BOUNDARY RATIONALITY AND LIMITED DATASETS*

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Abstract

Theories of bounded rationality are typically characterized over an exhaustive data set. This paper develops a methodology that can be helpful to understand if and how a theory of bounded rationality can be tractably tested when the available data is limited, as is the case in most practical settings. We apply our approach to several leading theories. We also point out that the recent literature on bounded rationality has overlooked a methodological pitfall that can lead to ‘false positives’ and ‘empty’ out-of-sample predictions in the presence of limited data.

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1. Introduction

The recent literature has proposed insightful and plausible choice procedures in response to mounting evidence against rational choice. Great progress has been made to understand the set of choice functions that these new theories generate, and which characteristics of the Decision Maker (DM) can be identified. Though theoretically insightful, such results do not apply to typical situations, in which only some choices are observed.\(^1\) For the theory of Rationality, the problems of testable implications for limited data, out-of-sample prediction, and identification are all well understood.\(^2\) We explore how these ideas can be brought into the discourse on bounded rationality.

A DM’s observed choices are consistent with a theory if they can be extended to a complete choice function arising under the theory. Previous attempts to study bounded rationality theories under limited data focus on explaining only observed choices, without considering out-of-sample implications.\(^3\) Such an approach may at first seem natural, since there is no need to worry about out-of-sample problems when testing for Rationality in its standard description. Indeed, if one can find a preference ordering for which observed choices are maximal, then choices may be defined for out-of-sample problems simply by maximizing that same preference. We show in Section 3 that such extensibility need not hold in general, resulting in a problem of false positives: a DM’s choices may be incorrectly attributed to a theory for which no extension of those choices can arise. A first contribution of this paper is thus to move the goalpost for consistency tests to the proper location.

We build on the classic testing approach for Rationality to understand how other theories of choice may be tested, and delineate which are similarly easy to test. The insightful methodology developed for Rationality can be described as follows: first, infer key preference comparisons under the presumption that the DM is rational; next, note that the theory requires these comparisons to be acyclic; and finally, check that these are all the comparisons that may be inferred, so that being acyclic is not just necessary, but also sufficient for consistency. This approach is summarized by the Strong Axiom of Revealed Preference (SARP). We show in Section 4 that the

\(^1\)In empirical settings, the modeler cannot control the choice problems faced by individuals. In experimental settings, generating a complete dataset requires an overwhelming number of decisions by subjects: 26 choice problems when the space of alternatives contains 5 elements, 1,013 choice problems when it contains 10 elements, and 32,752 choice problems when it contains 15 elements.


\(^3\)These include Manzini and Mariotti (2007), Manzini and Mariotti (2012) and Tyson (2013).
empirical content of various theories is naturally captured by the *acyclic satisfiability* of restrictions summarizing key information revealed by choices. Acyclic satisfiability generalizes SARP, as the restrictions may be more complex than the simple comparisons revealed under Rationality. We show that when the restrictions to be tested all pertain to *lower contour sets* (LCS), or all pertain to *upper contour sets* (UCS), there is a simple algorithm to determine whether some acyclic relation satisfies them. We also delineate the limits of tractable testing. We prove that testing acyclic satisfiability of restrictions beyond the LCS (or UCS) class is generally NP-hard, and apply this result to identify some theories that are NP-hard to test. For such theories, our results can help pinpoint classes of datasets that are easy to test, namely those where the corresponding restrictions fall into the LCS (or UCS) class.

Most theories of bounded rationality assume strict preferences. In Section 5, we take some initial steps to advance the methodology to accommodate indifferences, and discuss interesting new features that arise. In Section 6, we discuss limitations of our approach, and outline several further applications of the methodology to other types of bounded rationality theories and datasets. Some of these applications are carried out in separate works, while others are developed here. Almost all proofs in the paper are relegated to the Appendix.

2. Framework

Consider a finite set $X$ of alternatives. A *choice problem* is a nonempty subset of $X$ and represents those alternatives that are feasible. The set of all conceivable choice problems is denoted $\mathcal{P}(X)$. A *choice function* $c : \mathcal{P}(X) \to X$ associates an element $c(S) \in S$ to each choice problem $S$.

A theory $\mathcal{T}$ formally describes the DM’s choice procedure. For instance, the classic theory of Rationality posits that the DM uses a single preference ordering $P$ to select the best element from any choice problem: $c(S) = \arg \max_P S$ for all $S \in \mathcal{P}(X)$.

The literature has also proposed interesting and plausible choice procedures that depart from Rationality. As a start, consider the following two theories that will serve as illustrative examples for Section 3. In the theory of *Limited Attention* by Masatlioglu, Nakajima and Ozbay (2012), a DM facing a choice problem $S$ maximizes

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4We use the term *relation* to mean a (possibly incomplete or cyclic) binary relation, while we use the term *ordering* to mean a complete, asymmetric and transitive relation. For any relation $P$ and any $S \subseteq X$, we denote $\arg \max_P S = \{x \in S \mid xPy, \forall y \in S \setminus \{x\}\}$.  

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a preference ordering $P$ over a consideration set $\Gamma(S) \subseteq S$, with the restriction that consideration sets don’t change when removing ignored alternatives:

\begin{align*}
&\text{(1a)} \quad c(S) = \arg\max_P \Gamma(S), \text{ for all } S \in \mathcal{P}(X), \text{ and} \\
&\text{(1b)} \quad \Gamma(S) \subseteq T \subseteq S \Rightarrow \Gamma(T) = \Gamma(S), \text{ for all } S, T \in \mathcal{P}(X)
\end{align*}

In Manzini and Mariotti (2007)’s theory of Shortlisting, the DM makes a shortlist of undominated options using an asymmetric relation $P_1$. She then chooses the undominated element in the shortlist according to an asymmetric preference relation $P_2$:

\begin{equation}
\{c(S)\} = \max(\max(S, P_1), P_2), \text{ for all } S \in \mathcal{P}(X).^{5}
\end{equation}

This paper develops a methodology for characterizing the testable implications of choice theories. Importantly, only the DM’s choices, not her thought process or choice method, are observable. Each theory $\mathcal{T}$ yields a collection $\mathcal{C}(\mathcal{T})$ of possible choice functions. In the presence of limited data, one observes the DM’s choices only for problems in a dataset $\mathcal{D} \subseteq \mathcal{P}(X)$. An observed choice function $c_{obs} : \mathcal{D} \to X$ associates to each choice problem $S \in \mathcal{D}$ the alternative in $S$ that the DM selected.

3. Testing for Consistency Using Limited Data

We aim to understand when observed choices are consistent with (that is, do not refute) a given theory. This means that the theory must yield at least one choice function that coincides with $c_{obs}$ on $\mathcal{D}$, guaranteeing the ability to make coherent predictions for unobserved choice problems.

**Definition 1** (Consistency) An observed choice function $c_{obs} : \mathcal{D} \to X$ is consistent with a theory $\mathcal{T}$ if there is $c \in \mathcal{C}(\mathcal{T})$ such that $c_{obs}(S) = c(S)$ for every $S \in \mathcal{D}$.

**Definition 2** (Prediction) The set of predictions for $S \notin \mathcal{D}$ under a theory $\mathcal{T}$ is given by $\{c(S) \mid c \in \mathcal{C}(\mathcal{T}) \text{ and } c \text{ coincides with } c_{obs} \text{ on } \mathcal{D}\}$.

Addressing the question of out-of-sample predictions thus reduces to identifying the testable implications of a theory on limited data: $x$ is a possible choice for $S \notin \mathcal{D}$ if and only if the expanded observed choice function, derived from $c_{obs}$ by adding the counterfactual that the DM picks $x$ from $S$, is consistent with $\mathcal{T}$.

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5Following Manzini and Mariotti (2007)’s notation, $\max(S, R) = \{x \in S \mid \nexists y \in S \text{ s.t. } yRx\}$. Their notation $\{c(S)\}$ requires the undominated set to be a singleton.
The recent literature on bounded rationality has overlooked a potential pitfall one should keep in mind in the presence of limited data. To test Shortlisting with limited data, Manzini and Mariotti (Corollary 1, 2007) study when there exist asymmetric relations $P_1, P_2$ such that (2) holds for $S \in \mathcal{D}$. Manzini and Mariotti (Definition 4, 2012) take a similar approach for Categorization, discussed further below. To test Limited Attention, Tyson (2013) seeks conditions guaranteeing the existence of an ordering $P$ and a consideration set mapping defined on $\mathcal{D}$ such that (1a) and (1b) hold for $S, T \in \mathcal{D}$. In other words, the theory’s conditions describing how choices emerge are checked only over observed problems. Such an approach may seem natural at first. Taking Rationality as a benchmark, if there is an ordering $P$ such that $c_{\text{obs}}(S)$ is the $P$-maximal element for all $S \in \mathcal{D}$, then it is trivial to extend $c_{\text{obs}}$ to a rational choice function $c$ by letting $c(S)$ be the $P$-maximal element for all $S \in \mathcal{P}(X)$. In general, however, such an approach may yield ‘false positives,’ as it may be impossible to extend observed choices into a complete choice function under the theory. This extensibility issue affects prevalent theories of bounded rationality, leading to a potentially dangerous methodological pitfall.

The datasets below illustrate the possible issues that may arise. According to the prior literature’s approach, $c_{\text{obs}1}$ should be consistent with Limited Attention, and $c_{\text{obs}2}$ should be consistent with Shortlisting.\footnote{Indeed, (1a) and (1b) hold for $S, T \in \mathcal{D}$ using the ordering $P$ defined by $aPdPcPbPf$, with $\Gamma(S)$ given by $c_{\text{obs}1}(S)$ and its $P$-lower counter set for $S \in \mathcal{D}$; moreover, (2) holds for $S \in \mathcal{D}$ using $P_1$ given by $eP_1d$, $fP_1a$ and $gP_1b$, and $P_2$ given by $aP_2bP_2dP_2a$ and $xP_2y$ for $x \in \{a, b, d\}$, $y \in \{e, f, g\}$.}

In fact, neither is true.

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Consider Limited Attention. Suppose, by contradiction, that some $\Gamma$ satisfying (1b) and some ordering $P$ generate an extension of $c_{\text{obs}1}$ under the theory. By (1b), $d$ must be considered in $\{a, d, e\}$, since its removal changes the choice. As $a$ is chosen from $\{a, d, e\}$, we learn $aPd$. Similarly, we conclude $b \in \Gamma(\{b, e, f\})$ and $ePb$. Now consider the out-of-sample problem $\{b, d\}$. The ranking $aPd$ implies $a \not\in \Gamma(\{a, b, d\})$, thus (1b) requires $\Gamma(\{b, d\}) = \Gamma(\{a, b, d\})$. At the same time, the ranking $ePb$ implies $e \not\in \Gamma(\{b, d, e\})$, thus (1b) also requires $\Gamma(\{b, d\}) = \Gamma(\{b, d, e\})$. This is impossible, as the choices from $\{a, b, d\}$ and $\{b, d, e\}$ differ. The problem here, quite simply, is that a mapping $\Gamma$ satisfying the required property over $\mathcal{D}$ need not satisfy it elsewhere.
Next take Shortlisting. Suppose, by contradiction, that some asymmetric relations $P_1$ and $P_2$ yield an extension of $c_{obs}$ under the theory. As $a$ is chosen from \{a, b, d, e\}, it must be $P_1$-undominated in \{a, b, d, e\} and subsets thereof. Thus the choice of $d$ from \{a, d\} implies $dP_2a$. Symmetric reasoning for $b$ and $d$ yields the preference cycle $aP_2bP_2dP_2a$, and none of these elements can $P_1$-dominate one of the two others. But then contrary to the theory’s requirements, choice cannot arise from max(max($S, P_1$), $P_2$) for $S = \{a, b, d\}$. The data requires $P_2$ to be cyclic over \{a, b, d\}, and yet prevents $P_1$ from eliminating any from the shortlist. The problem here is that not all combinations of asymmetric $P_1, P_2$ are valid inputs to the theory, and checking validity requires thinking about both observed and unobserved choice problems.

4. Methodology and Applications

How does one test Rationality with limited data? One could naively traverse all the possible preference orderings to see whether any generate the observed choices. One could also check whether any completions of observed choices satisfy the axiom of Independence of Irrelevant Alternatives (IIA). Such mechanical approaches are tedious and impractical. Seminal contributions by Samuelson (1948), Houthaker (1950), Richter (1966), Afriat (1967) and Varian (1982) develop a more insightful approach, summarized as follows: (1) assuming the DM is rational, infer key preference comparisons from the data ($x$ is revealed preferred to $y$ if $x$ is picked in the presence of $y$, etc.); (2) notice these inferred preference comparisons must be acyclic if observed choices are consistent with Rationality; and (3) check that this necessary condition is also sufficient, to ensure no key insights have been overlooked in the data.

This approach makes the testing of Rationality simple and tractable. The Strong Axiom of Revealed Preference (SARP), which requires Samuelson (1948)’s revealed preference relation to be acyclic, is checkable through a simple procedure. First, one finds an option (call it $x_1$) which isn’t at the top of any revealed preference comparison. Failure to find one implies the revealed preference is cyclic, and SARP fails. Otherwise, $x_1$ is a fine candidate for the least-preferred option and the test continues. Next, one finds an option (call it $x_2$) which is never on top of a revealed preference comparison among the remaining elements $X \setminus \{x_1\}$. Again, failure to find one means SARP fails; else the process continues. One can enumerate all of $X$ in this way if and only if SARP holds. Notice that this test is simple, in the sense that it can be carried out in a number of steps that is at most polynomial in the dataset.
As a benefit, computers can carry out the testing quickly for any dataset (and for relatively small datasets, answers can even be found by hand).

We build on this classic methodology to understand how theories beyond Rationality may be tested, and which theories (and datasets) are also easy to test. As a natural first step, Section 4.1 demonstrates how the testing approach, and types of comparisons revealed under Rationality, generalize when the DM maximizes a single preference ordering over a consideration set. Section 4.2 formalizes the notions of restrictions on a relation and acyclic satisfiability, and illustrates these with the results of Section 4.1. The practical matters of testing are discussed in Section 4.3: Section 4.3.1 presents a tractable enumeration procedure for testing acyclic satisfiability of lower (upper) contour set restrictions, and Section 4.3.2 considers the sense in which we have identified the frontier of tractable testing. Applications of these results to other types of theories and datasets are considered in Sections 5 and 6.

4.1 Bounded Rationality via Consideration Sets

The maximization of a single preference ordering remains a central feature of several theories. Bounded rationality emerges because the DM maximizes her preference over consideration sets that may overlook some feasible options. Such theories are distinguished by how the consideration set mapping \( \Gamma : \mathcal{P}(X) \rightarrow \mathcal{P}(X) \) crystallizes. This structured departure from Rationality offers a natural starting point to gain intuition on testing beyond Rationality.

Samuelson’s revealed preference no longer applies, because unchosen options may have been overlooked. Similarly to Rationality, however, it is clear that we should be able to write the testable implications of such theories in terms of the existence of an acyclic relation satisfying some ‘restrictions,’ although this insight is not necessarily helpful without further analysis. To see why, observe that each theory requires the consideration set mapping \( \Gamma \) to belong to some set of possible mappings \( \{\Gamma_k\}_{k=1,\ldots,N} \) (which may be very large). Observed choices are consistent with the theory if and only if there exists an acyclic relation \( P \) satisfying Samuelson’s revealed preference over some possible consideration set mapping \( \Gamma_k \). Formally, this corresponds to the existence of an acyclic relation \( P \) such that: either \( [c_{obs}(S)Py \text{ for all } y \in \Gamma_1(S) \setminus \{c_{obs}(S)\} \text{ and all } S \in \mathcal{D}] \), or \( [c_{obs}(S)Py \text{ for all } S \in \mathcal{D} \text{ and } y \in \Gamma_2(S) \setminus \{c_{obs}(S)\}] \), \ldots , or \( [c_{obs}(S)Py \text{ for all } S \in \mathcal{D} \text{ and } y \in \Gamma_N(S) \setminus \{c_{obs}(S)\}] \). The difficulty, of course, is that we don’t know which consideration set mapping \( \Gamma_k \) the DM applies. The test
above thus not only lacks insight, but also likely lacks tractability (as will be seen from Section 4.3.2). Our aim is thus to see how the modeler can more cleverly make inferences about preference from the theory’s requirements on $\Gamma$.

The information potentially revealed about preferences may be more complex than the comparisons arising from Rationality, as will be seen from the following different properties on $\Gamma$. The first,

\[
S \subset T \Rightarrow \Gamma(T) \cap S \subseteq \Gamma(S), \text{ for all } S, T \in \mathcal{P}(X),
\]

characterizes the consideration sets in Cherepanov, Feddersen and Sandroni (2013)’s theory that the DM considers only rationalizable alternatives: those that are top-ranked in the choice problem for at least one of her rationales (transitive and asymmetric relations); their theory of Order Rationalization corresponds to applying a preference ordering to such a $\Gamma$. Property (3) also emerges from Manzini and Mariotti’s (2012) theory of categorization; their Transitive Categorize-Then-Choose theory corresponds to applying a preference ordering to such a $\Gamma$. Property (3) is directly imposed by Lleras et al. (2017), who consider preference orderings. Second, we consider property (1b), which characterizes Limited Attention. Third, we consider a theory of Spotlighting defined by the following property, capturing a form of the attraction effect:

\[
\forall x \neq y : x \in \Gamma(S \cup \{y\}) \setminus \Gamma(S) \Rightarrow x \in \Gamma(T) \text{ whenever } x, y \in T,
\]

which identifies $y$ as an alternative that calls the DM’s attention to $x$.

Consider (3) first. If the DM considers an option in a set, then she must consider it in all subsets. Preference comparisons are thus inferred from IIA violations:

\[
(5) \quad \text{For all } S, T \in \mathcal{D} \text{ with } c_{\text{obs}}(S) \neq c_{\text{obs}}(T) \in S \subset T : c_{\text{obs}}(S) P c_{\text{obs}}(T).
\]

IIA violations are also informative under Limited Attention’s property (1b). Indeed, if the choice from $T$ is available but not chosen from $S \subset T$, then the DM must have considered at least one alternative in $T \setminus S$ when choosing from $T$; otherwise, (1b) would require $\Gamma(T) = \Gamma(S)$, contradicting that the choices differ. The IIA violation thus informs the modeler that there exists $z \in T \setminus S$ such that $c_{\text{obs}}(T) P z$.

\footnote{Note that in this theory, violations of the ‘weak WARP’ axiom of Manzini and Mariotti (2012) and Cherepanov et. al (2013) are possible, and in fact contribute to revealed-preference inferences.}
In contrast to the simple preference comparisons discussed above, these restrictions have a new feature: we only learn the choice must be preferred to some alternative in a set, without knowing exactly which one. IIA violations, however, do not provide all the key comparisons that may be inferred from the data. In fact, under (1b), any WARP violation is informative, and reveals a yet more general form of restriction:

\begin{equation}
\text{For all } S, T \in D \text{ with } c_{\text{obs}}(S) \neq c_{\text{obs}}(T) \text{ and } c_{\text{obs}}(S), c_{\text{obs}}(T) \in S \cap T : \\
\text{either } c_{\text{obs}}(S) P z \text{ for some } z \in S \setminus T \text{ or } c_{\text{obs}}(T) P z' \text{ for some } z' \in T \setminus S.
\end{equation}

Otherwise, the choice from $S \cap T$ would be ill defined by a similar argument as above.

Finally, consider property (4). For any $S \in D$, either $c_{\text{obs}}(S)$ is considered in all subsets of $S$ in which it is contained, or it is not. In the former case, collect in the set $Y(S) = \{ c_{\text{obs}}(T) \mid T \subseteq S, c_{\text{obs}}(S) \in T \setminus \{ c_{\text{obs}}(T) \} \}$ all such observed choices differing from $c_{\text{obs}}(S)$; then every element in $Y(S)$ is revealed preferred to $c_{\text{obs}}(S)$. In the latter case, in which $c_{\text{obs}}(S)$ is not considered in all subsets of $S$, there must exist some $y \in S \setminus \{ c_{\text{obs}}(S) \}$ and some $R \subseteq S$ such that $x$ is considered in $R \cup \{ y \}$, but not in $R$, implying by (4) that the DM considers $c_{\text{obs}}(S)$ in all choice problems where $y$ also belongs. For such choice problems, collect in the set $Z(S, y) = \{ c_{\text{obs}}(T) \mid \{ c_{\text{obs}}(S), y \} \subseteq T, c_{\text{obs}}(S) \neq c_{\text{obs}}(T) \}$ all observed choices differing from $c_{\text{obs}}(S)$; then every element of $Z(S, y)$ is revealed preferred to $c_{\text{obs}}(S)$. In summary,

\begin{equation}
\text{For all } S \in D : \text{ either } x P c_{\text{obs}}(S) \text{ for all } x \in Y(S), \text{ or there exists } y \in S \setminus \{ c_{\text{obs}} \} \text{ such that } x P c_{\text{obs}}(S) \text{ for all } x \in Z(S, y).
\end{equation}

Unlike in Limited Attention, deciding which possibility holds for one of the restrictions in (7) may simultaneously fix multiple preference comparisons. This feature also arises in the preference restrictions found by Barberà, de Clipped, Neme and Rozen (2018), who model a DM willing to choose an option as long as it is ‘good enough’ according to his preference ordering; and, as they show, when imposing both (3) and a lower bound on the number of elements a DM considers.

For the above theories, the existence of a preference ordering satisfying the corresponding requirements is necessary for consistency. Typically, the more challenging aspect of extending the testing methodology for Rationality to other theories is ensuring we have found all the necessary conditions. We now confirm this.

**Proposition 1** Observed choices $c_{\text{obs}}$ are consistent with applying a strict prefer-
ence ordering to a consideration set mapping \( \Gamma \) satisfying:

(a) Property (3), if and only if the relation defined by (5) is acyclic.
(b) Property (1b), if and only if there exists an acyclic relation satisfying (6).
(c) Property (4) if and only if there exists an acyclic relation satisfying (7).

These theories show how the insightful approach pioneered by Samuelson for Rationality can be fruitfully pursued for other theories of choice involving a preference ordering (and beyond, as we will later see). For some theories, like Order Rationalization, the key information from choice data continues to come in the form of simple preference comparisons (just as in Rationality, albeit with a different definition of revealed preference). For other theories, one must consider preference restrictions that take a more complex logical form. In the next subsection, we formalize the commonalities and differences among the variety of restrictions illustrated here.

### 4.2 Formalizing Restrictions and Acyclic Satisfiability

For any \( x, y \in X \), define the function \( 1_{(x,y)} \) that takes as input a strict relation \( P \), and outputs ‘true’ if \( xP y \) and ‘false’ otherwise. The function \( 1_{(x,y)} \) thus tests whether a simple comparison holds. More generally, a restriction takes as input a strict relation \( P \), and outputs ‘true’ or ‘false’ based on logical conjunctions (‘and’, \( \land \)) and/or disjunctions (‘or’, \( \lor \)) of simple comparisons under \( P \).

Given an option \( x \), we say that a restriction pertains to the lower-contour set (LCS) of \( x \) if all the comparisons in it take the form \( 1_{(x,\cdot)} \). As is well known, any logical formula can be written in disjunctive normal form, i.e., as a disjunction of conjunctions. Thus a restriction pertaining to the LCS of \( x \) can be described by a family of sets \( \Sigma \subseteq P(X) \) and denoted \( 1_{(x,\Sigma)} \). A relation \( P \) satisfies \( 1_{(x,\Sigma)} \) if and only if some set in \( \Sigma \) belongs to the \( P \)-lower contour set of \( x \).\(^8\) For notational convenience, we assume throughout that LCS restrictions are given using the notation \( 1_{(x,\Sigma)} \). Similar definitions apply for upper-contour set (UCS) restrictions, which we denote \( 1_{(\Sigma,x)} \).

\(^8\)Slightly more generally, an LCS restriction may also pertain to a set \( T \subseteq X \): a relation \( P \) satisfies \( 1_{(T,\Sigma)} \) if and only if some member of \( \Sigma \) is contained in the \( P \)-lower contour set of every \( x \in T \) (and similarly for a UCS restriction pertaining to a set). One can immediately see that all our enumeration-related results (Lemmas 1-2, and Propositions 2 and 4) also hold for LCS (UCS) restrictions pertaining to sets. We avoid this more cumbersome notation in the text, since our only example of an application with LCS restrictions pertaining to sets appears in our study of correspondences in Barberà et al (2018).
We illustrate these definitions with the results of Section 4.1. The restrictions under Order Rationalization, corresponding to property (3), will be simple comparisons of the type $1_{(a,b)}$, which qualify both as UCS and LCS restrictions. For Limited Attention, corresponding to property (1b), each restriction takes a disjunctive form: e.g., the choices $c_{obs}(\{a, b, d\}) = b$ and $c_{obs}(\{b, d, e, f\}) = d$ generate the restriction $1_{(b,a)} \lor 1_{(d,e)} \lor 1_{(d,f)}$. This is neither an UCS restriction nor a LCS restriction, since there isn’t one alternative which is always at the top, or always at the bottom, of each component comparison. Under Spotlighting, corresponding to property (4), each restriction is a disjunction of conjunctions, and is moreover of the UCS type. For instance, if $S = \{a, b, d\}$, $c_{obs}(S) = a$, $Y(S) = \{b\}$, $Z(S, b) = \{d, e\}$ and $Z(S, d) = \{f, g\}$, then the restriction for $S$ takes the form $1_{(b,a)} \lor (1_{(d,a)} \land 1_{(e,a)}) \lor (1_{(f,a)} \land 1_{(g,a)})$, which is denoted by $1_{(\Sigma, a)}$ for $\Sigma = \{Y(S), Z(S, b), Z(S, d)\}$.

A set of restrictions is **acyclically satisfiable** if there exists an acyclic relation satisfying them. Note that restrictions pin down a (typically incomplete) relation when all comparisons are simple, and in this case acyclic satisfiability just boils down to this relation being acyclic. As illustrated by our results in Section 4.1 and what follows, the empirical content of various theories is naturally captured through acyclic satisfiability of restrictions summarizing key information revealed by choices. Rationality being equivalent to SARP is now understood as just one instance of this broader approach.

### 4.3 Testing Acyclic Satisfiability

Remember how testing SARP is tractable by enumeration. We show how this idea extends to perform similarly well for testing acyclic satisfiability of a collection of LCS restrictions (or a collection of UCS restrictions). For intuition, remember that SARP can be tested by identifying, in every step, some candidate for the worst remaining option, namely, one that does not appear at the top of any Samuelson revealed preference with other remaining options. Lower-contour set restrictions may not precisely identify *which* alternative must be ranked below an option, but they do reveal that *some* alternative must be ranked below it, which is all that is needed to rule out candidate-worst options. Section 4.3.1 formalizes this intuition, while Section 4.3.2 shows how testing acyclic satisfiability can become much harder to perform (NP-hard) when considering wider classes of restrictions.\(^9\)

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\(^9\)Testing acyclic satisfiability can be seen as an extension of the topological sort problem in
4.3.1 Easy To Test with LCS or UCS Restrictions

The extension of the classic enumeration test for SARP when considering a collection of LCS restrictions rests on two observations, presented as lemmas. The first lemma echoes the intuition from the previous paragraph.

**Lemma 1** Let \( X \) be a set of options and \( \mathcal{R} \) be a set of LCS restrictions defined on \( X \). If \( \mathcal{R} \) is acyclically satisfiable, then there exists an option \( x \in X \) such that no restriction in \( \mathcal{R} \) pertains to the lower-contour set of \( x \).

Indeed, any acyclic relation satisfying \( \mathcal{R} \) can be completed into an ordering satisfying \( \mathcal{R} \), and \( x \) may be taken to be its minimal element. This provides a first, simple necessary condition for acyclic satisfiability: traverse elements of \( X \) to find one that does not appear at the top of a restriction. Let \( x_1 \) be an element with this property (if one exists). Because \( x_1 \) will be treated as the bottom element of the ordering, any restriction \( 1_{(x,\Sigma)} \) such that \( \{x_1\} \in \Sigma \) is now satisfied, and may be eliminated; all other restrictions \( 1_{(x,\Sigma)} \) simplify to \( 1_{(x,\Sigma')} \), where \( \Sigma' = \{S \setminus \{x_1\} \mid S \in \Sigma\} \). Let \( \mathcal{R}_1 \) be this reduced set of LCS restrictions over \( X \setminus \{x_1\} \).

**Lemma 2** Let \( x_1 \) satisfy the property of Lemma 1. Then \( \mathcal{R} \) (defined over \( X \)) is acyclically satisfiable if and only if \( \mathcal{R}_1 \) (defined over \( X \setminus \{x_1\} \)) is acyclically satisfiable.

Necessity obtains by considering the restriction of the acyclic relation satisfying \( \mathcal{R} \) to the set \( X \setminus \{x_1\} \). Sufficiency obtains by augmenting the acyclic relation satisfying \( \mathcal{R}_1 \) by placing \( x_1 \) at the bottom of any pairwise comparison.

The two Lemmas hold independently of the set \( X \) and the set of LCS restrictions \( \mathcal{R} \), so the reasoning may be iterated. For instance, acyclic satisfiability of \( \mathcal{R}_1 \) means we can find an option \( x_2 \in X \setminus \{x_1\} \) that does not appear at the top of any restriction in \( \mathcal{R}_1 \). As \( x_2 \) will be treated as the bottom element among \( X \setminus \{x_1\} \), \( \mathcal{R}_1 \) can be reduced to a set of restrictions \( \mathcal{R}_2 \) over \( X \setminus \{x_1, x_2\} \). The Lemmas thus provide a conceptual roadmap for defining the enumeration procedure for LCS restrictions. The first step follows as in Lemma 1. If there has been no failure thus far, then in each step \( k > 1 \) one chooses an element \( x_k \in X \setminus \{x_1, \ldots, x_{k-1}\} \) that does not appear at

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*computer science. Some extensions have been studied in problems of job-scheduling with waiting conditions; see Möhring et al. (2004) who provide a fast scheduling algorithm given conditions “job \( i \) comes before at least one job in a set \( J \),” which corresponds to a special case of \( 1_{(x,\Sigma)} \) with every \( S \in \Sigma \) a singleton. They show scheduling is NP-hard for the generalization “some job in a set \( I \) comes before some job in a set \( J \);” we show the problem is already NP-hard with simpler restrictions.  

\(^{10}\) Without using the notation \( 1_{(x,\Sigma)} \) for restrictions, this is equivalent to replacing functions of the form \( 1_{(y,x_1)} \) for any \( y \in X \) by the logical value ‘true’.
the top of any LCS restriction in the reduced set of restrictions $R_{k-1}$ (or equivalently, does not appear at the top of an LCS restriction in the original set $R$, while ignoring all restrictions $1_{(x, \Sigma)}$ such that $S \subseteq \{x_1, \ldots, x_{k-1}\}$ for some $S \in \Sigma$). The procedure *fails* if it is impossible to find $x_k$ in some step; but if one can enumerate all of $X$ in this way, then the enumeration procedure *succeeds*. Our reasoning thus far has shown that success of the enumeration algorithm is a necessary condition for $R$ to be acyclically satisfiable. Vice versa, ranking options in opposite order from a successful enumeration will satisfy $R$ by construction. We have thus shown the following.

**Proposition 2** A set of LCS restrictions $R$ is acyclically satisfiable if and only if the enumeration procedure succeeds.

The procedure can also be used to check acyclic satisfiability of a set of UCS restrictions. The only difference is that one seeks candidate *maximal* options in each step, i.e., options that do not appear at the bottom of any remaining UCS restrictions.

How easy is testing LCS (or UCS) restrictions through enumeration? The procedure requires at most $|X| - 1$ iterations, so testing is tractable if finding a candidate for minimal element is tractable in each iteration. Testing a theory using data also requires constructing the restrictions in the first place, so one should confirm that doing so is easy. We check these steps for all theories noted as tractable in this paper.

Consider the theories discussed in Section 4.1. The testable implications of Order Rationalization are captured through LCS (and UCS) restrictions, and the testable implications of Spotlighting are captured through UCS restrictions. Proposition 2 applies in those cases, and can be used to show that, like Rationality, these theories are easily tested (i.e., polynomial time in $|X|$ and $|\mathcal{D}|$). Indeed, with Order Rationalization, a very loose upper bound for the number of IIA violations is the square of $|\mathcal{D}|$; and for Spotlighting, each observation leads to one restriction, which is itself easy to compute (polynomial in $|X|$ and $|\mathcal{D}|$). The restrictions for Limited Attention may be neither LCS nor UCS; we discuss the complexity of testing such restrictions in the next subsection. However, only LCS restrictions matter whenever $\mathcal{D}$ is closed under

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11While there may exist multiple candidates for the minimal element in a step, a different selection among these would not convert a failure of the algorithm to a success, or vice versa. This property of path independence ensures that success and failure are definitive outcomes.

12In fact, for many theories (including Rationality), a restriction pertains to the lower contour set of an option only if the DM chose it in some problem. In this case, options that are never chosen can be randomly enumerated in a preliminary step of the enumeration procedure, and the cardinality of the image of $c_{obs}$ bounds the number of nontrivial choices to be made.
intersection (or at least contains the intersection of any two choice problems causing a WARP violation).13 Thus, for such datasets, testing consistency with Limited Attention is also easy.

Proposition 2 also sheds light on preference identification. If the DM’s choices are consistent with a theory and every preference ordering \( P \) that gives rise to these choices under the theory has \( xPy \), then we say that the data identifies a preference for \( x \) over \( y \). To test this, we add the opposite preference restriction \( 1_{(y,x)} \) to the restrictions characterizing consistency. If the augmented restrictions are acyclically satisfiable, then the DM could potentially prefer \( y \) to \( x \); but if acyclic satisfiability fails, then the data identifies a preference for \( x \) over \( y \). Conveniently, if the original restrictions are of the LCS (UCS) type, then adding a simple restriction does not change this. Thus preference identification is performable through enumeration.

4.3.2 Hard to Test Otherwise

As we are expanding the realm of the classic testing methodology beyond Rationality, it is natural to ask whether we have reached the frontier of ‘tractable testing’ for theories whose empirical content is naturally captured through acyclic satisfiability. Is there another clever extension of the enumeration procedure, or an entirely different algorithm, that would make acyclic satisfiability easy to test more generally? The answer is essentially negative, in a sense we now make precise.

So far we analyzed cases where all restrictions pertain to lower contour sets, or all restrictions pertain to upper contour sets. To make the negative result most striking, say a collection of restrictions is a mixed set of binary restrictions if each restriction is either of the form \( 1_{(x,{\{}y,{\{}z}{\}}} \) or of the form \( 1_{({\{}y,{\{}z}{\}},x)} \) for some \( x,y,z \in X \). We view such collections as a minimal departure from those considered thus far, as they contain only LCS and UCS restrictions, and each restriction can fail to be a simple comparison because of at most one disjunction between two options. We show that this small generalization of the problem becomes NP-hard; as a corollary, testing acyclic satisfiability with more permissive classes of restrictions is NP-hard as well.14

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13If \( S \) and \( S' \) cause a WARP violation, then \( S \cap S' \) causes a WARP violation with \( S \) or \( S' \). Suppose it occurs with \( S \). Then \( c_{obs}(S) \) must be preferred to some element of \( S \setminus S' \), automatically satisfying the ‘or’ condition from the WARP violation between \( S \) and \( S' \). Finally, confirming \( D \) is closed under intersection takes a polynomial number of steps in \( |D| \), as it only requires checking pairs \( S,S' \in D \).

14\( P \) is the set of problems solvable in polynomial time; \( NP \) is the set of problems that may or may not be solvable in polynomial time, but for which any conjectured solution can be checked in polynomial time. A problem is NP-hard if solving it is at least as complex as solving the most difficult problems in NP. Finding a polynomial-time solution for some NP-hard problem would have
**Proposition 3** The problem of checking acyclic satisfiability for mixed sets of binary restrictions is NP-hard.

If \( P \neq NP \), as first conjectured by Nash (1955) in the context of cryptography and now widely believed to be true, no algorithm solves every instance of an NP-hard problem in polynomial time. To prove Proposition 3, we show that every instance of SAT3 (a classic NP-hard problem\(^{15}\)) has a polynomial-time reduction to an equivalent problem of checking acyclic satisfiability of mixed sets of binary restrictions.

Proposition 3 illustrates how small departures from only LCS or only UCS restrictions can make acyclic satisfiability much harder to test. Showing that the empirical content of a theory involves such restrictions is suggestive that testing is likely to be NP-hard. Proposition 3 can be used to prove this formally; to do this, one must show that for an arbitrary mixed set of binary restrictions \( \mathcal{R} \), there exists a dataset (constructed in polynomial time given \( \mathcal{R} \)) such that the theory is consistent with the choices if and only if \( \mathcal{R} \) is acyclically satisfiable.

Our first of several applications of Proposition 3 is to Limited Attention: Proposition 5 in the Appendix shows that testing consistency with this theory is NP-hard. Limited Attention clearly generates LCS restrictions from IIA violations, but doesn’t exactly generate UCS restrictions. For intuition on how Proposition 5 is proved, note that WARP violations can give rise to restrictions of the form \( aPb \) or \( a'P'b' \), with \( a \neq a' \) and \( b \neq b' \). One can add another option \( x \) and construct observed choices to make sure that both simple comparisons \( bPx \) and \( b'Px \) hold, which implies the UCS restriction that either \( a \) or \( a' \) is better than \( x \).

Being NP-hard to test means that there exist datasets for which testing consistency is intractable. A first implication is that it may be possible to prove a theory is NP-hard even with a partial understanding of its testable implications. A second implication is that there may be some large classes of datasets for which testing remains tractable; and that these datasets can be identified by when the restrictions take only the LCS, or only the UCS, form. For instance, we argued in the previous section that the enumeration procedure applies to Limited Attention when the dataset has the intersection property. As seen in Section 6, a similar conclusion applies to two other theories, which are NP-hard to test in general, but easy to test by enumeration if the

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\(^{15}\)Given any set of ‘clauses’ that are disjunctions of three ‘literals’ (variables or their negations), the question is whether there is a truth assignment for the variables that makes all clauses true. No one has found such a solution thus far.
5. Indifferences

Most preference-based theories in bounded rationality assume strict preferences. As such, the previous section focused on restrictions pertaining to orderings (which are strict). Though it may be slightly precocious to advance our testing methodology to accommodate indifferences prior to further developments in bounded rationality, this section nonetheless offers some steps in this direction. The methodology we advocate is the same: (a) identify necessary restrictions on the preference; (b) ensure one has gained a full understanding of the theory by checking that acyclic satisfiability of these restrictions is sufficient for consistency; (c) evaluate whether testing acyclic satisfiability is tractable. However, a couple of interesting new features arise.

First, there is a new form of revealed (in)attention in theories where the DM maximizes a weak preference ordering over a consideration set. Witnessing the DM select both $x$ and $y$ in a choice problem reveals that she is indifferent between them. Using Richter (1966)’s ‘strong’ consistency notion (defined below), seeing her select $x$ but not $y$ in a different problem reveals she did not consider $y$ there. The application we study in Section 5.3 shows this creates nontrivial new restrictions on preferences, and is an additional cause for the extensibility issue identified in Section 3.

Second, we may need to test acyclic satisfiability of restrictions where some must hold strictly while others may hold weakly. As we know how to test strict restrictions, it may be tempting to treat the weak ones as strict. This can yield false negatives: $1_{(x,y)}$, $1_{(y,z)}$, and $1_{(z,x)}$ can be satisfied weakly (with $x, y, z$ in one equivalence class) but not strictly. In settings with simple weak restrictions and LCS (UCS) strict restrictions, as in the application we study, Section 5.2 shows how testing can be performed by using the enumeration procedure to enumerate indifference classes.

5.1 Definitions

When indifferences are possible, the modeler’s data becomes an observed choice correspondence $C_{obs}: \mathcal{D} \to \mathcal{P}(X)$. Two possible definitions of consistency come naturally to mind. Observed choices are strongly consistent with a theory if there is a correspondence $C$ under the theory such that $C_{obs}(S) = C(S)$ for each $S \in \mathcal{D}$. This definition is used by Richter (1966) to study Rationality with weak preferences. It assumes the modeler observes the DM choose all her acceptable options in a set. If
the modeler does not trust that this is the case, then choices are said to be weakly consistent with a theory if there is a correspondence \( C \) under the theory such that \( C_{obs}(S) \subseteq C(S) \) for each \( S \in D \). This definition is used by Afriat (1967) to study Rationality of consumers with nonsatiated preferences.

As is well-known, any \( C_{obs} \) will be weakly consistent with Rationality if the DM is allowed to be indifferent among all options. Bounded rationality theories typically contain Rationality as a special case, so the same caveat applies. In some contexts, it is natural to impose greater regularity on preferences. Afriat (1967) advances this approach in consumer theory, where monotonicity is a natural property. Similarly, one might restrict attention to preferences respecting FOSD when \( X \) contains lotteries, or preferences where delays are worse when \( X \) contains payment streams. In general, for any strict transitive relation \( Q \) over \( X \), and any preference-based theory, one may consider the theory’s \( Q \)-variant, where the preference must respect \( Q \)-comparisons.

### 5.2 Enumerating Indifference Classes

A restriction is weak if it pertains to the weak component of a relation, and strict otherwise. The preference restrictions one identifies in (a) and (b) of the methodology may be of both kinds. One must thus better understand procedures to test acyclic satisfiability in these contexts. As a step in that direction, we show how enumeration extends to problems with simple weak restrictions and LCS (UCS) strict restrictions.

Fix a collection \( R^s \) of all LCS (or all UCS) strict restrictions, as in the rest of the paper, and a collection \( R^w \) of simple weak restrictions. Say \((R^s, R^w)\) is acyclically satisfiable if there exists an acyclic relation satisfying the restrictions in \( R^s \) strictly, and those in \( R^w \) weakly. Let \( R^w \) be the transitive closure of comparisons listed in \( R^w \): \( xR^w y \) if there is a sequence \( (x_k)_{k=1}^{K} \) with \( x_1 = x, \ x_K = y, \) and \( 1(x_k, x_{k+1}) \in R^w \) for all \( k = 1, \ldots, K - 1 \). Say \( x \) and \( y \) are equivalent, denoted \( x \sim y \), if \( xR^w y \) and \( yR^w x \). Note that \( \sim \) partitions \( X \), and any relation \( R \) satisfying \( R^w \) has \( xRy \) and \( yRx \) if \( x \sim y \).

Having identified options belonging to the same equivalence class for any relation satisfying \( R^w \), we rephrase the restrictions onto \( \sim \)-equivalence classes. Importantly, we may now treat weak restrictions between different equivalence classes as if they are strict.\(^{16}\) For any \( x \in X \), let \( \bar{x} \subseteq X \) be its \( \sim \)-equivalence class. Let \( \bar{R} \) be the following strict restrictions on a relation over \( X/\sim \): (i) \( 1(x, y) \in \bar{R} \), for each \( 1(x, y) \in R^w \) such that \( x \sim y \).

\(^{16}\)Indeed, suppose an acyclic relation \( R \) satisfies \((R^s, R^w)\), with \( xRy, yRx \), \( 1(x, y) \in R^w \) and \( x \not\sim y \). The relation obtained by dropping \( yRx \) remains acyclic and satisfies \((R^s, R^w)\).
that \( x \not\sim y \); and (ii) \( 1_{(x, \Sigma)} \in \mathcal{R} \), where \( \Sigma = \{ \{ \bar{y} \mid y \in S \} \mid S \in \Sigma \} \), for each \( 1_{(x, \Sigma)} \in \mathcal{R}^s \) (similarly for UCS restrictions). After constructing \( X/\sim \) and \( \mathcal{R} \) (which takes polynomial time), we apply our enumeration procedure to this new space.

**Proposition 4** \((\mathcal{R}^s, \mathcal{R}^w)\) is acyclically satisfiable if and only if the set of strict LCS (UCS) restrictions \( \bar{\mathcal{R}} \), defined over the quotient set \( X/\sim \), is acyclically satisfiable. The latter is testable by enumeration.

### 5.3 An application

We now illustrate these ideas by adding the possibility of indifference to the first theory discussed in Section 4.1: the DM maximizes a weak preference ordering over a consideration set mapping satisfying (3). In addition to (3) being a focal property,\(^{17}\) Lleras et al (2017)’s full-data axiomatic characterization (under strong consistency) provides a benchmark against which to contrast our results.

Like in Section 4.1, IIA violations reveal preference comparisons. Generalizing condition (5) to correspondences, the DM must weakly prefer \( x \) over \( y \) if \( x \) is among her choices from \( S \), and \( y \in S \) is among her choices in \( T \supseteq S \).\(^{18}\) This holds under both weak and strong consistency. We denote by \( \mathcal{R}^w_{IIA} \) these (simple) weak restrictions.

Under strong consistency, the data further reveals that the DM strictly prefers \( x \) over \( y \) if \( y \not\in C_{obs}(S) \). The set of all such (simple) strict restrictions is denoted \( \mathcal{R}^s_{IIA} \). Following Section 5.1, strict restrictions only arise under weak consistency from considering the \( Q \)-variant of the theory, for a strict transitive relation \( Q \). Let \( \mathcal{R}^s_Q \) be the set of strict restrictions \( 1_{(x,y)} \) for each \( x, y \) such that \( xQy \). Proposition 6(a) in the Appendix establishes that weak consistency is equivalent to acyclic satisfiability of \((\mathcal{R}^s_Q, \mathcal{R}^w_{IIA})\). One might be tempted to conjecture that strong consistency holds if and only if \((\mathcal{R}^s_{IIA}, \mathcal{R}^w_{IIA})\) is acyclically satisfiable, since this seemingly generalizes Proposition 1(a) to weak preferences, and encapsulates all the restrictions that can be teased out from Lleras et al.’s (2017) axiomatic full-data result.

Such a conjecture would be incorrect, as additional restrictions arise from choice configurations that reveal inattention. To see this, consider the dataset:

\[
\begin{array}{c|cccccccc}
S & wx & vxz & wxy & wxz & wxy & wxyz & xyz \\
\hline
C_{obs}(S) & wx & yz & w & x & y & z
\end{array}
\]


\(^{18}\)Applying \( S = T \), the DM is revealed indifferent among all her choices from a set.
Then \((R^s_{IIA}, R^w_{IIA})\) is acyclically satisfiable (as \(R^s_{IIA} = \emptyset\) and \(R^w_{IIA}\) consists of \(1_{(w,x)}, 1_{(x,w)}, 1_{(y,z)}\) and \(1_{(z,y)}\)), but the data is not strongly consistent with the theory.\(^{19}\) Incidentally, notice that \(C_{obs}\) would arise from maximizing a preference (with \(w, x, y, z\) in one indifference class above \(v\)) over consideration sets equal to \(C_{obs}\) (and thus satisfying (3)). Hence this theory is subject to the extensibility issue identified in Section 3, even though it is not subject to it when restricting to strict preferences.

The problem is that \(R^s_{IIA}\) is missing some non-simple LCS restrictions coming from choice problems that reveal inattention. In the Appendix we detail these restrictions and define \(\bar{R}^s_{IIA}\) to be the set obtained by appending these missing ones to \(R^s_{IIA}\).\(^{20}\) Proposition 6(b) in the Appendix establishes that strong consistency is equivalent to acyclic satisfiability of \((\bar{R}^s_{IIA}, R^w_{IIA})\). Thus Proposition 4 applies for both weak and strong consistency, making testing tractable by enumeration.

6. Discussion and Further Applications

The empirical content of Rationality is captured through the acyclicity of Samuelson’s revealed preference. The methodology we propose shows how to fruitfully expand the scope of this approach. First, one can explore restrictions the data reveals about any acyclic relation that is (explicitly or implicitly) part of the model. Second, these restrictions can be more complex than simple comparisons between two options, in which case the notion of acyclicity must be generalized to acyclic satisfiability. Third, testing acyclic satisfiability is tractable by enumeration in the presence of all LCS (or all UCS) restrictions, and is otherwise likely to be NP-hard.

Beyond the motivating applications already covered, this section illustrates the surprising versatility and generality of our approach. Restrictions can pertain to orderings other than preference (e.g. salience or priority), and to relations that may be incomplete (as in problems of just-noticeable differences). They can capture satisficing instead of maximizing behavior. The methodology applies to settings where \(X\) is not finite (e.g. consumer theory). Beyond bounded rationality, it even provides useful insight in some classic problems of interactive decision making with rational agents.

\(^{19}\)It is impossible to define choices in the out-of-sample set \(\{w, x, y, z\}\) (e.g., choosing \(w\) implies it is considered in \(\{w, x, y\}\), yet only \(x\), which must be indifferent to \(w\), is picked).

\(^{20}\)For the example above, the missing strict restrictions are \(1_{(w,\{x\},\{y,z\})}\) and \(1_{(x,\{w\},\{y,z\})}\), as \(w\) is chosen from \(\{w, x, y\}\) but not \(\{w, x, z\}\) and vice-versa for \(x\); and similarly \(1_{(y,\{w\},\{x,z\})}\) and \(1_{(z,\{w\},\{x,y\})}\). These indeed lead to a failure of acyclic satisfiability.
Versatility, however, does not mean that there is no limitation. First, we do not have a one-size-fits-all result to find the restrictions capturing an arbitrary theory’s empirical content through acyclic satisfiability. For each application we provide, we had to think carefully about the theory to identify the full set of restrictions. These results are nontrivial. Second, while many theories (explicitly or implicitly) involve a relation to examine, it is possible for a theory to have a sufficiently different structure that capturing its empirical content through acyclic satisfiability is unfruitful. Third, finding choice patterns yielding mixed sets of restrictions is suggestive that testing is NP-hard, but applying Proposition 3 still requires some work. Indeed, it is conceivable for seemingly complex restrictions to combine into a simpler form. One must show that consistency with the theory is reducible to the acyclic satisfiability problem of Proposition 3. We did this exercise for all the theories noted in this paper as NP-hard to test. Proposition 3 conveniently allows one to begin the analysis from a problem very close to the one hand; the main remaining challenge is that complex restrictions may not immediately present in the exact same format as in the proposition.21

With these caveats in mind, we briefly discuss further applications that illustrate the wide scope of our methodology. Some are carried out in separate papers, while others are detailed in the Appendix.

Satisficing heuristics. Barberà, de Clippel, Neme and Rozen (2018) propose a class of theories, Order-$k$ Rationality, capturing a variety of choice heuristics whereby a DM picks an option that may only be ‘good enough’ instead of optimal. For example, a DM may stop searching when her consideration set is sufficiently large, or be satisfied choosing a sufficiently well-ranked option (e.g., in the top two, or top quintile of a set). They show the empirical content of Order-$k$ Rationality is captured by acyclic satisfiability of LCS restrictions on the DM’s preference, and thus testable by enumeration. They also study settings where observed choice correspondences arise from contextual effects, and show the tractability extends. By contrast, they show Sen (1993)’s theory of choosing the second best, which falls outside the Order-$k$ Rationality class as the DM cannot choose her first best, is reducible to mixed sets of binary restrictions and thus NP-hard to test.

Interactive settings. While motivated by developments in bounded rationality, the methodology we propose is applicable to testing other theories of choice as well. de

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21For instance, in Limited Attention, no single choice directly reveals that $a \succ c$ or $a' \succ c$, but a choice can reveal $a \succ b$ or $a' \succ b'$, and another choice can reveal $b \succ c$ and $b' \succ c$. 

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Clippel and Rozen (2018b) study the testable implications induced by rational agents implementing some classic assignment/matching methods. Under serial dictatorship, for instance, agents are ranked according to a power relation, and can pick their most preferred option among those that haven’t been picked by more powerful agents. The modeler observes final allocations as a function of some collective endowments, but does not observe, and tries to identify, preferences and the power relation. The unobserved constraints from other individuals’ choices imply that even the choices of rational agents can violate SARP. The data generates LCS restrictions on the unknown power relation, and testing is thus tractable by enumeration. Stability is also easy to test for an interesting class of many-to-one matching problems. By contrast, the core of Shapley and Scarf (1974)’s housing market is NP-hard to test. This is shown by identifying a subclass of problems where core consistency amounts to acyclic satisfiability of a mixed set of binary restrictions on a single agent’s preference.

**Misperception in consumer theory.** de Clippel and Rozen (2018a) study misperceived preferences in consumer theory. They consider a consumer who has a true utility function, but may fail to perfectly maximize it. In each consumption problem, she may misperceive her marginal rate of substitution, and behave as if she is maximizing one of (infinitely) many possible ‘nearby’ utility functions instead. They show it is easy to test consistency of demand data with misperception, as it amounts to acyclic satisfiability of LCS restrictions on the consumer’s preference over bundles.

**Reference dependence.** Reference effects provide an important context in which multiple preferences play a role. Without further restrictions, any choices can be explained by the maximization of a reference-dependent preference. Rubinstein and Salant (2006b)’s theory of Triggered Rationality structures reference-point formation, by positing a salience ordering $\succ_\sigma$ over the alternatives and a collection of preference orderings $\{P_x\}_{x \in X}$. The most salient element in a set is the DM’s reference point, anchoring the preference $P_x$ maximized. While there are multiple preference orderings in this theory, applying our methodology to the salience ordering yields a tractable test. Indeed, Proposition 7 in the Appendix shows that the empirical content of this theory is captured by UCS restrictions on $\succ_\sigma$. These restrictions guarantee that the salience ordering does not lead to cycles in the DM’s preference relations. The test

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22In addition, studying choice from lists, Rubinstein and Salant (2006a) propose a model (their Example 4) where the preference depends on the first element presented. Triggered Rationality can be seen as the case where the list is unknown, or subjectively determined.
is tractably implemented using one enumeration procedure nested in another.

Shortlisting. Noteworthily, our methodology applies to acyclic relations, not just orderings. This proves useful in studying Manzini and Mariotti (2007)’s Shortlisting theory. Observe that the shortlisting relation $P_1$ must be acyclic for choices to be well defined, else the DM’s shortlist would be empty in the choice problem corresponding to the cycle. It is easy to see that UCS restrictions on $P_1$ are generated by certain choice configurations. For instance, suppose $c_{obs}(\{a, x\}) = x$ but $c_{obs}(\{a, x, y, z\}) = a$. Then either $y$ or $z$ must eliminate $x$ from consideration under $P_1$ (note that $a$ cannot eliminate $x$, as $x$ is chosen in $a$’s presence). However, non-UCS restrictions on $P_1$ can be generated as well: e.g., either $b$ eliminates $z$, or $d$ eliminates $y$, if $c_{obs}(\{b, y, z\}) = y$ and $c_{obs}(\{d, y, z\}) = z$. Additional data may require a third alternative to eliminate both $b$ and $e$, leading to something resembling a binary LCS restriction. This lights a path towards an application of Proposition 3. Indeed, Proposition 8 in the Appendix shows that testing consistency with Shortlisting is NP-hard. This remains true for Order Shortlisting, which requires the DM’s preference to be an ordering.

Categorization and Rationalization. Manzini and Mariotti (2012)’s Categorization and Cherepanov et al (2013)’s Rationalization are known to be observationally equivalent to maximizing a complete, asymmetric $P$ over a consideration set mapping $\Gamma$ satisfying IIA. While neither theory has an explicit acyclic relation, the IIA property on $\Gamma$ generates an implicit one. Suppose $c_{obs}$ extends to a choice function $c$ under the theory. If $x = c(X)$, then IIA implies $x$ is considered in all choice problems (it is ‘most salient’); if $x' = c(X \setminus \{x\})$, then $x'$ is considered in all choice problems not containing $x$ (it is ‘second-most salient’); and so on. Choices reveal restrictions on this relation: e.g., $c_{obs}(\{x, y\}) = x$ and $c_{obs}(\{x, y, z\}) = y$ leads to the UCS restriction that either $y$ or $z$ is more salient than $x$ (else $x$ would be chosen from $\{x, y, z\}$). Proposition 9(a) in the Appendix shows that when a revealed preference $P^*$ is either complete over observed choices (e.g., $\mathcal{D}$ contains all binary problems) or simply acyclic, then the empirical content of these theories is captured by UCS restrictions on the more-salient-than relation, and enumeration applies. Proposition 9(b) shows testing is otherwise NP-hard, as trying to complete $P^*$ while avoiding the creation of top cycles in consideration sets yields LCS restrictions as well.

\footnote{\(P^*\) is generated by appending information from pairwise choices to the revealed preference of Cherepanov et al (2013): i.e., \(xP^*y\) if $x = c_{obs}(\{x, y\})$, or if $x = c_{obs}(S)$ and $y = c_{obs}(T) \in S \subset T$.}
Undominated alternatives. We consider a DM who chooses the set of undominated alternatives according to an acyclic (possibly incomplete) relation. Again, our methodology accommodates this, as acyclic satisfiability does not require the relation satisfying the restrictions to be complete. This choice procedure, which yields an observed choice correspondence, has several interesting interpretations. It has the same testable implications as Masatlioglu and Nakajima (2013)’s Markovian Choice by Iterative Search, when the starting point of the DM’s search process is unobserved. It corresponds to the shortlist in Manzini and Mariotti (2012)’s Shortlisting, which one might observe from alternatives considered by a committee, or interviewees for a position. It also relates to theories of just-noticeable differences. Following Luce (1956) and ensuing works, insights from psychology that people have difficulty discerning differences in stimuli (e.g., the Weber-Fechner law) may apply to preference comparisons. Consider a DM with a utility function \( u \) and a threshold function \( \tau : X \times X \to \mathbb{R}_+ \). She discerns \( y \) is preferable to \( x \) if \( u(y) > u(x) + \tau(x, y) \) (i.e., when the utility difference is big enough given the options); and selects alternatives for which she cannot discern anything better. This theory, too, is equivalent to choosing undominated elements. Proposition 10 in the Appendix establishes that all these theories are tractable to test: the empirical content of choosing undominated elements according to an acyclic relation amounts to acyclic satisfiability of UCS restrictions.

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24 Of course, the existence of an acyclic relation satisfying the restrictions, also implies the existence of an ordering satisfying the restrictions. However, that ordering need not yield the observed choices, generating for instance a single-valued choice function instead of a correspondence. Thus allowing the relation to be incomplete in the definition of acyclic satisfiability is useful indeed.

25 See Fechner (1860)’s seminal work and the large literature that follows.

26 A perceived-preference cycle \( x_1 \succ x_2 \succ \ldots \succ x_n \succ x_1 = x_1 \) implies \( \sum_{i=1}^{n} \tau(x_{i+1}, x_i) < 0 \), yet \( \tau(\cdot, \cdot) \geq 0 \). Conversely, for an acyclic \( \succ \), take \( u : X \to \mathbb{R} \) respecting \( \succ \)-comparisons, with \( \tau(a, b) = 0 \) if \( a \succ b \) and \( \tau(a, b) = \max_y u(y) - \min_x u(x) \) otherwise. Then \( y \succ x \) iff \( u(y) > u(x) + \tau(x, y) \). This theory includes Luce (1956)’s theory with constant thresholds as well as generalizations proposed in the large ensuing literature, which examines how the threshold varies with the alternatives compared, and the impact on perceived preferences.
REFERENCES


**APPENDIX A: PROOFS FOR SECTION 4**

The proofs of Lemmas 1-2 and Proposition 2 are provided in the text.

**Proof of Proposition 1(a): Consistency for IIA Property on Attention**

Necessity was given in the text. For sufficiency, suppose there is an acyclic relation satisfying (5), and let \( P \) be a transitive completion (hence \( P \) still satisfies (5)). Define \( \Gamma_P \) by \( \Gamma_P(S) = \{ \text{arg min}_P S \} \cup \{ \text{c}_\text{obs}(T) \mid S \subseteq T, T \in \mathcal{D}, \text{c}_\text{obs}(T) \in S \} \) for all \( S \in \mathcal{P}(X) \). This \( \Gamma_P \) satisfies (3) and thus by CFS13 (Section 4.1), it is the set of rationalizable elements for some rationales \( \{R_k\}_k \). Let \( c \) be the choice function arising from \( (P, \{R_k\}_k) \) under the theory. For any \( S \in \mathcal{D} \), we show \( c(S) = \text{c}_\text{obs}(S) \). Suppose otherwise; then \( \Gamma_P(S) \) contains at least two elements, and \( c(S) \) must be the observed choice from some \( T \in \mathcal{D} \) with \( S \subset T \). This implies \( \text{c}_\text{obs}(S) \) is revealed preferred to \( c(S) \), contradicting \( P \)-maximality of \( c(S) \) in \( \Gamma_P(S) \) for a \( P \) satisfying (5). \( \text{Q.E.D.} \)
Proof of Proposition 1(b): Consistency for Limited Attention

The argument for necessity appears in the main text. For sufficiency, suppose an acyclic relation satisfying (6) exists, and let $P$ be a transitive completion (so $P$ still satisfies (6)). We define $\Gamma : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ as follows. For $S \in \mathcal{D}$, $\Gamma(S) = \{c_{obs}(S)\} \cup \{x \in S | c_{obs}(S)Px\}$; for $S \notin \mathcal{D}$,

$$\Gamma(S) = \begin{cases} 
\Gamma(T) & \text{if } S \subseteq T, \; T \in \mathcal{D}, \text{ and } \Gamma(T) \subseteq S \\
S & \text{otherwise.}
\end{cases}$$

Clearly $\Gamma(S) \neq \emptyset$ for any $S \in \mathcal{P}(X)$ and the $P$-maximal element in $\Gamma(S)$ is $c_{obs}(S)$ for any $S \in \mathcal{D}$. We show $\Gamma$ is well-defined and satisfies (1b).

Suppose by contradiction that $\Gamma$ is not well-defined for some $S$. This means that for some $S \notin \mathcal{D}$, there exist $T, T' \in \mathcal{D}$ such that $S \subseteq T \cap T'$ with $\Gamma(T) \cup \Gamma(T') \subseteq S$, but $\Gamma(T) \neq \Gamma(T')$. This implies $c_{obs}(T) \neq c_{obs}(T')$. Consider any $y \in T \setminus T'$. Then, since $S \subseteq T'$, $y \in T \setminus S$. Moreover, since $\Gamma(T) \subseteq S$, we know $y \in T \setminus \Gamma(T)$. By definition of $\Gamma(T)$ for $T \in \mathcal{D}$, this means $yPc_{obs}(T)$. Similarly, if $y \in T' \setminus T$, we conclude $yPc_{obs}(T')$, contradicting that $P$ satisfies (6). To show $\Gamma$ satisfies (1b), consider $S \in \mathcal{P}(X)$ and $x \in S \setminus \Gamma(S)$. We prove $\Gamma(S \setminus \{x\}) = \Gamma(S)$ in each of the four possible cases:

Case 1: $S \setminus \{x\}, S \in \mathcal{D}$. Since $S \in \mathcal{D}$, and $x \notin \Gamma(S)$, we know $xPc_{obs}(S)$. Suppose that $\Gamma(S \setminus \{x\}) \neq \Gamma(S)$. Then $c_{obs}(S) \neq c_{obs}(S \setminus \{x\})$. Applying (6) for choice problems $S$ and $S \setminus \{x\}$, we conclude $c_{obs}(S)Px$, a contradiction.

Case 2: $S \setminus \{x\} \notin \mathcal{D}, S \notin \mathcal{D}$. Since $S \setminus \{x\} \notin \mathcal{D}$, we know $\Gamma(S \setminus \{x\}) = c_{obs}(S \setminus \{x\}) \cup \{y \in S | c_{obs}(S \setminus \{x\})Py\}$. Since $S \setminus \Gamma(S) \neq \emptyset$, there exists $T \in \mathcal{D}$ with $S \subseteq T$ and $\Gamma(T) \subseteq S$. Because $T \in \mathcal{D}$, $zPc_{obs}(T)$ for all $z \in T \setminus S$. Since $\Gamma(S) = \Gamma(T)$, we know $x \in T \setminus \Gamma(T)$. Hence $xPc_{obs}(T)$. If $\Gamma(S \setminus \{x\}) \neq \Gamma(S) = \Gamma(T)$, then $c_{obs}(S \setminus \{x\}) \neq c_{obs}(T)$ contradicting (6) for the pair of sets $T$ and $S \setminus \{x\}$.

Case 3: $S \setminus \{x\} \notin \mathcal{D}, S \in \mathcal{D}$. Since $S \in \mathcal{D}$, $\Gamma(S) = c_{obs}(S) \cup \{y \in S | c_{obs}(S)Py\}$. If $x \in S \setminus \Gamma(S)$ then $\Gamma(S) \subseteq S \setminus \{x\}$, so by construction $\Gamma(S \setminus \{x\}) = \Gamma(S)$.

Case 4: $S \setminus \{x\}, S \notin \mathcal{D}$. Since $S \setminus \Gamma(S) \neq \emptyset$, there exists $T \in \mathcal{D}$ with $S \subseteq T$ and $\Gamma(T) \subseteq S$. Since $x \in S \setminus \Gamma(S)$, then $\Gamma(T) = \Gamma(S) \subseteq S \setminus \{x\}$ and so $\Gamma(S \setminus \{x\}) = \Gamma(T)$ by construction, and equals $\Gamma(S)$ by transitivity.

Q.E.D.
Proof of Proposition 1(c): Consistency for Spotlighting

Necessity follows from the discussion above. As for sufficiency, let $P$ be an (strict) acyclic relation satisfying the restrictions in (7). We can assume without loss of generality that $P$ is complete, that is, an ordering.

Let $S$ be the set of $S \in D$ for which all elements of $Y(S)$ are $P$-superior to $c_{obs}(S)$. For all other choice problem $S \in D$, let $y(S)$ be an element of $S \setminus \{c_{obs}(S)\}$ such that all elements of $Z(S, y(S))$ are $P$-superior to $c_{obs}(S)$. For each $R \subseteq X$, let

$$\Gamma_1(R) = \{c_{obs}(S) | R \subseteq S \in S, c_{obs}(S) \in R\},$$
$$\Gamma_2(R) = \{c_{obs}(S) | S \in D \setminus S, \{y(S), c_{obs}(S)\} \subseteq R\},$$
$$\Gamma(R) = \Gamma_1(R) \cup \Gamma_2(R) \cup \arg \min P R.$$

The correspondence $\Gamma$ has nonempty values since $P$ admits a minimum for each $R$. We now show it satisfies (4). Let $x \neq y$ such that $x \in \Gamma(R \cup \{y\}) \setminus \Gamma(R)$. Both $\Gamma_1$ and being $P$-minimal satisfy IIA, and hence $x \in \Gamma_2(R \cup \{y\})$. Thus $x = c_{obs}(S)$ for some $S \in D \setminus S$ such that $\{y(S), c_{obs}(S)\} \subseteq R$. Since $x \notin \Gamma_2(R)$ and $x \in R$, it must be that $y(S) = y$. Then $x \in \Gamma_2(R) \subseteq \Gamma(R)$ for all $R$ containing $\{x, y\}$, as desired.

Finally, one must check that $c_{obs}(R) = \arg \max P \Gamma(R)$ for all $R \in D$. By definition, $c_{obs}(R) \in \Gamma_1(R) \subseteq \Gamma(R)$ if $R \in S$, and $c_{obs}(R) \in \Gamma_2(R) \subseteq \Gamma(R)$ if $R \not\in S$, as $y(R) \in R$. Now suppose, on the contrary that there exists $y \in \Gamma(R)$ such that $yPC_{obs}(R)$. In that case, $y$ is not $P$-minimal in $R$, and hence $y$ belongs to either $\Gamma_1(R)$ or $\Gamma_2(R)$. If $y \in \Gamma_1(R)$, then $y = c_{obs}(S)$, for some superset $S \in S$ of $R$ that contains $y$. In that case, $c_{obs}(R) \in Y(S)$. We reach a contradiction since the restrictions on $P$ imply that $c_{obs}(R)Py$. Finally, if $y \in \Gamma_2(R)$, then $y = c_{obs}(S)$, for some $S \in D \setminus S$ such that $\{y(S), c_{obs}(S)\} \subseteq R$. In that case, $c_{obs}(R) \in Z(S, y(S))$. Once again we reach a contradiction since the restrictions on $P$ imply that $c_{obs}(R)Py$. Q.E.D.

Proof of Proposition 3: Testing mixed binary restrictions is NP-hard

Fix an instance of SAT3 with a set $V$ of variables and a set $C$ of clauses. The three literals (a variable or its negation) involved in a clause $C$ are denoted $\ell^C_i$ for $i = 1, 2, 3$. Consider the set of options $X$ that contains all variables $v$ and their negations ($\bar{v}$), all clauses $C$, an option $x_C$ for each clause $C$, and an option $t$. Let $R$ be the following mixed set of binary restrictions: the restriction $1_{\{v, \bar{v}, t\}}$ for each $v \in V$ and the
restrictions $1_{(t,C)}$, $1_{(x,C,(\ell^1_C,\ell^2_C))}$ and $1_{(C,(x,C,\ell^0_C))}$ for each clause $C$.

We show the instance of SAT3 has a truthful assignment if and only if $\mathcal{R}$ is acyclically satisfiable. Given a truthful assignment for SAT3, an ordering constructed as follows will satisfy $\mathcal{R}$: place from worst to best, first all variables $v$ that are true, then $\bar{v}$ for each false variable $v$, then $x_C$ for all clause $C$ such that either $\ell^1_C$ or $\ell^2_C$ is true, then all clauses $C$, then $t$, then all remaining $x_C$'s, then $v$ for all false variables $v$, and finally $\bar{v}$ for all true variables $v$. Conversely, let $P$ be an acyclic relation satisfying $\mathcal{R}$. We can assume without loss of generality that $P$ is an ordering (otherwise take a completion of $P$; this will still satisfy $\mathcal{R}$). All variables ranked below $t$ are declared true, while all others are declared false. It is easy to check that this defines a truthful assignment for the instance of SAT3.

**Proposition 5**  
Testing consistency with Limited Attention is NP-hard.

**Proof.** Fix a mixed set $\mathcal{R}$ of binary restrictions defined on a set $X$. For each restriction $r$, let $x_r$ be the option whose contour set is being restricted, and let $y_r$ and $z_r$ ($y_r = z_r$ is allowed) be the two options potentially included in the upper (or lower) contour set of $x_r$ if $r$ is an UCS (or LCS) restriction. Consider the set of options $X'$ that contains all options in $X$, plus a new option $t_r$ for each LCS restriction $r$ and new options $u_r$, $v_r$, and $w_r$ for each UCS restriction $r$, and the following observed choices:

<table>
<thead>
<tr>
<th>$S$</th>
<th>$t_r, x_r$</th>
<th>$u_r, v_r$</th>
<th>$u_r, w_r$</th>
<th>$u_r, v_r, x_r$</th>
<th>$u_r, w_r, v_r, x_r$</th>
<th>$u_r, w_r, x_r$</th>
<th>$v_r, y_r, z_r$</th>
<th>$w_r, y_r, z_r$</th>
<th>$t_r, x_r, y_r, z_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{obs}(S)$</td>
<td>$t_r$</td>
<td>$u_r$</td>
<td>$u_r$</td>
<td>$v_r$</td>
<td>$w_r$</td>
<td>$y_r$</td>
<td>$z_r$</td>
<td>$x_r$</td>
<td></td>
</tr>
</tbody>
</table>

for each restriction $r$. By Proposition 1(b), $c_{obs}$ is consistent with Limited Attention if and only if there is an acyclic relation $P$ satisfying the following restrictions $\mathcal{R}'$:

(i) $x_rPy_r$ or $x_rPz_r$, from $c_{obs}\{\{t_r, x_r\}\} = t_r$ and $c_{obs}\{\{t_r, x_r, y_r, z_r\}\} = x_r$,

for each LCS $r$, and the following three restrictions for each UCS $r$:

(ii) $y_rPv_r$ or $z_rPw_r$, from $c_{obs}\{\{v_r, y_r, z_r\}\} = y_r$ and $c_{obs}\{\{w_r, y_r, z_r\}\} = z_r$,
(iii) $v_rPx_r$, from $c_{obs}\{\{u_r, v_r\}\} = u_r$ and $c_{obs}\{\{u_r, v_r, x_r\}\} = v_r$,
(iv) $w_rPx_r$, from $c_{obs}\{\{u_r, w_r\}\} = u_r$ and $c_{obs}\{\{u_r, w_r, x_r\}\} = w_r$.

We show acyclic satisfiability holds for $\mathcal{R}$ if and only if it holds for $\mathcal{R}'$. Let $P$ be an acyclic relation satisfying the restrictions in $\mathcal{R}$. We can assume without loss of generality that $P$ is an ordering (otherwise take a completion of $P$ that still satisfy
restrictions in $\mathcal{R}$). Then extend $P$ to any ordering on $X'$ that ranks the new options $v_r$ and $w_r$ for each $r$ directly above the corresponding $x_r$, and below any other $y \in X$ such that $yPx_r$. It is easy to check that any such extension satisfies $\mathcal{R}'$. Conversely, let $P'$ be an acyclic relation satisfying the restrictions in $\mathcal{R}'$. We can assume without loss of generality that $P'$ is an ordering (otherwise take a completion of $P'$; that will still satisfy $\mathcal{R}'$). Clearly, the restriction of $P'$ to $X$ satisfies $\mathcal{R}$.

Q.E.D.

APPENDIX B: PROOFS FOR SECTION 5

Proof of Proposition 4: Enumerating Indifference Classes

For necessity, let $R$ be an acyclic relation satisfying $(\mathcal{R}^s, \mathcal{R}^w)$. Whenever $1_{(x,y)} \in \mathcal{R}^w$ but $x \not\sim y$, then we may assume $R$ does not include $yRx$; indeed, the relation constructed by dropping $yRx$ is still acyclic and satisfies $(\mathcal{R}^s, \mathcal{R}^w)$. Define $\bar{R}$ by $\bar{x}\bar{y}$ if $xRy$ but not $yRx$. Then $\bar{R}$ inherits acyclicity from $R$ and satisfies $\bar{R}$ by construction.

For sufficiency, let $\bar{R}$ be an acyclic relation satisfying $\bar{R}$. Define $R$ on $X$ by (i) if $\bar{x} = \bar{y}$, then $xRy$ and $yRx$ and (ii) if $\bar{x}\bar{y}$ then $xRy$ for all $x \in \bar{x}$ and all $y \in \bar{y}$. Notice that $R$ inherits acyclicity from $\bar{R}$. By construction and the fact that $\bar{R}$ satisfies $\bar{R}$, each restriction $1_{(x,y)} \in \mathcal{R}^w$ such that $x \not\sim y$ is satisfied, as $\bar{x}\bar{y}$; and each $1_{(x,y)} \in \mathcal{R}^w$ such that $x \sim y$ is satisfied, as $\bar{x} = \bar{y}$. Finally, each $1_{(x,\Sigma)} \in \mathcal{R}^s$ is satisfied since there exists $\bar{S} \in \bar{\Sigma}$ with $\bar{x}\bar{y}$ for all $\bar{y} \in \bar{S}$, and $S \subseteq \{\bar{y} \mid \bar{y} \in \bar{S}\}$ for some $S \in \Sigma$ by construction of $\bar{\Sigma}$. Thus $(\mathcal{R}^s, \mathcal{R}^w)$ is acyclically satisfiable. Q.E.D.

Consider the theory that the DM maximizes a weak preference ordering over a consideration set mapping $\Gamma$ satisfying (5). We have already discussed necessary conditions for weak consistency. We now develop the remaining ones for strong consistency. Recall that $\bar{x}$ is the $\sim$-indifference class of $x$. Given a set $Y \subseteq X$, define

$$\mathcal{D}(Y, \bar{x}) = \{S \in \mathcal{D} \mid S \subseteq Y, C_{obs}(S) \subseteq \bar{x}\}$$

to be those observed choice problems contained in $Y$ for which choices belong to $\bar{x}$. There must be a worst indifference class intersecting $Y$, but the data may rule out some indifference classes from this role. Consider an indifference class $\bar{x}$ intersecting $Y$, and suppose there is a subset $A_Y$ of $\bar{x}$ such that for every $a \in A_Y$, there is an observed choice problem $S_a \in \mathcal{D}(Y, \bar{x})$ with $a \in S_a$ and $C_{obs}(S_a) \subseteq A_Y \setminus \{a\}$. In other words, $A_Y$ is a collection of indifferent elements, each of which is unchosen in some.
choice problem in $\mathcal{D}(Y, \bar{x})$ for which all observed choices belong to $A_Y$. If such $A_Y$ exists, then $\bar{x}$ cannot be the worst indifference class in $Y$. For if $\bar{x}$ were the worst, then the consideration set $\Gamma(\cup_{a \in A_Y} S_a)$ would be empty, because (i) if $y \in \Gamma(\cup_{a \in A_Y} S_a)$ then $y \in \Gamma(S_a)$, so $y$ must be in $\bar{x}$ and thus $y \in C_{obs}(S_a) \subset A_Y$; and (ii) finally, for $y \in A_Y$, if $y \in \Gamma(\cup_{a \in A_Y} S_a)$, then $y$ would be chosen in $S_y$, a contradiction. Thus, when such $A_Y$ exists, each $x \in \bar{x}$ (or equivalently given the weak restrictions, a fixed representative element $x$ of $\bar{x}$) must be strictly preferred to some other element in $Y$.

Observe that such a set $A_Y$ exists if and only if acyclic satisfiability fails for the following strict LCS restrictions $R^*(Y, \bar{x})$ defined over elements of $\bar{x} \cap Y$ only:

1. $(a, \{\{z\} \mid z \in C_{obs}(S)\})$ for each $S \in \mathcal{D}(Y, \bar{x})$ such that $a \in \bar{x}$ and $a \in S \setminus C_{obs}(S)$. Indeed, if such $A_Y$ exists, then no element $a \in A_Y$ can rank worst in $A_Y$ given $R^*(Y, \bar{x})$; and conversely, if acyclic satisfiability of $R^*(Y, \bar{x})$ fails, let $A_Y$ be the set of unranked elements when the enumeration fails (each $a \in A_Y$ must be at the top of a remaining restriction from some $S \in \mathcal{D}(Y, \bar{x})$ such that $a \in S \setminus C_{obs}(S)$ and $C_{obs}(S) \subseteq A_Y$, otherwise the restriction would have been removed in a prior step of the enumeration).

To incorporate the restrictions discussed above, let $\bar{R}^*_I$ append to $\bar{R}^*_{IIA}$ the following strict LCS restrictions: $1_{\{x, \{y\} \mid y \in Y\}}$ for every $x \in X$ and $Y \in \mathcal{P}(X)$ such that $\bar{R}^*(Y, \bar{x})$ is not acyclically satisfiable.

**Proposition 6** Consider the theory that the DM maximizes a weak preference ordering over a consideration set mapping $\Gamma$ satisfying (5).

(a) An observed choice correspondence $C_{obs}$ is weakly consistent with the $Q$-variant of the theory if and only if $(\bar{R}^*_Q, \bar{R}^*_I)$ is acyclically satisfiable.

(b) An observed choice correspondence $C_{obs}$ is strongly consistent with the theory if and only if $(\bar{R}^*_I, \bar{R}^*_IIA)$ is acyclically satisfiable.

Note that the number of restrictions in $\bar{R}^*_I$ may at first seem overwhelming, as one must potentially check each $Y \in \mathcal{P}(X)$. The beauty of the enumeration procedure is the ability to implement this tractably: one need only consider the $|X| - 1$ nonsingleton sets in the sequence $X, X \setminus \{x_1\}, X \setminus \{x_1, x_2\}, \ldots$. For any set $Y$ along this sequence, any element ruled out as a candidate for worst in a subset $Y'$ of $Y$ would also be ruled out as a candidate for worst in $Y$ itself. Finally, checking acyclic satisfiability of the LCS restrictions $\bar{R}^*(Y, \bar{x})$ fails, and thus knowing when a restriction applies, is easily implemented with a nested enumeration procedure.
PROOF OF Proposition 6(a): Weak Consistency

It remains to show sufficiency. For each $S \in \mathcal{D}$, define

$$\Gamma^*(S) = \{x \in S | x \in C_{\text{obs}}(T) \text{ for some } T \in \mathcal{D} \text{ containing } S\}.$$ 

Let $R$ be an acyclic completion of an acyclic relation satisfying $(\mathcal{R}_Q^s, \mathcal{R}_{I1A}^w)$. We prove that $C_{\text{obs}}(S) \subseteq \arg \max_R \Gamma^*(S)$ for all $S \in \mathcal{D}$. Otherwise, there exist $S \in \mathcal{D}$, $x \in C_{\text{obs}}(S)$ and $y \in \Gamma^*(S)$ such that $yPx$. This configuration of choices implies that $1_{(x,y)} \in \mathcal{R}_{I1A}^w$, contradicting that $R$ satisfies the restrictions in $\mathcal{R}_{I1A}^w$. Define $\Gamma(S) = \Gamma^*(S) \cup \arg \min_P S$, for all $S \subseteq X$. Note $\Gamma$ is non-empty valued and satisfies (3), as it is derived from $\Gamma^*$ only by adding $P$-minimal element. Finally, $Q$-comparisons are respected since $R$ satisfies $\mathcal{R}_Q^s$.

Q.E.D.

PROOF OF Proposition 6(b): Strong Consistency

It remains to show sufficiency. Let the DM’s preference be a weak ordering $R$ satisfying the restrictions, and let $P$ be its strict component. Next, we define a consideration set mapping. To do this, we introduce the following ordering $O$ on $X$. For each $\bar{x} \in X/\sim$, let $P_\bar{x}$ be an ordering on $\bar{x}$ satisfying the restrictions in $\mathcal{R}(U_\bar{x}, \bar{x})$ where $U_\bar{x}$ is the set of indifference classes ranked at least as high $\bar{x}$ under $R$. For each $y, z \in X$, say that $yOz$ if $yPz$, or if $\bar{y} = \bar{z}$ and $yP_{\bar{y}}z$. Thus $O$ agrees with $P$, and breaks the indifference in each indifference class $\bar{x}$ of $R$ by using $P_\bar{x}$. For each $S \subseteq X$, let

$$\Gamma^*(S) = \{x \in S | x \in C_{\text{obs}}(T) \text{ for some } S \subseteq T \in \mathcal{D}\},$$

and define $\Gamma = \Gamma^* \cup \arg \min_O$. Note $\Gamma$ is non-empty valued and satisfies (3).

It remains to show $C_{\text{obs}}(S) = \arg \max_R \Gamma(S)$ for $S \in \mathcal{D}$. Suppose there is $S \in \mathcal{D}$, $x \in C_{\text{obs}}(S)$ and $y \in \Gamma(S)$ such that $yPx$. Then $yOx$ as $O$ agrees with $P$; and thus $y \in \Gamma^*(S)$. By definition of $\Gamma^*$, there is a superset $T \in \mathcal{D}$ of $S$ such that $y \in C_{\text{obs}}(T)$. As $R$ satisfies $\mathcal{R}_{I1A}^s$, it must be that $xPy$, a contradiction. Thus $C_{\text{obs}}(S) \subseteq \arg \max_R \Gamma(S)$. For the opposite inclusion, suppose there is $x \in \arg \max_R \Gamma(S) \setminus C_{\text{obs}}(S)$. If $x$ is not $O$-minimal in $S$, then $x \in \Gamma^*(S)$, and there is $z \in C_{\text{obs}}(S) \subseteq \Gamma(S)$ such that $zPx$, contradicting the definition of $x$. But if $x$ is $O$-minimal element in $S$, then it is also $R$-minimal in $S$, and all $R$-best elements in $\Gamma(S)$ are $R$-minimal in $S$. Since $C_{\text{obs}}(S) \subseteq \arg \max_R \Gamma(S)$, $\bar{y} = \bar{x}$ for all $y \in C_{\text{obs}}(S)$; hence $P_\bar{x}$ satisfies $1_{(x,(\bar{y})|y \in C_{\text{obs}}(S))}$. This contradicts $O$-minimality of $x$, since $O$ agrees with $P_\bar{x}$ on $\bar{x}$.

Q.E.D.
**APPENDIX C: PROOFS FOR SECTION 6**

**C.1 Reference Dependence**

Denote by $\succ_\sigma$ the salience ranking in Triggered Rationality. If $x \in S$ is $\succ_\sigma$-maximal in $S$, then it is $\succ_\sigma$-maximal in subsets of $S$. Thus the DM’s revealed preference given the reference point $x$ in $S$ is $\succ_\sigma$ if $c_{\text{obs}}(R) = a$ for some $R \subseteq S$ such that $b, x \in R$. Thus $x$ cannot be most salient in $S$ if $P_{S,x}$ is cyclic. Let $\mathcal{R}_{\text{ref}}$ be the following necessary UCS restrictions on $\succ_\sigma$: for each $S \in \mathcal{P}(X)$ and $x \in S$ such that $P_{S,x}$ is cyclic, there is $y \in S \setminus \{x\}$ with $y \succ_\sigma x$. These restrictions are also sufficient:

**Proposition 7** Observed choices $c_{\text{obs}}$ are consistent with Triggered Rationality if and only if $\mathcal{R}_{\text{ref}}$ is acyclically satisfiable.

**Proof.** It remains to show sufficiency. Suppose an acyclic relation satisfying $\mathcal{R}_{\text{ref}}$ exists, and let $\succ_\sigma$ be a transitive completion (hence $\succ_\sigma$ still satisfies $\mathcal{R}_{\text{ref}}$). Let $x_i$ denote the $i$-th maximal element according to $\succ_\sigma$. For each $i$, let $P_{x_i}$ be a transitive completion of $P_{X_i,x_i}$. Such a completion exists, because $x_i$ being $\succ_\sigma$-maximal in $X_i = \{x_i, x_{i+1}, \ldots, x_n\}$ implies $P_{X_i,x_i}$ is acyclic. The choices generated by these primitives will now be shown to coincide with $c_{\text{obs}}$ on $\mathcal{D}$. Take any $S \in \mathcal{D}$. Let $k$ be the smallest index such that $x_k \in S$. Then $S \subseteq X_k$. By definition of $P_{x_k}$, $c_{\text{obs}}(S) \succ_{x_k} y$ for all $y \in S \setminus \{c_{\text{obs}}(S)\}$. Q.E.D.

Similarly to our discussion after Proposition 6, the number of restrictions in $\mathcal{R}_{\text{ref}}$ may seem overwhelming, as one must potentially check each $S \in \mathcal{P}(X)$. Again, the enumeration procedure implements this tractably, since one need only consider the $|X| - 1$ nonsingleton sets $X$, $X \setminus \{x_1\}$, $X \setminus \{x_1, x_2\}$, $\ldots$. Indeed, if $P_{S,x}$ is acyclic then so is $P_{S',x}$ for $x \in S' \subset S$. Moreover, within each such set, there is no need to consider all the restrictions: one may move to the next set upon finding any $x$ for which $P_{S,x}$ is acyclic. Finally, a nested enumeration procedure tests if $P_{S,x}$ is acyclic.

**C.2 Shortlisting**

**Proposition 8** Testing consistency with Shortlisting is NP-hard. This remains true for Order Shortlisting (which requires the DM’s preference to be an ordering).
Proof. Fix a mixed set \( \mathcal{R} \) of binary restrictions defined on a set \( X \). For each restriction \( r \), let \( x_r \) be the option whose contour set is being restricted; and without loss of generality assume \( y_r \neq z_r \) are the two options potentially included in the upper (or lower) contour set of \( x_r \) if \( r \) is an UCS (or LCS) restriction. Consider the set of options \( X' \) that contains all options in \( X \), plus the options \( a_r, b_r, c_r \) for each UCS restriction \( r \) and the options \( d_r, e_r, f_r, g_r \) for each LCS restriction \( r \). Take the following data:

\[
\begin{array}{c|ccccc}
S & a_r b_r & c_r x_r & b_r c_r x_r & a_r b_r y_r z_r \\
c_{\text{obs}}(S) & b_r & x_r & c_r & a_r \\
\end{array}
\]

for each UCS restriction \( r \), and

\[
\begin{array}{c|ccccc}
S & d_r f_r & e_r g_r & d_r f_r x_r & e_r g_r x_r & d_r y_r z_r & e_r y_r z_r \\
c_{\text{obs}}(S) & d_r & e_r & f_r & g_r & z_r & y_r \\
\end{array}
\]

for each LCS restriction \( r \).

Define the following set of restrictions \( \mathcal{R}' \) over \( X' \), which pertain to the shortlisting relation \( P_1 \) and are necessary conditions for consistency of \( c_{\text{obs}} \) with Shortlisting:

(i) \( 1_{(b_r, x_r)} \) from the two middle observed choices for UCS restrictions \( r \), since \( c_r \) cannot eliminate \( x_r \) and \( x_r \) is revealed preferred to \( c_r \).

(ii) \( 1_{(y_r, z_r), b_r} \) from the first and last observed choices for UCS restrictions \( r \), since \( a_r \) cannot eliminate \( b_r \) and \( b_r \) is revealed preferred to \( a_r \).

(iii) \( 1_{(x_r, d_r)} \) and \( 1_{(x_r, e_r)} \) from the first four choices for LCS restrictions \( r \), since by adding \( x_r \), the DM chooses the revealed-worse options \( f_r \) and \( g_r \), respectively.

(iv) The restriction \( 1_{(d_r y_r)} \lor 1_{(e_r z_r)} \) from the last two choices for the LCS restrictions \( r \), since neither \( y_r \) nor \( z_r \) can eliminate each other, yet one is more preferred.

Acyclic satisfiability is also sufficient for consistency with Shortlisting. Let \( P' \) be an ordering on \( X' \) satisfying \( \mathcal{R}' \), which exists by acyclic satisfiability of \( \mathcal{R}' \). The restriction of \( P' \) to \( X \), denoted \( P \), satisfies the restrictions in \( \mathcal{R} \). Let then \( P_1 \) be the relation on \( X' \) defined as follows: (a) for each LCS restriction \( r \): \( x_r P_1 d_r, x_r P_1 e_r, f_r P_1 x_r, g_r P_1 x_r, d_r P_1 y_r \) if \( x_r P y_r \), and \( e_r P_1 z_r \) if \( x_r P z_r \), and (b) for each UCS restriction \( r \): \( a_r P_1 y_r, a_r P_1 z_r, b_r P_1 x_r, y_r P_1 b_r \) if \( y_r P x_r \) and \( z_r P_1 b_r \) if \( z_r P x_r \). Notice that \( P_1 \) is
acyclic.\textsuperscript{27} Take $P_2$ as a preference ordering on $X'$ such that $b_r P_2 a_r$ and $x_r P_2 c_r$, for each UCS restriction $r$, and $y_r P_2 e_r$, $z_r P_2 d_r$, $f_r P_2 e_r$, $e_r P_2 g_r$, $y_r P_2 z_r$ if $z_r P y_r$, and $z_r P 2 y_r$ if $y_r P z_r$, for each LCS restriction $r$.\textsuperscript{28} It is easy to check that the choice function associated to $(P_1, P_2)$ coincides with $c_{\text{obs}}$ on $D$. Given that the constructed $P_2$ is an ordering, this proof will also establish that Order Shortlisting is NP-hard.

Finally, we claim $\mathcal{R}'$ is acyclically satisfiable if and only if $\mathcal{R}$ is acyclically satisfiable. Necessity follows from the argument at the beginning of the previous paragraph. For the converse, let $P$ be an acyclic relation satisfying $\mathcal{R}$. Then, just as in Footnote 27, we may derive from $P$ an acyclic relation $P'$ over $X'$ that satisfies $\mathcal{R}'$. Q.E.D.

\subsection*{C.3 Categorization and Rationalization}

Manzini and Mariotti (2012) and Cherepanov et al (2013) show that for full datasets, Categorization and Rationalization are observationally equivalent to maximizing a complete, asymmetric $P$ over a consideration set mapping $\Gamma$ satisfying (3) and the feature that the DM pays attention to both elements in binary choice problems. By Definition 1, the observational equivalence extends to limited datasets. Thus at least two types of revealed preference, denoted $P^*$, can be inferred from limited data. Indeed, $x P^* y$ if either $x = c_{\text{obs}}(\{x,y\})$ (as the DM considers both options in binary sets) or if $x = c_{\text{obs}}(S)$ and $y = c_{\text{obs}}(T) \in S \subset T$ (due to the IIA property on $\Gamma$).

If $c_{\text{obs}}$ is consistent with these theories, then it can be extended to a complete choice function $c$, with $x_1 = c(X)$, $x_2 = c(X \setminus \{x_1\})$, $x_3 = c(X \setminus \{x_1, x_2\})$, etc. Define the ordering $\succ_s$, to be read as ‘more salient than’, by $x_1 \succ_s x_2 \succ_s \cdots$. By the IIA property on $\Gamma$, the option $x_k$ belongs to $\Gamma(S)$ for any $S$ such that $x_k \in S \subseteq X \setminus \{x_1, \ldots, x_{k-1}\}$. As the DM chooses the best alternative in her consideration set, a necessary condition for consistency with these theories is that $x$ cannot be more salient than the choice from a set $S$ if $x$ is preferred to the choice: that is, for each $S \in D$, if $x P^* c_{\text{obs}}(S)$ then $y \succ_s x$ for some $y \in S$. Moreover, even though the preference may have cycles, it must still be possible to identify a best element for each consideration set. Thus, another necessary condition for consistency is that if $P^*$ is cyclic on the set $\{x, c_{\text{obs}}(S), c_{\text{obs}}(T)\}$

\textsuperscript{27}Indeed, one can derive from $P$ an ordering on $X'$ satisfying these restrictions, simply by placing $a_r$ above all options in $X$ and $b_r$ right above $x_r$, for each UCS restriction $r$, and by placing $f_r$ and $g_r$ above all options in $X$, and $d_r$ and $e_r$ below $x_r$ but above any other element of $X$, for each LCS restriction $r$, and $c_r$ at the bottom for each UCS restriction.

\textsuperscript{28}For instance, rank from bottom to top: $a_r$, $b_r$, $c_r$, for all UCS restriction $r$, then $f_r$, $g_r$, $d_r$, $e_r$ for all LCS restriction $r$, then options in $X$ in opposite order than $P$. 

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and all these elements are contained in $S \cap T$ for some $S, T \in \mathcal{D}$, then to avoid a cycle when facing this set, $x$ cannot be more salient than $c_{\text{obs}}(S)$ and $c_{\text{obs}}(T)$, i.e., either $c_{\text{obs}}(S) \succ_s x$ or $c_{\text{obs}}(T) \succ_s x$. Call this collection of UCS restrictions $\mathcal{R}_{\text{Cat/Rat}}(P^*)$.

Acyclic satisfiability of $\mathcal{R}_{\text{Cat/Rat}}(P^*)$ is clearly necessary for consistency; it also sufficient under certain circumstances (e.g., $\mathcal{D}$ contains all binary choice problems).

**Proposition 9** Consider Categorization and Rationalization, and an observed choice function $c_{\text{obs}}$. There is a set of UCS restrictions $\mathcal{R}_{\text{Cat/Rat}}(P^*)$ such that:

(a) If $P^*$ is either (1) complete over the image of $c_{\text{obs}}$, or (2) acyclic, then $c_{\text{obs}}$ is consistent with these theories if and only if $\mathcal{R}_{\text{Cat/Rat}}(P^*)$ is acyclically satisfiable.

(b) Otherwise, testing consistency is NP-hard.

A corollary is that for any $P^*$, $c_{\text{obs}}$ is consistent with these theories if and only if $\mathcal{R}_{\text{Cat/Rat}}(\bar{P}^*)$ is acyclically satisfiable for some $\bar{P}^*$ completing $P^*$ over chosen elements.

**Proof of Proposition 9(a).**

**Lemma 3** For any complete asymmetric preference $P$, there exists a consideration set mapping $\Gamma$ satisfying (3) such that $c_{\text{obs}}$ is explained by $(\Gamma, P)$ if and only if there exists an ordering $O$ such that $c_{\text{obs}}$ is explained by $(\Gamma_O, P)$, where $\Gamma_O$ is given by

$$\Gamma_O(S) = \{\arg \max_O S\} \cup \{c_{\text{obs}}(T) \mid S \subseteq T, T \in \mathcal{D}, c_{\text{obs}}(T) \in S\}, \quad \forall S \in \mathcal{P}(X).$$

*Proof.* It is easy to check that $\Gamma_O$ satisfies (3), hence sufficiency follows by taking $\Gamma = \Gamma_O$. For necessity, let $c$ be the extension of $c_{\text{obs}}$ emerging under $(\Gamma, P)$. Define the ordering $O$ by $x_1O \cdots Ox_n$, where $x_k = c(X \setminus \{x_1, \ldots, x_{k-1}\})$ for each $k = 1, \ldots, n$. By (3) and the fact that one pays attention to chosen elements, $\Gamma_O(S) \subseteq \Gamma(S)$ for all $S$. This implies that for any $S \in \mathcal{D}$, the observed choice $c_{\text{obs}}(S)$ remains $P$-maximal in $\Gamma_O(S)$. To conclude, we show $(\Gamma_O, P)$ deliver a well-defined choice everywhere. Fix any $T \subset X$. Applying $\Gamma_O \subseteq \Gamma$ to $S = \Gamma_O(T)$ yields $\Gamma_O(\Gamma_O(T)) \subseteq \Gamma(\Gamma_O(T)) \subseteq \Gamma_O(T)$.Now using the property (3), $\Gamma_O(\Gamma_O(T)) = \Gamma_O(T)$, and by squeezing, $\Gamma(\Gamma_O(T)) = \Gamma_O(T)$. Then $\Gamma_O(T)$ has a $P$-maximal element since $\Gamma(\Gamma_O(T))$ does. $Q.E.D.$

For sufficiency, suppose an acyclic relation satisfying $\mathcal{R}_{\text{Cat/Rat}}(P^*)$ exists. Let $O$ be a transitive completion of this relation (hence $O$ still satisfies the restrictions). Next, we complete $P^*$ as follows (and henceforth use that completion). If there are $x, y \in X$ such that neither $xP^*y$ nor $yP^*x$ hold, then by assumption on $P^*$, it cannot
be that both $x, y$ are in the image of $c_{\text{obs}}$. If one of these, say $x$, satisfies $x = c_{\text{obs}}(S)$ for some $S \in \mathcal{D}$, then we add $xP^*y$; otherwise, add exactly one of $xP^*y$ or $yP^*x$.

Using Lemma 3, it remains to show: (i) $P^*$ is asymmetric, (ii) $P^*$ has a well-defined choice on $\Gamma_O(S)$ for each $S \subseteq X$, and (iii) this choice is $c_{\text{obs}}(S)$ for $S \in \mathcal{D}$. For (i), suppose $yP^*z$ and $zP^*y$. Then there must exist $R, R' \in \mathcal{D}$ such that $y = c_{\text{obs}}(R)$, $z = c_{\text{obs}}(R')$, and $\{y, z\} \subseteq R \cap R'$. Taking $x = y$ in $\mathcal{R}_{\text{Cat/Rat}}(P^*)$ implies $zOy$; but taking $x = z$ implies $yOz$, a contradiction. For (ii), suppose $P^*$ has a top-cycle on $\Gamma_O(S)$. By completeness, there must be a 3-cycle $xP^*yP^*zP^*x$. Suppose without loss $x$ is $O$-maximal in $\{x, y, z\}$. As $y, z \in \Gamma_O(S)$, they are the respective choices from some $T_y, T_z \supseteq S$; then $xP^*y$ cannot be from our completion. Hence $x$ is in the image of $c_{\text{obs}}$, so $zP^*x$ cannot be from our completion. But $\{x, y, z\} \subseteq S \subseteq T_y \cap T_z$ contradicts $O$ satisfying $\mathcal{R}_{\text{Cat/Rat}}(P^*)$. For (iii), note $c_{\text{obs}}(S) \in \Gamma_O(S)$ whenever $S \in \mathcal{D}$. Suppose by contradiction $c_{\text{obs}}(T)P^*c_{\text{obs}}(S)$, for some $T \in \mathcal{D}$ such that $c_{\text{obs}}(T) \in S \subseteq T$. Then $c_{\text{obs}}(S)P^*c_{\text{obs}}(T)$, contradicting (i). Finally, suppose $xP^*c_{\text{obs}}(S)$ for $x = \text{arg max}_O S$. This cannot be from our completion and contradicts $\mathcal{R}_{\text{Cat/Rat}}(P^*)$. Q.E.D.

**Proof of Proposition 9(b).**

Fix a mixed set $\mathcal{R}$ of binary restrictions defined over $X$. For each restriction $r$, let $x_r$ be the option whose contour set is being restricted, and let\(^{29}\) $y_r \neq z_r$ be the two options potentially included in the upper (or lower) contour set of $x_r$ if $r$ is an UCS (or LCS) restriction. Take the set of options $X'$ that contains all options in $X$, plus options $w, w'$, and new options $a_r$ and $b_r$ for each restriction $r$, and the following data:

\[
\begin{array}{ccc}
S & a_rx_r & b_rx_r & a_rb_rx_r \\
c_{\text{obs}}(S) & a_r & b_r & x_r \\
\end{array}
\]

for each restriction $r$,

\[
\begin{array}{cccccccc}
S & a_ry_r & a_rz_r & b_ry_r & b_rz_r & a_rb_ry_r & a_rbw_r & b_rw'z_r & a_rb_wz_r & a_rb'w'z_r \\
c_{\text{obs}}(S) & a_r & z_r & y_r & b_r & a_r & b_r & a_r & b_r \\
\end{array}
\]

for each LCS restriction $r$, and

\(^{29}\)We assume that $y_r \neq z_r$ for all $r$. This is without loss of generality as it is easy to check that Proposition 3 holds with that additional assumption on restrictions (simply by capturing a restriction $xPy$ by the combination of $(x, \{a, b\})$ and $(\{a, b\}, y)$ for some two new options $a$ and $b$).
for each UCS restriction $r$.

Recall that $xP^*y$ if either $x = c_{obs}(\{x, y\})$ or there exist $S \subset T$ such that $c_{obs}(S) = x$ and $c_{obs}(T) = y \in S$. The above observed choices generate the following $P^*$ comparisons: $x, P^*y_r$ for each UCS restriction $r$, $a_r, P^*y_r b_r, P^*z_r, P^*a_r$, for each LCS restriction $r$, and $a_r, P^*x_r, b_r, P^*x_r$ for each restriction $r$. Proposition 9 implies that there must exist an acyclic relation $O$ satisfying the following restrictions:

(i) $x, Oa_r$ and $x, Ob_r$, which follows from the first type of restriction in $R_{Cat/Rat}(c_{obs})$

applied to $c_{obs}(\{a_r, x_r\}) = a_r$, $c_{obs}(\{b_r, x_r\}) = b_r$, and $c_{obs}(\{a_r, b_r, x_r\}) = x_r$,
for each restriction $r$, and

(ii) $y_r, Ox_r$ or $z_r, Ox_r$, from (i) and $c_{obs}(\{a_r, x_r, y_r\}) = x_r$, $c_{obs}(\{a_r, x_r, y_r, z_r\}) = y_r$,
for each UCS restriction $r$. These are all the necessary conditions from Proposition 9(a). However, $P^*$ is incomplete, and hence they need not be sufficient. Indeed, here is an extra restriction that must hold for all LCS restrictions $r$:

(iii) $a_r, Oy_r$ or $b_r, Oy_r$ or $a_r, Oz_r$ or $b_r, Oz_r$: If $c(\{a_r, b_r\}) = a_r$, the extended $P^*$ is cyclcic on $\{a_r, b_r, z_r\}$. Since $a_r$ and $b_r$ must be considered from $\{a_r, b_r, z_r\}$ (given $c_{obs}(\{a_r, b_r, w, z_r\}) = a_r$ and $c_{obs}(\{a_r, b_r, w', z_r\}) = b_r$), $z_r$ cannot be considered, and thus $a_r, Oz_r$ or $b_r, Oz_r$. Similarly, if $c(\{a_r, b_r\}) = b_r$, then $a_r, Oy_r$ or $b_r, Oy_r$.

Let $\mathcal{R}'$ be the set of restrictions listed in (i) to (iii). We now show that $c_{obs}$ is consistent with Categorization and Rationalization if and only if $\mathcal{R}'$ is acyclically satisfiable. We already proved necessity. As for sufficiency, let $O$ be an acyclic relation (assumed to be an ordering without loss of generality) satisfying $\mathcal{R}'$. Options $w$ and $w'$ are not involved in any restriction in $\mathcal{R}'$. Thus we can assume without loss of generality that $w$ and $w'$ are the two top options according to $O$. Let $P$ be any asymmetric completion of $P^*$ such that $a_r, Pw$ and $b_r, Pw'$ for each UCS restriction $r$, and $y_r, Pz_r$ for each LCS restriction $r$. We conclude by showing that the choice function generated by $(\Gamma_O, P)$, where $\Gamma_O$ is defined as in (8), coincides with $c_{obs}$ on $\mathcal{D}$. This follows by definition of $P$ on pairs. By definition of $\Gamma_O$ and the properties of $O$, we have:

$$
\begin{array}{c|cccccc}
S & a_r b_r x_r & a_r b_r w y_r & a_r b_r w y_r & a_r b_r w z_r & a_r b_r w' z_r \\
\Gamma_O(S) & x_r & a_r w & b_r w' & a_r w & b_r w'
\end{array}
$$
for each LCS restriction $r$, and

$$
\begin{array}{c|ccc}
S & a_r b_r x_r & a_r x_r y_r & a_r x_r y_r z_r \\
\Gamma_O(S) & x_r & x_r \text{ or } x_r y_r & y_r \text{ or } y_r z_r \\
\end{array}
$$

for each UCS restriction $r$. Maximizing $P$ indeed delivers $c_{obs}$ for $S \in \mathcal{D}$. Now consider $S \notin \mathcal{D}$. If $S$ is binary then asymmetry of $P$ ensures a $P$-maximal element. Note also that $\Gamma_O(S)$ is singleton for any $S$ such that $S \subset T$ for any $T \in \mathcal{D}$, so again there is a $P$-maximal element. The only remaining $S$ are the triples $\{a_r, b_r, w\}$, $\{a_r, b_r, w'\}$, $\{a_r, b_r, y_r\}$ and $\{a_r, b_r, z_r\}$. For the first (second) $\Gamma_O$ is $\{a_r, w\}$ ($\{b_r, w'\}$). For the third and fourth, $\Gamma_O$ is $\{a_r, b_r\}$. Asymmetry then ensures a $P$-maximal element.

Finally, we show acyclic satisfiability holds for $\mathcal{R}$ if and only if it holds for $\mathcal{R}'$. Let $O$ be an acyclic relation on $X$ satisfying $\mathcal{R}$. We can assume without loss that $O$ is an ordering (otherwise take an acyclic completion of $O$). Extend $O$ to any ordering on $X'$ ranking $w$ and $w'$ on top, and $a_r$ and $b_r$ for each $r$ directly below the corresponding $x_r$, and above any other $y \in X$ such that $x_r O y$. It is easy to see this extension satisfies $\mathcal{R}'$. Conversely, let $O'$ be an acyclic relation satisfying the restrictions in $\mathcal{R}'$. Again, we can assume without loss that $O'$ is an ordering. Then the restriction of $O'$ to $X$ satisfies $\mathcal{R}$ (the LCS restrictions follow by combining (i) and (iii)). Q.E.D.

C.4 Undominated Alternatives

For this theory, we mainly discuss the notion of strong consistency, as the testable implications under weak consistency are fairly trivial (for the $Q$-version of the theory, the only testable implications are that a $Q$-dominated element cannot be chosen).

Suppose that $x \notin C_{obs}(S)$ for some $S \in \mathcal{D}$ with $x \in S$. Then, there must be another $y \in S$ such that $y$ dominates $x$. However, observed choices may disqualify some $y$’s from playing this role. This would be the case, for instance, if $x$ is ever picked in the presence of $y$. Define $Z(x)$ to be the union of all the choice problems $T \in \mathcal{D}$ such that $x \in C_{obs}(T)$. Thus, there must exist some $y$ in $S \setminus Z(x)$ that dominates $x$.

Formally, let $\mathcal{R}_{UD}(C_{obs})$ be the set of of strict UCS restrictions $I_{(\Sigma,x)}$, where $\Sigma = \{\{y\}|y \in S \setminus Z(x)\}$, derived by varying $S \in \mathcal{D}$ and $x \in S \setminus C_{obs}(S)$. If some $S \subseteq Z(x)$, then satisfying $\mathcal{R}_{UD}(C_{obs})$ is clearly impossible, and observed choices must be inconsistent with choosing the undominated elements for some acyclic relation. Thus, we henceforth assume that $S \setminus Z(x)$ is nonempty for all restrictions in $\mathcal{R}_{UD}(C_{obs})$.$^{30}$

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$^{30}$This assumption is only to simplify notation; one can instead create an auxiliary option $\diamond$. 

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These strict UCS restrictions capture the empirical content of the theory.

**Proposition 10** The observed choice correspondence $C_{obs}$ is strongly consistent with choosing undominated elements according to an acyclic relation if and only if $R_{UD}(C_{obs})$ is acyclically satisfiable.

**Proof.** Necessity follows from the above discussion. As for sufficiency, let $\succ$ be acyclic relation satisfying the restrictions in $R_{UD}(C_{obs})$. As $\succ$ may be overly complete, construct $\succ^*$ by $y \succ^* x$ if $y \succ x$ and there is $S \in D$ such that $x \in S \setminus C_{obs}(S)$ and $y \in S \setminus Z(x)$. Then $\succ^*$ inherits the property of acyclicity from $\succ$. Let $T \in D$. We have to check that $C_{obs}(T) = \{x \in T \mid \nexists y \in T : y \succ^* x\}$. Suppose that $x \in C_{obs}(T)$ and that $y \in T$. Then $y \in Z(x)$, and thus it is not the case that $y \succ^* x$. Conversely, let $x \in T \setminus C_{obs}(T)$. Then there is $y \in T \setminus Z(x)$ such that $y \succ x$ since $\succ$ satisfies the restrictions in $R_{UD}(C_{obs})$, and moreover $y \succ^* x$ holds. \(Q.E.D.\)

---

augment $R_{UD}(C_{obs})$ by including $\diamond$ in every $Z(x)$ and adding the restrictions $1_{(y, \diamond)}$ for all $y \in X$, and then test acyclic satisfiability of these restrictions over $X \cup \{\diamond\}$. 38