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Cores of combined games ☆

Francis Bloch [a], Geoffroy de Clippel [b,*]

[a] *Ecole Polytechnique, France*
[b] *Brown University, Department of Economics, 64 Waterman Street, Providence, RI, United States*

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Abstract

This paper studies the core of combined games, obtained by summing different coalitional games when bargaining over multiple independent issues. It is shown that the set of balanced transferable utility games can be partitioned into equivalence classes of component games to determine whether the core of the combined game coincides with the sum of the cores of its components.

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1. Introduction

The broad subject of the paper is the study of bargaining and cooperation when multiple issues are at stake. We have two complementary objectives in mind:

1. Identify conditions under which negotiating over different issues separately is equivalent to negotiating over these issues simultaneously;
2. Identify situations in which combining issues reduces conflict in bargaining.

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* Corresponding author.
E-mail addresses: francis.bloch@polytechnique.edu (F. Bloch), declippel@brown.edu (G. de Clippel).

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We use games in coalitional form, a classical model to study cooperation, to tackle these two questions. The coalitional function specifies for each coalition the surplus to be shared should its members cooperate. The simplicity of this reduced-form approach, making no direct reference to the underlying social or economic alternatives, comes at a cost. Indeed, relating the cooperative opportunities associated to different issues to the cooperative opportunities of the combined issues is possible in this framework only if the different issues are independent. In such cases, the coalitional function associated to the combined issues is simply the sum of the coalitional functions associated to each issue taken separately. Spillovers are certainly an important feature of multi-issue bargaining, and further analysis of nonwelfarist models is needed to understand their implication. The present paper illustrates that *bargaining over multiple issues may have relevant implications even in the absence of such spillovers*.

Multi-issue bargaining was of central importance to Professor Shapley when studying values for games in coalitional form, as illustrated by his motivation for the additivity axiom: "*The third axiom ("law of aggregation") states that when two independent games are combined, their values must be added player by player*" (Shapley, [27, page 309]).

Put differently, additivity implies that the outcome of multi-issue negotiations does not depend on the agenda chosen by the negotiators. Whether issues are discussed separately or "packaged" in different ways does not affect the result of the negotiation. In Professor Shapley's view, this agenda independence is a natural requirement to impose on a solution concept.

However, the Shapley value is the only solution concept for which additivity is posited as an axiom. Other solution concepts, whether they are based on alternative axiomatizations, like the Nash bargaining solution, or more positive considerations, like the core, do not satisfy this property of agenda independence. In this paper, we focus attention on the core primarily because of its importance in economic theory. Other solution concepts are briefly discussed in Section 5.

It is well known that the core is superadditive (see for example, Peleg's [20] axiomatization of the core), so that the core of the combination of two games is always larger than the sum of the core of the two components. Intuitively, by combining two negotiation processes, and forcing players to make coalitional objections on the issues simultaneously, it is easier to sustain an imputation than when players can make separate objections on the two issues. Hence, the specific question we tackle in this paper is the following: For which pairs of games is the core of the combination of the two games exactly equal to the sum of the core of the component games? This offers a formal statement to the first objective listed at the beginning of the paper.

Our main result shows that the core of the sum of two games v and w is equal to the sum of the cores of v and w if and only if the extreme points of the cores of v and w are defined by the same set of coalitional constraints. Because the latter property defines an equivalence relation among games, we conclude that *the set of all balanced transferable utility games can be partitioned into equivalence classes such that the core of the combination of two games is equal to the sum of the cores of the components if and only if the two games belong to the same class*. One of these equivalence classes (where the extreme points are determined by any increasing sequence of coalitions) is the set of convex games introduced by Shapley [29]. Hence, the combination of two convex games does not result in an expansion of the set of core allocations. By contrast, whenever two games v and w are taken from two different equivalence classes, the core of the combined game is strictly greater than the sum of the core of its components. When v and w are close, a simple continuity argument shows that the difference between the core of $v + w$ and the sum of the cores of v and w is small. In other cases, the difference can be extremely large, as the dimension of the core of $v + w$ may exceed the dimension of the sum of the cores (for example, even when the cores of v and w are singletons, the core of $v+w$ may be a set of full dimension in the set of imputations).

To the best of our knowledge, the only previous studies of the additivity of the core in the cooperative game theoretic literature are due to Tijs and Branzei [30]. They identify three subclasses of games on which the core is additive (including the class of convex games). Our results complement and extend their analysis by showing that in fact *the entire set of balanced games can be partitioned into subclasses on which the core correspondence is additive.* The literature on noncooperative games has paid more attention to simultaneous, multi-issue bargaining. In a two-player setting, Fershtman [12] and Busch and Hortsmann [4] extend Rubinstein's [26] alternating offers game to a multi-issue setting, where players bargain over each issue in a predefined sequence. They show that the equilibria of this multi-issue bargaining differ considerably from the single-issue model. In later contributions to this literature, Bac and Raff [1], Inderst [15] and In and Serrano [14] allow players to endogenously choose on which issue to bargain, and show that players have an incentive to manipulate strategically the agenda. Issue linkage has also been studied in noncooperative games representing international negotiations across countries. It has long been argued that combining negotiations over different dimensions (trade, protection of the environment) may have beneficial effects (see for example Carraro and Siniscalco, [5]). Conconi and Perroni [7] propose a model of issue linkage and evaluate this argument using a parameterized model of international trade and environmental negotiations. Issue linkage also appears implicitly in the literature on mergers in Industrial Organization (e.g. Perry and Porter [22] and Farrell and Shapiro [11]). In order to be profitable, a merger must involve two dimensions — both a cost and a market dimensions — and result in cost synergies as well as market concentration.

As an exact characterization of situations where the core of a combined game equals the sum of the cores of its components, our main result is also useful to determine when combining issues reduces conflict (cf. the second objective listed in the first paragraph), namely when the core of the sum of games is strictly larger than the sum of their separate cores. Perhaps even more interestingly, it is easy to construct examples where the core of the sum of two games with an empty core is nonempty. In such cases, bargaining over each component would lead to an impasse or to partial cooperation, but efficiency can be recovered (on both components) by combining the issues. Unfortunately, our characterization of the set of games for which the core is additive does not carry over to games with empty cores. The binary relation associating two games v and w whose combination has an empty core is not transitive. This is easily understood: for two games v and w to be such that the combined game $v+w$ has an empty core, it is sufficient that one of the balanced[1] collections of coalitions has a worth exceeding the worth of the grand coalition in both games v and w. Now consider a triple of games v, w, z. The worth of the balanced collection $\mathcal{C}$ may exceed the worth of the grand coalition in both v and w and the worth of the balanced collection $\mathcal{D}$ may exceed the worth of the grand coalition in both w and z. However, v and z may very well not share any balanced collection whose worth exceeds the grand coalition, and be such that the core of $v+z$ is nonempty. Put differently, for a game to be unbalanced, one only requires one of the balanced collection to have a greater worth than the grand coalition, so that the set of games with empty cores is not defined by a set of linear inequalities, and is in fact typically *not convex*. In spite of this, it is possible identify a convex subset of the class of unbalanced games which has the following property: for any game in that class, the combination of this game with any other game with empty core also has an empty core. Intuitively, this subset

[1] As in Bondareva [3] and Shapley [28] — the reader is reminded of the formal definition in Section 2.

contains those games which are hardest to "balance" with other games, and may create the more difficulties in negotiations. The interested reader is referred to the working paper version of this article (Bloch and de Clippel [2, Section 4]).

The rest of the paper is organized as follows. In the next section, we recall the standard definitions of coalitional games and the core. In Section 3, we analyze the combination of games with nonempty cores. We state and prove our main characterization result. In Section 4, we illustrate the result by looking at four-player symmetric games. Section 5 contains our final remarks and conclusions.

2. Preliminaries

Let N be a set of players. A *cooperative game* is described by a *coalitional function* v which assigns to every nonempty subset S of N a real number, $v(S)$, called the *worth of the coalition*. Games will be assumed to be *superadditive*: $v(S \cup T) \geqslant v(S) + v(T)$, for any two disjoint coalitions S and T. We denote the set of all such n-player games by $\Gamma(n)$. A game is *convex* if the players' marginal contributions are nondecreasing: $v(S \cup \{i\}) - v(S) \leqslant v(T \cup \{i\}) - v(T)$, for each pair (S, T) of coalitions such that $S \subseteq T$.

An *imputation* is a vector $x \in \Re^N$ that is feasible, efficient, and individually rational: $\sum_{i \in N} x_i = v(N)$ and $x_i \geqslant v(\{i\})$, for each $i \in N$. The *core* of a cooperative game v is the set of payoff vectors $x \in \Re^N$ that are feasible when all the players cooperate, and which cannot be improved upon by any coalition: $\sum_{i \in N} x_i \leqslant v(N)$ and $\sum_{i \in S} x_i \geqslant v(S)$ for each coalition S. Let A be the $(2^n - 1) \times n$ matrix encoding coalitional membership: $A_{S,i} = 1$ if $i \in S$ and $A_{S,i} = 0$ if $i \notin S$, for each coalition S and each player i. Then,

$$C(v) = \left\{ x \in \Re^N \,\middle|\, \sum_{i \in N} x_i = v(N),\ Ax \geqslant v \right\}.$$

This rewriting highlights the fact that *the core is a bounded convex polyhedron* defined by a system of linear inequalities. As any such set, the core is characterized by its set of extreme points — points which cannot be obtained as convex combinations of other points in the set. Equivalently, a payoff vector x is an extreme point of the core of v if there exists a collection $(S_k)_{k=1}^n$ of coalitions such that $\sum_{i \in S_k} x_i = v(S_k)$, for each k, and these n equations are linearly independent.

The system of linear inequalities defining the core may be inconsistent, in which case the core is empty. Bondareva [3] and Shapley [28] proposed a characterization of games with nonempty core based on balanced collections of coalitions. A collection $(S_k)_{k=1}^K$ of coalitions is *balanced* if there exists a collection $(\delta_k)_{k=1}^K$ of real numbers between 0 and 1 (called *balancing weights*) such that $\sum_{k | i \in S_k} \delta_k = 1$, for each $i \in N$. A game v is *balanced* if and only if $\sum_k \delta_k v(S_k) \leqslant v(N)$, for each balanced collection $(S_k)_{k=1}^K$ of coalitions and each collection $(\delta_k)_{k=1}^K$ of balancing weights. The core of a game v is nonempty if and only if the game v is balanced. The set of all balanced superadditive n-player games is denoted $\beta(n)$.

3. Combining balanced games

In this section, we consider two balanced, superadditive games v and w and investigate conditions under which the core of $v + w$ is equal to the sums of the cores of v and w. We show that the set of transferable utility games can essentially[2] be partitioned into equivalence classes of games, such that the core of the combined game is equal to the sum of the cores of the component games if and only if the two component games belong to the same class. We will prove this statement as a corollary to a general result on convex polyhedra.

We define an equivalence relation between two bounded convex polyhedra $P(A,b)=\{x\in\Re^N \mid Ax\geqslant b\}$ and $P(A,b')=\{x\in\Re^N \mid Ax\geqslant b'\}$ if the extreme points of the two polyhedra are defined by the same constraints. To gain some intuition, we consider the simpler and generic[3] case where every extreme point of $P(A,b)$ and $P(A,b')$ is characterized by exactly N equalities. If $P(A,b)+P(A,b')=P(A,b+b')$, then any extreme point of $P(A,b+b')$ can be decomposed as the sum of two elements of $P(A,b)$ and $P(A,b')$. These vectors have to be extreme points of the polyhedra $P(A,b)$ and $P(A,b')$, and furthermore neither $P(A,b)$ nor $P(A,b')$ can possess additional extreme points. This shows that, whenever $P(A,b)+P(A,b')=P(A,b+b')$, the extreme points of $P(A,b)$ and $P(A,b')$ must be defined by the same constraints. To prove the converse statement, we need to show that, when extreme points are defined by the same constraints, $P(A,b+b')\subset P(A,b)+P(A,b')$ (the other inclusion being always trivially true). This is proven by using duality theory, and proving that the support function of the convex set $P(A,b+b')$ is everywhere below the sum of the support functions of $P(A,b)$ and $P(A,b')$.[4]

The equivalence relation described in the previous paragraph captures most of the cases where the additivity property holds, but not all. The general result states that $P(A,b)+P(A,b')=P(A,b+b')$ if and only if one can construct sequences of b^k and b'^k converging to b and b' such that $P(A,b^k)$ and $P(A,b'^k)$ are equivalent for all k. Applying this lemma to the core of cooperative games, we obtain the following result.

Proposition. *Consider the equivalence relation $\mathcal{R}$ on $\beta(n)$, where $v\mathcal{R}w$ if and only if the extreme points of $C(v)$ and $C(w)$ are defined by the same constraints. Then $C(v)+C(w)=C(v+w)$ if and only if there exist two sequences of games v^k and w^k in $\beta(n)$ that converge to v and w respectively, and such that $v^k\mathcal{R}w^k$, for all k. In the generic case where exactly n coalitional constraints are binding at each extreme point of the core of both v and w, we have that $C(v)+C(w)=C(v+w)$ if and only if $v\mathcal{R}w$.*[5]

The proposition is a direct corollary of a more general result on the sum of convex polyhedra that we state and prove in the next lemma. For each positive integers M and N, let $\mathcal{A}_{M,N}$ be the set of couples (A,b), where A is an $(M\times N)$-matrix and b is an M-vector such that $P(A,b)=\{x\in\Re^N \mid Ax\geqslant b\}$ is nonempty and bounded. For each extreme point of $P(A,b)$, let $M_e(A,b)$ be the set of binding constraints at e, i.e. $M_e(A,b)=\{m\in\{1,\dots,M\}\mid A_m e=b_m\}$. Two vectors b and b' are *equivalent* (given A), $b\sim b'$, if there exists a bijection f between the set of extreme points of $P(A,b)$ and the set of extreme points of $P(A,b')$ such that $M_e(A,b)=M_{f(e)}(A,b')$, for each extreme point e of $P(A,b)$.

2 This statement only holds for generic games, as discussed below.

3 If $P(A,b)$ has an extreme point with more than N binding inequalities, then at least one of these equations can be written as a linear combination of the other equations, which implies that b satisfies at least one affine equation and is thus contained in a hyperplane, a nongeneric feature.

4 See Rockafellar [25, Section 13, p. 112] for a description of support functions and their usefulness in duality theory.

5 For each set $\mathbb{S}$ of coalitions, the set of games in $\beta(n)$ that have an extreme point of the core for which the set of binding constraints is exactly $\mathbb{S}$ forms a convex cone in $\mathbb{R}^{2^n-1}$. The equivalence classes defined by $\mathcal{R}$ are thus the intersection of convex cones, and thus form cones as well.

Lemma 1. *Let (A, b) and (A, b') be two elements of $\mathcal{A}_{M,N}$, for some integers M and N. The two following properties are equivalent*:

1. $P(A, b + b') = P(A, b) + P(A, b')$.
2. *There exist two sequences $(b^k)_{k\in\mathbf{N}}$ and $(b'^k)_{k\in\mathbf{N}}$ in $\Re^N$ such that $(b^k)_{k\in\mathbf{N}}$ converges to b, $(b'^k)_{k\in\mathbf{N}}$ converges to b', (A, b^k) and (A, b'^k) belong to $\mathcal{A}_{M,N}$, and $b^k \sim b'^k$ for each $k \in \mathbf{N}$.*

The proof of Lemma 1 itself requires another lemma.

Lemma 2. *Let α be a strictly positive real number, and let (A, b) and (A, b') be two elements of $\mathcal{A}_{M,N}$, for some integers M and N. If $P(A, b+b') = P(A, b) + P(A, b')$, then $P(A, \alpha b+b') = P(A, \alpha b) + P(A, b')$.*

Proof of Lemma 2. It is always true that $P(A, \alpha b) + P(A, b') \subseteq P(A, \alpha b + b')$. So we have to prove the other inclusion. We first assume that $\alpha > 1$. Let x be an element of $P(A, \alpha b + b')$. Consider the correspondence $F : P(A, b') \to 2^{P(A,b')}$ defined as follows:

$$F(y') = \left\{ z' \in P(A, b') \;\middle|\; \big(\exists z \in P(A, b)\big)\colon\ z + z' = \frac{x - y'}{\alpha} + y' \right\},$$

for each $y' \in P(A, b')$. Observe that $A(\frac{x-y'}{\alpha} + y') \geqslant b + b'$ (the total coefficient of y', $\frac{\alpha-1}{\alpha}$, is positive because $\alpha > 1$). Hence F is nonempty valued. It is easy to check that it is also convex-valued, and has a closed graph. Kakutani's fixed point theorem implies that there exists y' in $P(A, b')$ such that $y' \in F(y')$. Hence $\frac{x-y'}{\alpha} \in P(A, b)$, and $x = (x - y') + y' \in P(A, \alpha b) + P(A, b')$.

Suppose now that $\alpha < 1$. We have: $P(A, \alpha b) + P(A, b') = \alpha P(A, b) + \alpha P(A, \frac{b'}{\alpha}) = \alpha[P(A, b) + P(A, \frac{b'}{\alpha})] = \alpha P(A, b + \frac{b'}{\alpha}) = P(A, \alpha b + b')$. The penultimate equality follows from the previous paragraph. The other equalities are straightforward. □

Proof of Lemma 1. $(1 \Rightarrow 2)$ For each $k \in \mathbb{N}$, let $b^k = \frac{k}{k+1}b + \frac{1}{k+1}b'$ and $b'^k = \frac{1}{k+1}b + \frac{k}{k+1}b'$. Notice that if e^k is an extreme point of $P(A, b^k)$, then there exists a unique extreme point x of $P(A, b)$ and a unique extreme point x' of $P(A, b')$ such that $e^k = \frac{k}{k+1}x + \frac{1}{k+1}x'$. In addition, $M_{e^k}(A, b^k) = M_x(A, b) \cap M_{x'}(A, b')$. Indeed, if e^k is an extreme point of $P(A, b^k)$, then there exists a set L of N independent lines such that $A_L e^k = b^k_L$. By Lemma 2, there exist $x \in P(A, b)$ and $x' \in P(A, b')$ such that $e^k = \frac{k}{k+1}x + \frac{1}{k+1}x'$. It must be that $A_L x = b_L$ and $A_L x' = b'_L$. So x and x' are the unique vectors in $P(A, b)$ and $P(A, b')$ whose weighted sum coincides with e^k. It must also be that x and x' are extreme points of $P(A, b)$ and $P(A, b')$, respectively. Finally, $A_m e^k = b^k_m$ if and only if $A_m x = b_m$ and $A_m x' = b'_m$ (the necessary condition follows from the fact that $x \in P(A, b)$ and $x' \in P(A, b')$). Conversely, observe that if there exists an extreme point x of $P(A, b)$ and an extreme point x' of $P(A, b')$ such that $M_x(A, b) \cap M_{x'}(A, b')$ contains N independent lines, then $\frac{k}{k+1}x + \frac{1}{k+1}x'$ is an extreme point of $P(A, b^k)$. A similar argument holds to show that $\frac{1}{k+1}x + \frac{k}{k+1}x'$ is an extreme point of $P(A, b'^k)$.

For each extreme point e^k of $P(A, b^k)$, let $f(e^k)$ be the vector $\frac{1}{k+1}x + \frac{k}{k+1}x'$, where x is the unique extreme point of $P(A, b)$ and x' is the unique extreme point of $P(A, b')$ such that $e^k = \frac{k}{k+1}x + \frac{1}{k+1}x'$. The previous paragraph implies that $f(e^k)$ is an extreme point of $P(A, b'^k)$. It also implies that f is a bijection, and that $M_{e^k}(A, b^k) = M_{f(e^k)}(A, b'^k)$, for each extreme point

e^k of $P(A, b^k)$. We thus have established Condition 2, since $(b^k)_{k\in\mathbb{N}}$ converges to b, and $(b'^k)_{k\in\mathbb{N}}$ converges to b'.

($2 \Rightarrow 1$) Consider the correspondence ϕ associating to any vector b the nonempty bounded convex polyhedron $P(A, b)$. Because $\lambda P(A, b) + (1 - \lambda)P(A, b') \subseteq P(A, \lambda b + (1 - \lambda)b')$, the graph of ϕ is convex, and by Corollary 9.2.3 in Peleg and Sudhölter [21], the correspondence ϕ is lower hemi continuous. Because $P(A, b)$ is defined by a set of inequalities, the correspondence ϕ is clearly upper hemi continuous, and hence fully continuous. A simple limit argument thus implies that we will be done with the proof of the sufficient condition after showing that $P(A, b + b') \subseteq P(A, b) + P(A, b')$ for each pair (b, b') of M-vector such that $b \sim b'$. It is always true that $P(A, b) + P(A, b') \subseteq P(A, b + b')$. So we have to prove the other inclusion.[6]

Recall that, for any p in the barrier cone of a convex set C, the support function of C is defined by

$$\sigma_C(p) = \sup_{x\in C} p \cdot x$$

and that a convex set is fully characterized by its support function so that $C \subseteq D$ if and only if $\sigma_C(p) \leqslant \sigma_D(p)$ for all p in the barrier cone of C and D (Rockafellar [25, Corollary 13.1.1, p. 113]).

Let $Q = P(A, b)$, $Q' = P(A, b')$ and $Q^* = P(A, b + b')$. Because the polyhedra Q, Q', $Q + Q'$ and Q^* are bounded, their barrier cones are identical and equal to $\Re^n$. We will show

$$\sigma_{Q^*}(p) \leqslant \sigma_{Q+Q'}(p) \quad \text{for all } p \in \Re^n.$$

First note that $\sigma_{Q+Q'}(p) = \sigma_Q(p) + \sigma_{Q'}(p)$ (Rockafellar [25, p. 113]), so it suffices to show that $\sigma_{Q^*}(p) \leqslant \sigma_Q(p) + \sigma_{Q'}(p)$, which will follow from LP duality.

Let x be an extreme point solution to the problem: Maximize $p \cdot z$ subject to $Az \geqslant b$. Obviously such a solution exist. Let $I = M_x(A, b)$ denote the set of constraints satisfied with equality at the extreme point x. Applying LP duality, there exists a vector $y \in \Re^M$ (a solution to the LP dual) such that:

$$\begin{aligned}
&A_m x = b_m \quad \text{for each } m \in I,\\
&A_m x > b_m \quad \text{for each } m \in M \setminus I,\\
&y_m \leqslant 0 \quad \text{for each } m \in I,\\
&y_m = 0 \quad \text{for each } m \in M \setminus I,\\
&A^T y = p,
\end{aligned}$$

and $\sigma_Q(p) = b \cdot y$. Since $b \sim b'$, there exists an extreme point x' of Q' for which the pair (x', y) satisfies precisely the same conditions when b is replaced with b' and x is replaced with x'. Applying LP duality again, it follows that y also solves the dual to the problem maximize $p \cdot z$ subject to $Az \geqslant b'$ and $\sigma_{Q'}(p) = b' \cdot y$.

Now consider the dual of the problem: Maximize $p \cdot z$ subject to $Az \geqslant b + b'$. Clearly, y is feasible for the dual, and since $\sigma_{Q*}(p)$ is the optimal value of that dual problem,

$$\sigma_{Q^*}(p) \leqslant (b + b') \cdot y = \sigma_Q(p) + \sigma_{Q'}(p),$$

concluding the proof. □

[6] We thank the associate editor for providing a shorter argument to prove this inclusion, applying the duality principle of linear programming. The interested reader will find an alternative inductive argument that does not require any prior knowledge of linear programming in Bloch and de Clippel [2].

4. Four-player symmetric games

We illustrate the partition of the set of balanced games into equivalence classes by considering normalized four-player symmetric games — $N = \{1, 2, 3, 4\}$, $v(N) = 1$ and $v(\{i\}) = 0$, for each $i \in N$. Let v_2 denote the value of two-player coalitions and v_3 the value of three-player coalitions. Superadditivity requires that $v_2 \in [0, 1/2]$ and $v_3 \in [v_2, 1]$.

We characterize (up to a permutation) the different categories of extreme points, and the conditions on the games for which those extreme points belong to the core. We restrict attention to the generic case where each extreme point is characterized by a set of three coalitions (in addition to N) for which the inequalities are binding. By superadditivity, we can restrict attention to coalitions which have a nonempty intersection — if two coalitions S and T with $S \cap T = \emptyset$ are used, this must imply that $v(S \cup T) = v(S) + v(T)$, a nongeneric condition. Furthermore, we only have to consider collections of coalitions for which the conditions are independent. This leaves us with the following possible extreme points:

E1 Coalitions $\{1\}$, $\{1, 2\}$, and $\{1, 2, 3\}$ lead to the extreme point $(0, v_2, v_3 - v_2, 1 - v_3)$. This vector belongs to the core if and only if $v_3 \geqslant 2v_2$ and $1 \geqslant 2v_3 - v_2$.

E2 Coalitions $\{1\}$, $\{1, 2\}$, and $\{1, 3\}$ lead to the extreme point $(0, v_2, v_2, 1 - 2v_2)$. This vector belongs to the core if and only if $v_2 \leqslant \frac{1}{3}$ and $2v_2 \geqslant v_3$.

E3 Coalitions $\{1\}$, $\{1, 2, 3\}$, and $\{1, 2, 4\}$ lead to the extreme point $(0, 2v_3 - 1, 1 - v_3, 1 - v_3)$. This vector belongs to the core if and only if $v_3 \leqslant \frac{2}{3}$ and $2v_3 \geqslant v_2 + 1$.

E4 Coalitions $\{1, 2\}$, $\{1, 3\}$, and $\{1, 2, 3\}$ lead to the extreme point $(2v_2 - v_3, v_3 - v_2, v_3 - v_2, 1 - v_3)$. This vector belongs to the core if and only if $2v_2 \geqslant v_3$ and $v_2 + 1 \geqslant 2v_3$.

E5 Coalitions $\{1, 2\}$, $\{1, 3\}$, and $\{1, 4\}$ lead to the extreme point $(\frac{3v_2-1}{2}, \frac{1-v_2}{2}, \frac{1-v_2}{2}, \frac{1-v_2}{2})$. This vector belongs to the core if and only if $v_2 \geqslant \frac{1}{3}$ and $v_2 + 1 \geqslant 2v_3$.

E6 Coalitions $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ lead to the extreme point $(\frac{v_2}{2}, \frac{v_2}{2}, \frac{v_2}{2}, 1 - 3\frac{v_2}{2})$. This vector belongs to the core if and only if $\frac{3}{2}v_2 \geqslant v_3$.

E7 Coalitions $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{1, 3, 4\}$ lead to the extreme point $(3v_3 - 2, 1 - v_3, 1 - v_3, 1 - v_3)$. This vector belongs to the core if and only if $\frac{3}{4} \geqslant v_3 \geqslant \frac{2}{3}$ and $2v_3 \geqslant 1 + v_2$.

Fig. 1 depicts the subsets of games where the extreme points of the cores are defined by the same constraints. Games in region A (resp. F; G) have extreme points of the E1-type (resp. E3-; E7-type) only. Games in region B have extreme points of both E2- and E4-type. Games in region C have extreme points of both E2- and E6-type. Games in region D have extreme points of both E4- and E5-type. Games in region E have extreme points of both E5- and E6-type.

The seven regions labeled from A to G correspond to the partition induced on the class of generic games for which exactly n constraint are binding at each extreme point of the core. The equivalence relation leads to the lines (e.g. the line between regions A and B) and intersecting points (e.g. the point that falls next to all seven regions) separating these regions when considering nongeneric games.

As explained in the previous section, the additivity property holds if one chooses two component games that fall in the same equivalence class, but not only in those cases. It would also hold for instance if we combine a game that falls on the line between A and F with a game that falls on the line between A and B, since both games can be approximated by games that belong to A. This extended property with limits, on the other hand, characterizes all the cases where the additivity property holds. The core of the sum of a game that belongs to A with a game that belongs to E is strictly larger than the sum of the cores, or the core of the sum of a game that falls

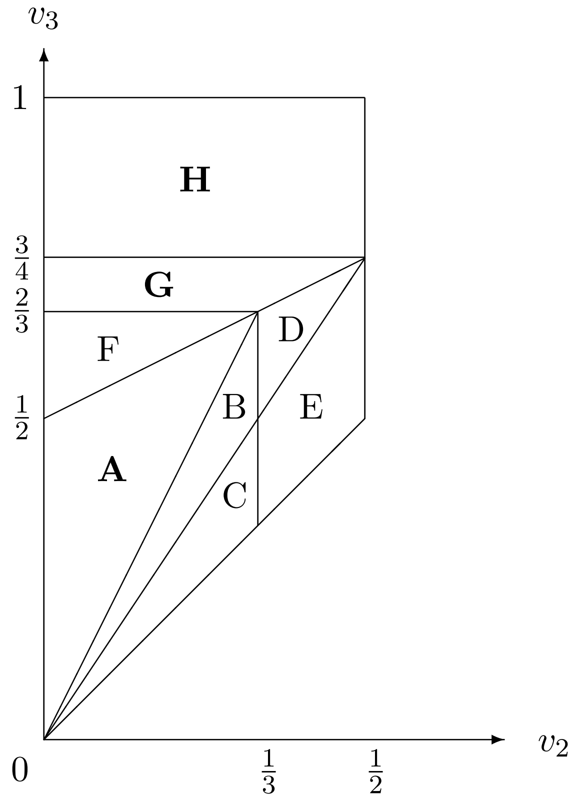

Fig. 1. Equivalence classes of four-player symmetric games.

on the line between A and F with a game that falls on the line between C and E is strictly larger than the sum of the cores. The difference between the core of the combined game and the sum of the cores of the component games can be extremely large. In fact, it is possible to combine two component games where the core collapses to a single point, and obtain a full-dimensional core. For example, pick two games v and w such that $v_2 = \frac{1}{2}$ and $w_3 = \frac{3}{4}$. For each of these games, the core is a single point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. However, the sum of the two games can belong to any of the regions A, B, D, E, F or G, where the core is a full-dimensional set.

Three classes of games stand out. Region A and its closure corresponds to the class of *convex games*. The work of Shapley [29] and Ichiishi [13] imply that a game is convex if and only if the extreme points of the core coincide with the vectors of marginal contribution. Our proposition confirms the known-result that the core of the sum of any two convex games is equal to the sum of the cores (see also Tijs and Branzei [30] on that point). Region G and its closure corresponds to games where the extreme points of the core are characterized by constraints involving only three-player coalitions, or the dual imputation set. This is the class of games K_d introduced by Driessen and Tijs [10] — and for which Tijs and Branzei [30] also note that the core is additive. Finally, region H (for which $v_3 > \frac{3}{4}$) corresponds to games with empty cores.

5. Concluding remarks

In this paper, we characterize the classes of cooperative games on which the core is additive. In this concluding section, we briefly comment on the generalization of our results to other cooperative solution concepts, and discuss the existing literature on additivity axioms in cooperative game theory.

We first note that, whenever a solution is defined by a system of linear inequalities, a direct application of Lemma 1 shows that the set of cooperative games can be partitioned into equivalence classes where the solution is additive. For example, Laussel and Le Breton [17] analyze the Pareto frontier of sets $U(v) = \{(u_1, \ldots, u_n) \mid u_i \geqslant 0,\ \sum_{i \in S} u_i \leqslant v(N) - v(N \setminus S)\}$ for a given cooperative game v. From our analysis, it is clear that the convex polyhedron corresponding to the sum of two games v and w is equal to the sum of the convex polyhedra, $U(v+w) = U(v) + U(w)$ if and only if the extreme points of $U(v)$ and $U(w)$ are defined by the same coalitions.[7] On the other hand, the lemma does not apply if the solution concept is not a unique polyhedron but a finite union of polyhedra, like the $\mathcal{M}_1^i$ bargaining set (Davis and Maschler [8] and Maschler [18]), or the kernel (Davis and Maschler [9] and Maschler and Peleg [19]). Suppose for illustration that a solution can be written as the union of two polyhedra: $\mathcal{S}(v) = \mathcal{A}(v) \cup \mathcal{B}(v)$. Even if we consider two games v and w with the same binding coalitions in the two polyhedra $\mathcal{A}$ and $\mathcal{B}$, so that $\mathcal{A}(v+w) = \mathcal{A}(v) + \mathcal{A}(w)$ and $\mathcal{B}(v+w) = \mathcal{B}(v) + \mathcal{B}(w)$, there is no guarantee that $\mathcal{S}(v+w) = \mathcal{S}(v) + \mathcal{S}(w)$. In fact, it is easy to check that $(\mathcal{A}(v+w) \cup \mathcal{B}(v+w)) \subseteq (\mathcal{A}(v) \cup \mathcal{B}(v)) + (\mathcal{A}(w) \cup \mathcal{B}(w))$, with strict inclusion for generic games.

We next consider solutions defined as unique points rather than convex polyhedra. Of course, the Shapley value satisfies additivity. Peters [23] and [24] provides an axiomatic characterization of solutions to Nash's bargaining problem which satisfy additivity and variants of superadditivity. Charnes and Kortanek [6] and Kohlberg [16] prove that the nucleolus is piecewise linear in the following sense. For any imputation x, and any coalition S, compute the excess function $e(x, S) = v(S) - x(S)$, and order the coalitions, by decreasing values of the excess, to obtain an array of coalitions $\mathbf{b}(x, v) = (b_1(x, v), \ldots, b_{2^n-1}(x, v))$. Partition then the set of coalitional games in such a way that v and w belong to the same equivalence class if and only if, at the nucleolus of the two games, $\nu(v)$ and $\nu(w)$, the array of coalitions satisfy $\mathbf{b}(\nu(v), v) = \mathbf{b}(\nu(w), w)$. Then, for any two v and w in the same equivalence class, $\nu(v+w) = \nu(v) + \nu(w)$.

Finally, we would like to emphasize that, in our opinion, the study of the additivity of the core is only a first step in a research program on multi-issue cooperation. In the future, we hope to extend the analysis by studying alternative models of multi-issue bargaining in nonwelfarist environments.

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