's better off (or no worse off) by so doing.

But sometimes exchanges are forced: You give me your wallet and your watch, or I'll kill you! This is the "law of the jungle." The more powerful takes from the less powerful.

To model this kind of situation, we will assume there is a *power relation* among individuals $\{1, 2, \ldots, n\}$. That is, there is a strict ordering of the individuals from strongest to weakest. For example, suppose that person 1 is strongest, person 2 is second strongest, and so on. Then if person i meets person j in the jungle, and $i < j$, person i can take whatever he wishes from j, whether j agrees or not.

Let's pause for a moment to consider the plausibility, or lack of plausibility, of jungle exchange. Firstly, we do not intend to claim that there is more forcible taking in tropical rain forests than in New York City. The economic jungle is not a geographic locality. Secondly, we do not intend to claim that taking things based purely on power is pervasive, or even common, in everyday economic activity. Thankfully, it is not. But, thirdly, there are examples where the model might apply: in times of war, in some despotic states, and even, occasionally, in the distribution of certain goods in modern democracies. For example, in some academic departments, offices and/or parking places may be distributed on the basis of seniority. If professor i is senior to professor j, then i has the right to claim j's office, or his parking place.

Michele Piccione and Ariel Rubinstein, who wrote the seminal paper on this model, called the structure in which power and coercion govern the distribution of goods "the jungle." They might have as well called it "anarchy," but the important point is that there is a power ordering of the participants, and the more powerful take from the less.

Let us now consider what is taken in the jungle.

If we simply laid a power ordering over our barter and price exchange models, which involve allocations of m divisible goods among n people, and which assume monotonic self-interested utility functions, we wouldn't have much of interest. The most powerful person would just seize the goods of everyone else. This would produce a trivial and degenerate jungle equilibrium. In order to make the n-person m-goods jungle model interesting, with divisible goods, we would have to assume that individuals have satiation points, so the most powerful person could seize the consumption bundle that he most wants, but would leave something for the others to fight over.

However, we are not going to assume the usual consumption bundle of m (infinitely divisible) goods, in our jungle model.

2. A Model of Indivisible Objects

We will now assume that each person consumes *one and only one indivisible object*. (Think for example of a house.) We assume that there are n of these objects to be distributed, 1 each, to each of the n people. We will assume that there is some initial distribution of the objects.

The general economic exchange model where n people are allocated n indivisible objects and where each person only wants to consume 1 object, was first developed by Lloyd Shapley and Herbert Scarf. Shapley and Scarf proved that such economies have core allocations, that is, allocations which are unblocked by any coalitions. They also discussed related allocation problems, such as the "roommate problem" (how to pair up n people, who have preferences about *each other*, in some optimal way), and the "marriage problem" (how to pair up n men with n women, again in some optimal way.)

In this chapter, we are considering how to allocate n indivisible objects among n people. Under barter or market exchange, each person swaps his object, in a voluntary exchange, for someone else's, or he trades it in the market for a different one. But under jungle exchange, there is a power relation which permits the more powerful to take from the less powerful.

A *jungle equilibrium* is a distribution of the n objects among the n persons such that, for every i and j, if person i is more powerful than j, person i's utility from the object in his possession is greater than or equal to the utility he would get from j's object. If this is the case, the more powerful i does not bother to force an exchange on the less powerful j.

From this point onward, we will call the indivisible objects houses. As we have done before, we let u_i represent i's utility function. The houses are labeled $h_1, h_2, \ldots, h_n$, where h_i is the house that person i starts with. An allocation of houses is a permutation of the vector $(h_1, h_2, \ldots, h_n)$. The original allocation is called h. We will use g or g' to represent alternative allocations, for example $(h_2, h_1, h_3, \ldots, h_n)$, which switches houses between persons 1 and 2.

(To translate this notation back into the notation used in Chapters 2 and 3, we would proceed as follows: Call house h_1 good 1, call h_2 good 2, etc. Each person can consume a bundle x_i of goods, comprised of exactly one unit of one good, and zero of all the rest. For example, $x_i = (0, 1, 0, \ldots, 0)$ means i is consuming the house h_2. The initial allocation is the set of unit vectors: $\omega_1 = (1, 0, 0, \ldots, 0)$, $\omega_2 = (0, 1, 0, \ldots, 0)$, and so on. An allocation is any permutation of the houses, so $x = (x_1, x_2, \ldots, x_n)$ is an allocation if each x_i is a unit vector (but not necessarily the one with a 1 at the ith place), and the sum of the x_i's is $(1, 1, \ldots, 1)$. A price vector $p = (p_1, p_2, \ldots, p_n)$ attaches a price to each house: p_i is the price on h_i, the house originally owned by person i. If person i is buying or selling according to his budget constraint, the price of any house he buys would have to be less than or equal to the price of the house he is selling; and this would give $p \cdot x_i \leq p \cdot \omega_i$, as before.)

We now revert to our house allocation notation, and proceed with formal definitions.

Let g be an allocation of houses. Let S be a subset of the set of n individuals. We will say *S blocks g* if there is some redistribution of the houses that members of S started with, redistribution that we will call g', such that $u_i(g'_i) \geq u_i(g_i)$ for all i in S, and $u_i(g'_i) > u_i(g_i)$ for at least one i in S. If no subset of individuals blocks g, we will say g is in the *core*. (Note that the definition of the core is formally much the same as it was in Chapter 2.)

Let g be an allocation of houses. We will say g is *Pareto optimal*, or *optimal* for short, if there is no alternative allocation g' such that $u_i(g'_i) \geq u_i(g_i)$ for all i, and $u_i(g'_i) > u_i(g_i)$ for at least one i. That is, g is Pareto optimal if it is not blocked by the set of all individuals

$\{1, 2, \ldots, n\}$. As in Chapter 2, it is obviously the case that if g is in the core, it is also Pareto optimal.

Let g be an allocation of houses. Let $p = (p_1, p_2, \ldots, p_n)$ be a house price vector, with p_i, the price of h_i (the house originally allocated to person i). The house going to person i under allocation g, that is, g_i, is not necessarily h_i, the one going to him originally. We will write $p(g_i)$ for the price of that particular house: that is, if i gets house 5 under the allocation g, then $p(g_i) = p_5$.

We say (g, p) is a (competitive) *market equilibrium* if, for all i, g_i maximizes u_i subject to the budget constraint $p(g_i) \leq p_i$.

Finally, we call an allocation of houses g a *jungle equilibrium* if, whenever i is more powerful than j, $u_i(g_i) \geq u_i(g_j)$. For example, if the power ordering is 1 over 2, 2 over 3, etc., then g is a jungle equilibrium if, whenever $i < j$, $u_i(g_i) \geq u_i(g_j)$.

3. A 4-Person 4-Houses Example

We now turn to an example to illustrate.

We assume there are 4 people. The initial allocation of houses is $h = (h_1, h_2, h_3, h_4)$. An arbitrary allocation of houses is a permutation of h. For example, $g = (h_2, h_1, h_4, h_3)$ swaps houses between persons 1 and 2, and also between persons 3 and 4. The number of possible allocations is the number of permutations of $n = 4$ things, which equals $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

We assume the 4 individuals have the following preferences: (As in Chapter 1, we list the houses under person i's number, in person i's order of preference).

Table 4.1a.

1	2	3	4
h_2	h_4	h_1	h_2
h_1	h_3	h_3	h_4
h_3	h_2	h_2	h_1
h_4	h_1	h_4	h_3

Therefore, for example, person 1 has the following relative utility levels: $u_1(h_2) > u_1(h_1) > u_1(h_3) > u_1(h_4)$; he likes the house originally allocated to 2 best, his own house second best, and so on.

The reader can see that these preferences have some interesting characteristics. For example, if we focus on persons 1, 2, and 3, and pretend person 4 and his house are not there, the preferences reduce to:

Table 4.1b.

1	2	3
h_2	h_3	h_1
h_1	h_2	h_3
h_3	h_1	h_2

But this looks just like the Condorcet voting paradox preferences introduced in Chapter 1. In fact, there is a nice house swapping cycle here: If these 3 people get together and talk about their preferences they will realize that person 2 can transfer his house to person 1, person 3 can transfer his house to person 2, and person 1 can transfer his house to person 3. This swap would make these 3 better off. Since such a swap exists, it is clear that the original house allocation $h = (h_1, h_2, h_3, h_4)$ is *not* in the core and is *not* Pareto optimal. Interestingly, there is another swap available with these preferences. If we focus on persons 2 and 4, and pretend that persons 1 and 3, and their houses, are not there, the preferences reduce to:

Table 4.1c.

2	4
h_4	h_2
h_2	h_4

Now we can see another obvious swap, which would make 2 and 4 better off. This again proves $h = (h_1, h_2, h_3, h_4)$ is not in the core and is not Pareto optimal. Note also that the 3-person swap and the 2-person swap cannot *both* be done, because person 2 is in both cycles. If he swaps his house in a deal with 1 and 3, he no longer has it to swap in a deal with 4.

Now let's consider which of the 24 possible allocations are efficient. As it turns out, there are 5 Pareto optimal allocations. One is the allocation that results from the 3-way swap among persons 1, 2, and 3. This is (h_2, h_3, h_1, h_4). To get a clear picture of this allocation, it is helpful to return to the 4-person, 4-house preference picture, and indicate in bold non-italics the (h_2, h_3, h_1, h_4) allocation:

Table 4.2.

$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$
$\mathbf{h}_2$	h_4	$\mathbf{h}_1$	h_2
h_1	$\mathbf{h}_3$	h_3	$\mathbf{h}_4$
h_3	h_2	h_2	h_1
h_4	h_1	h_4	h_3

Note that with (h_2, h_3, h_1, h_4), 2 people (persons 1 and 3) are getting their favorite houses, and the remaining 2 cannot make a mutually agreeable swap.

Another optimal allocation is generated by starting with the initial house distribution, and letting 2 and 4 do their swap. This would produce (h_1, h_4, h_3, h_2). This swap would be particularly notable because 2 and 4 both would be getting their top choices. We will call such a swap a "top swap" below, and we will see how such swaps and cycles can be used to derive market equilibria.

The 5 Pareto optimal allocations in this example are as follows:

$$\begin{array}{c} (h_2, h_3, h_1, h_4) \\ (h_2, h_4, h_1, h_3) \\ (h_3, h_4, h_1, h_2) \\ (h_2, h_4, h_3, h_1) \\ (h_1, h_4, h_3, h_2) \end{array}$$

The interested reader can illustrate each one of these by reproducing the original set of preferences, and, as we did in Table 4.2 above, and underlining or highlighting the houses assigned to persons 1 through 4 under each of these 5 allocations.

In their seminal paper Shapley and Scarf proved that economies like this one have non-empty cores. Since a core allocation of houses also has to be Pareto optimal, the reader can simply check the 5 Pareto optimal allocations to discover which are core allocations. For instance, (h_2, h_3, h_1, h_4) is blocked by the set $S = \{2, 4\}$. These two people can swap their original houses between themselves, giving each his favorite. For person 2, $u_2(h_4) > u_2(h_3)$, and for person 4, $u_4(h_2) > u_4(h_4)$. Therefore, (h_2, h_3, h_1, h_4) is not in the core. A few minutes of examination of the Pareto optimal allocations should convince the reader that there is one and only one core allocation in this particular example. It is (h_1, h_4, h_3, h_2).

4. Finding a Market Equilibrium with Top Cycles

Now let's consider whether or not there exists a market equilibrium of houses, given our original assignment of house 1 to person 1, house 2 to person 2, and so on. Recall that a price vector $p = (p_1, p_2, p_3, p_4)$ and an allocation of houses g is a market equilibrium if, for every person i, g_i maximizes i's utility subject to his budget constraint. His budget constraint says he can afford house g_i if its price, what we call $p(g_i)$, is less than or equal to the price of the house he starts with, p_i.

We will show how to construct a market equilibrium allocation, and an equilibrium price vector. Looking back at Tables 4.1.b and 4.1.c, we recall that these are 2 potential trading cycles in our example. The subset {1,2,3} of individuals could swap houses among themselves, making all better off, and the same is true of {2,4}.

The {2,4} swap is especially interesting, because if person 2 gives h_2 to person 4, and person 4 gives h_4 to person 2, each party is receiving his *favorite* house. (This is not the case if a swap is made within the group {1,2,3}.)

If some subset of traders can execute a swap among themselves, so that each person gets his favorite house from among all the houses that are available, we call that subset a *top trading cycle*, and we call the swap a *top swap.* A look at Table 4.1.a should convince the reader that, in that example with 4 people and 4 houses in play, $\{2, 4\}$ is the top trading cycle, and the exchange between 2 and 4 is the top swap.

The idea of a top trading cycle is easily extended beyond our 4-person 4-house example. Suppose there are n people and n houses, and assume no one is even indifferent between 2 houses. Construct a table showing orders of preference similar to Table 4.1.a. Now ignore everything in the table except for the 1st row. In our example this is:

1	2	3	4
h_2	h_4	h_1	h_2

We claim that we can always pick out a top trading cycle; no matter what the preferences and what n may be. Here is how: Start with any person i. If i likes his own house best, we are done, person i himself, that is, the set $\{i\}$, is a top trading cycle. If not, i likes someone else's house best, say h_j. Start a list with i at the left, and j next. If j likes i's house best, we are done, the set $\{i, j\}$ is a top trading cycle. If not, there must be a new person, say k, such that j likes h_k best. Add k to the list, which now reads i, j, k. Continue in this fashion. Eventually, since there are only a finite number of people and houses, the list must loop back on itself, e.g., we must have a list like i, j, k, l, m, k. Once it loops back, we have a top trading cycle, e.g., k, l, m: person k likes h_l

best, person l likes h_m best, and person m likes h_k best. The top cycle $\{k, l, m\}$ can then execute the obvious top swap.

In our 4 person example, if we arbitrarily start with person 1, we would list our people 1,2,4,2, and $\{2, 4\}$ would be revealed as a top trading cycle.

A top trading cycle may have just one person in it; for instance, if the top row of the preference table is

1	2	3	4
h_2	h_3	h_4	h_4

the top trading cycle is $\{4\}$.

Also, there can be more than 1 top trading cycle; for example, if the first row is

1	2	3	4
h_2	h_1	h_4	h_3

there are 2 top cycles, and if each person starts with his favorite house

1	2	3	4
h_1	h_2	h_3	h_4

then there are 4 top cycles.

But in any case, given n persons and n houses, there must exist (at least one) top trading cycle.

Here is how to construct a competitive equilibrium. Start with all persons and all houses. Find a top trading cycle. Assign a (single) price to each house in that 1^{st} cycle, and choose a (relatively) high price. For our example, the top trading cycle is $\{2, 4\}$, and we will let $p_2 = p_4 = 3$. Next, remove persons 2 and 4 and houses h_2 and h_4 from the lists of persons and houses. Focusing on the remaining persons and houses, construct a preference table. For our example, this is

Table 4.3.

1	3
h_1	h_1
h_3	h_3

In this remaining population and set of houses, find a top trading cycle. In our example, it is $\{1\}$. Assign a (single) price to each house in this cycle, and choose a price lower than the price chosen previously. For instance, set $p_1 = 2$. Next, remove this person and house from the lists of persons and houses, and repeat. In the next round, choose a price lower than the previously chosen price. In our example, for instance, choose $p_3 = 1$.

Now let the above constructed prices be the market prices (e.g., set $p = (2, 3, 1, 3)$), and let traders "go to the market" with their original houses. That is, let them choose utility maximizing houses subject to their budget constraints based on these prices.

With respect to person 1, he starts with a house worth 2. Table 4.1.a shows he would most like h_2, but it costs 3, and he cannot afford it. Of the houses he can afford, he likes his original house h_1 best; he buys it. Persons 2 and 4 each start with houses worth 3. Person 2 likes h_4 best, he can afford it; he buys it. Person 4 likes h_2 best, he can afford it; he buys it. Person 3 starts with a house worth 1. He would prefer h_1, but he cannot afford it, in fact the only house he can afford is h_3. He buys it. The result is that each person is buying the house he likes best, subject to his budget constraint, and all the houses get allocated to all the people.

In short, $p = (2, 3, 1, 3)$ and the house allocation (h_1, h_4, h_3, h_2) comprise a market equilibrium.

5. Fundamental Theorems of Welfare Economics and Jungle Economics

Recall the first fundamental theorem of welfare economics from Chapter 3. It says that a competitive equilibrium allocation is in the core, and is therefore Pareto optimal. The Chapter 3 assumptions are somewhat different than the assumptions being made here, but the market-equilibrium–implies–core-implies-Pareto-optimality result survives.

For the purposes of the results to follow we are assuming that n houses are being distributed among n people, and that no person is indifferent between any pair of houses. (The assumption of no-indifference is crucial here; without it, we can construct market equilibrium allocations that are not Pareto optimal. See Roth and Postlewaite (1977).) We now have the following in the house allocation model:

> *First Fundamental Theorem of Welfare Economics.* Let (p, g) be a market equilibrium allocation of houses. Then g is in the core, and is Pareto optimal.
>
> *Proof*: Omitted.

Let us reflect for a moment on the fact that the market allocation of houses is in the core. This means that market exchange captures the essence of being voluntary, of being non-coercive. Every possible coalition has to acquiesce to a proposal for it to be in the core. Given the initial allocation of houses, and given the preferences, there are many Pareto optimal allocations (5 in our numerical example), but far fewer core allocations (1 in our numerical example). The competitive allocation must not only be Pareto optimal, it must also be in the core.

At this point we can return to our thoughts about the jungle.

Are there jungle equilibria? Of course there are. To find one, first determine the power relation. In our example, suppose the power relation is 1, 2, 3, 4; meaning 1 is strongest, 2 is second strongest, and so on. Consider an allocation constructed as follows: Ask person 1 which house he wants most. The answer is h_2. Assign h_2 to person 1 and remove person 1 and h_2 from the lists of persons and houses. Next ask person 2 which house he likes most, of the set of remaining houses, $\{h_1, h_3, h_4\}$. The answer is h_4. Assign h_4 to person 2 and remove person 2 and h_4 from the lists. Next, ask person 3 which house he likes most, of the remaining houses, $\{h_1, h_3\}$. The answer is h_1. Assign it to him, and remove person 3 and h_1 from the lists. Finally, ask person 4 which house he likes most of the one left, and the answer is h_3. Assign it to him. This process produces the allocation (h_2, h_4, h_1, h_3).

This is obviously a jungle equilibrium under the assumed power relation: person 1 likes his house h_2 more than the houses of his inferiors 2, 3, and 4. Person 2 likes his house h_4 more than the houses of his inferiors 3 and 4, and so on.

This procedure can easily be followed for any power relation among the 4 individuals. The number of such power relations, like the number of house allocations, is $4 \cdot 3 \cdot 2 \cdot 1 = 24$, since there are four ways to name the most powerful, and having named the most powerful there are 3 ways to name the second most powerful, and so on.

But the jungle equilibrium outcome is the same for many power relations. The reader can check, for example, that if the power relation is 1, 2, 3, 4, the jungle equilibrium is (h_2, h_4, h_1, h_3), and if the power relation is 1, 3, 2, 4, the equilibrium is the same.

In fact, a mechanical examination of all 24 power relations reveals that there are 5 jungle equilibria, and they are exactly the 5 Pareto optimal allocations. (It turns out that, in general, one way to identify all the Pareto optimal allocations is to follow this so-called serial dictatorship procedure; see, for example, Abdulkadiroglu and Sonmez (1998).)

There is a jungle theorem for the house allocation model that corresponds to, but is weaker than, the first fundamental theorem of welfare economics. Piccone and Rubinstein established that "efficiency also holds in the jungle." Their proof is for a divisible-goods model, and is more complex than the one that follows. We now turn to our version:

> *First Fundamental Theorem of Jungle Economics*. Let g be a jungle equilibrium allocation. Then g is Pareto optimal.
>
> *Proof*: Suppose not. Then g is Pareto-dominated by another allocation g'. In a hypothetical move from g to g' some individuals would be better off and others would stay the same. An individual would keep the same utility under our assumptions if and only if he kept the same house. We will ignore those individuals. Let V be the individuals who are made better off by a hypothetical move from g to g'. They are all evidently getting different houses. But their houses are coming from within the group.
>
> Since houses are being shifted within V itself, and since everyone in V is being made better off by the hypothetical move from g to g', there must exist a subset of V, which we will denote $\{a, b, c\}$, such that, in the move from g to g', a's house shifts to b; b's house shifts to c, and c's house shifts to a. (We are assuming a 3-person cycle to illustrate; the actual cycle has to have at least 2 people, and at most n. Our argument obviously applies to cycles of length 2 through n.) In this hypothetical 3 way switch, every person is getting a house he likes better (under $g\prime$) than the one he had (under g).
>
> Now consider the power relation. It must make one of $\{a, b, c\}$ the most powerful of that subset. Say it is a. But in the hypothetical shift from g to g', he's getting c's house, which he likes better than the one he has. But c is a's inferior in the power relation.
>
> This is a contradiction, because if g is a jungle equilibrium, and person c is person a's inferior in the power relation, a cannot possibly envy c's house: if he did envy it he would already have taken it under the law of the jungle. Q.E.D.

Note the difference between the market result and the jungle result. The market leads to the core, which is based on the initial allocation of houses plus voluntary trade. Since it leads to the core, it *ipso facto* also leads to a Pareto optimal outcome. The jungle also leads to a Pareto optimal outcome, but certainly not to the core, because the essence of the jungle is power, and power overrides voluntary transactions based on initial endowments.

At this point we consider whether or not there is a second fundamental theorem of jungle economics. Such a theorem would say that if g is any Pareto optimal allocation of houses, there must be a power relation such that g is a jungle equilibrium.

The interested reader can check the 5 Pareto optimal distributions of houses are also jungle equilibrium allocations for some power relation among the individuals in our example. To see that this is in fact the case, consider (h_2, h_3, h_1, h_4), for instance. For a power relation, let person 1 be most powerful (he chooses h_2); let person 3 be second most powerful (he chooses h_1); person 4 be third most powerful (he chooses h_4, since his superior 1 has already claimed h_2); and let person 2 be at the end of the list (he takes the remaining house h_3). So if the power relation is 1, 3, 4, 2, the jungle equilibrium will be the Pareto optimal house allocation (h_2, h_3, h_1, h_4).

While a Pareto optimal allocation is not necessarily a jungle equilibrium in general models with divisible goods, it turns out that in our house allocation model, every Pareto optimal allocation must in fact be a jungle equilibrium allocation. This is our second theorem of the jungle:

> *Second Fundamental Theorem of Jungle Economics.* Let g be a Pareto optimal allocation of houses. Then there exists a power relation such that, given that power relation, g is a jungle equilibrium allocation.

> *Proof*: (Following Piccione and Rubinstein). Assume g is an optimal allocation. Write iTj if, under the allocation g, person i would prefer person j's house, i.e. $u_i(g_j) > u_i(g_i)$. In this case we say "i envies j." (For more on the concept of envy, see our discussion of "fairness" in Chapter 10.)
>
> Note that the relation T cannot cycle. For instance, we cannot have iTj, jTk, and kTi. If we did, we could arrange a swap that benefits all people in the cycle. This would contradict the assumption that g is a Pareto optimal distribution of houses.
>
> We proceed in stages. In stage 1, we separate $\{1, 2, \ldots, n\}$ into two subsets; those who are envied by someone, and those whom no one envies. Call the latter set B_1 (for bottom-set 1). We claim that B_1 is nonempty. If not, any person k would be envied by at least one other person j. So, jTk. In turn, j would be envied by another person, say i. So, iTj. Repeating this argument would eventually produce a cycle, such as iTj, jTk, and kTi, which would be a contradiction. The individuals in B_1 will be placed (in any order) at the bottom of the power relation. In stage 2, we consider $\{1, 2, \ldots, n\} \setminus B_1$ (the set of remaining persons after removing those in B_1). We separate this set into two subsets: those who are envied by someone in this set, and those whom no one in this set envies. Call this latter set B_2 (for bottom-set 2). By arguments similar to those made above for B_1, B_2 must be nonempty. The individuals in B_2 will be placed (in any order) next to the bottom of the power relation (i.e., above the B_1 people, but below everyone else).

Continue in this fashion until everyone has been separated out. Given the power relation so defined, it is clear that iTj only if j is higher in the power relation than i. It follows that g is a jungle equilibrium allocation. Q.E.D.

We will finish this section by illustrating the logic of the second fundamental theorem of jungle economics with our example, as shown in Table 4.1.a.

Consider the Pareto optimal allocation (h_2, h_4, h_3, h_1). With this allocation, persons 2 and 4 are getting their favorite houses. Therefore they envy no one. Person 3 is getting h_3, and he envies person 4, who is getting h_1. Therefore $3T4$. Person 4 envies persons 1 and 2, and so $4T1$ and $4T2$. Nobody envies 3, but each of the others is envied by someone. Therefore place 3 at the bottom of the power relation. Then, looking only at persons 1, 2, and 4, we do this again. Again, 1 and 2 are getting their favorite houses, so they envy no one, and in particular no one envies person 4. But we still have $4T1$ and $4T2$. Now place 4 next to the bottom of the power relation, above person 3, but below the remaining pair 1 and 2. Then, looking only at persons 1 and 2, repeat the process. Neither envies the other; so place them at the top, in any order.

The conclusion of all of this is that the optimal house allocation (h_2, h_4, h_3, h_1) is a jungle equilibrium from any power relation consistent with the above argument. There are two such power relations. Going as usual from left (most powerful) to right (least powerful), they are 1, 2, 4, 3, and 2, 1, 4, 3.

6. Exercises

1 For the 4 person, 4 house example in the text, show why (h_2, h_4, h_1, h_3), (h_3, h_4, h_1, h_2), and $(h_2, h_4, h_3, h_1,)$ are not in the core.

2 For the 4 person, 4 house example in the text, show that if the power relation is 1,2,3,4, the jungle equilibrium is (h_2, h_4, h_1, h_3), and if the power relation is 1,3,2,4, the jungle equilibrium is again (h_2, h_4, h_1, h_3).

What is the jungle equilibrium if the power relation is 4,3,2,1?

3 Consider the 4 person, 4 house example in the text. For each of the 5 Pareto optimal house allocations, find a power relation for which the jungle equilibrium is the given Pareto optimal allocation.

7. Selected References

(Items marked with an asterisk (*) are mathematically difficult.)

*1. A. Abdulkadiroglu and T. Sonmez, "Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problems," *Econometrica* V. 66, 1998, pp. 689-701.

This is an interesting article that uses the characterization of Pareto optimal allocations in terms of outcomes of serial dictatorships to obtain its main result, which concerns the core of an economy in which endowments are allocated at random.

*2. M. Piccione and A. Rubinstein, "Equilibrium in the Jungle," Mimeo, 2005.

This is an innovative and thorough analysis of jungle economics in various alternative models, including the house allocation model.

*3. A. Roth and A. Postlewaite, "Weak versus Strong Domination in a Market for Indivisible Goods," *Journal of Mathematical Economics* V. 4, 1977, pp. 131-137.

This articles studies versions of Pareto optimality and the core based on strict improvements (if all members of the coalition improve) or weak improvements (if only some do). In particular, the authors construct examples showing violations of the conclusions of the first welfare theorem if indifference is allowed.

*4. L. Shapley and H. Scarf, "On Cores and Indivisibility," *Journal of Mathematical Economics*, V. 1, 1974, pp. 23-28.

A mathematically rigorous treatment which includes helpful non-mathematical discussions of several different exchange models with indivisibilities.