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The procedural value for cooperative games with non-transferable utility ☆

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Abstract

I adapt Raiffa's discrete bargaining solution in order to take the possibility of partial cooperation into account when there are more than two players. The approach is non-cooperative. I slightly modify the bargaining procedure proposed by Sjöström for supporting the Raiffa solution, exactly as Hart and Mas-Colell introduced the possibility of partial cooperation in (a slight variation of) the Rubinstein procedure. I characterize the unique subgame perfect equilibrium outcome and so justify a new value for cooperative games with non-transferable utility. The so-called procedural value is obtained by applying recursively the Raiffa solution to appropriate bargaining problems. It appears to satisfy nice properties.

Keywords: Value for NTU games; Nash program; Raiffa bargaining solution

JEL classification: C71; C72; C78

1. Introduction

Moulin (1984, Section 4) suggests the following variant of the Rubinstein alternating offers model (see also Binmore et al., 1986; Binmore, 1987). The game is composed of a succession of rounds. Each round is played as follows: a player is chosen at random according to a uniform probability distribution, he proposes a contract and the other players choose whether to accept the offer. If everybody accepts, then it is realized and the game stops. Otherwise, a lottery is drawn. With a small probability ϵ, there is full disagreement and the game stops. With probability $1-\epsilon$, a

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new round is played. The stationary subgame perfect equilibrium (SSPE) outcomes of this procedure converge towards the Nash bargaining solution when ϵ tends to zero. Hart and Mas-Colell (1996) adapt this procedure in order to take the possibility of partial cooperation into account. Instead of going to full disagreement with probability ϵ after a refusal, only the proposer has to quit the group and bargaining continues without him. The SSPE outcomes converge towards some element of the consistent Shapley value (see Maschler and Owen, 1992) when ϵ tends to zero.

Sjöström (1991) (see also Myerson, 1991, page 393) formalizes the procedural content implicit in Raiffa's discrete bargaining solution (see Luce and Raiffa, 1957, Section 6.7). The bargaining horizon of the players is finite. They can make proposals only for a finite number R of rounds. If no agreement is reached after this horizon, then full disagreement prevails. Each round is played as before except that a new round is played for sure after a refusal ($\epsilon=0$) if the current round is not the last one. The game stops on full disagreement ($\epsilon=1$) if refusal occurs at the last round. The Raiffa solution is the limit of the unique subgame perfect equilibrium (SPE) outcome when R tends to infinity. Following the idea of Hart and Mas-Colell, I slightly modify the procedure in order to take into account the possibility of partial cooperation. If no agreement is reached after R rounds of bargaining within a coalition, then the last proposer has to quit the group and bargaining continues without him.

I characterize the unique SPE outcome of the game for each R (see Proposition 1). The procedural value is defined as the limit of these SPE outcomes as R tends to infinity. It is obtained by applying recursively the Raiffa solution to appropriate bargaining problems (see Proposition 2). In addition, it satisfies nice properties: 1) it coincides with the Shapley value when utilities are transferable; 2) it coincides with the consistent Shapley value on the class of hyperplane games (see Maschler and Owen, 1989); 3) it coincides with the Raiffa solution on the class of bargaining problems; 4) it is scale covariant; 5) it is single-valued; 6) it specifies utility profiles that are Pareto optimal and individually rational. The procedural value is therefore an appealing alternative to the consistent Shapley value, as was the Raiffa bargaining solution with respect to the Nash bargaining solution.

2. Definitions

I start with some notation. Let n be a positive integer and let $N:=\{1,\ldots,n\}$ be the set of players. $P(N)$ denotes the set of coalitions. $P^*(N)$ is the set of coalitions with at least two members. Let S be a coalition. $\mathbb{R}^S$ is seen as a subset of $\mathbb{R}^N$: $\mathbb{R}^S:=\{x\in\mathbb{R}^N|(\forall i\in N\backslash S)\colon x_i=0\}$. When $x\in\mathbb{R}^N$, x_S denotes the projection of x on $\mathbb{R}^S$. When X is a subset of $\mathbb{R}^S$, $\partial_S X$ denotes the topological frontier of X in $\mathbb{R}^S$. Inequalities between vectors in $\mathbb{R}^S$ are of three kinds: $\geq_S$, $>_S$, $>>_S$ (usual definitions). If $i\in S$, then $S-i$ stands for $S\backslash\{i\}$. If $i\in N\backslash S$, then $S+i$ stands for $S\cup\{i\}$.

A *bargaining problem* for S is a pair (V, d), where $V\subseteq\mathbb{R}^S$ is the set of utility profiles that can be achieved through cooperation and $d\in V$ is the disagreement point, i.e. the utility profile that prevails in case of disagreement. I assume that V is compact, convex and non-level. Non-level means that the utilities are transferable above the disagreement point, but not necessarily at a one to one exchange rate. If a vector is orthogonal to V at some individually rational utility profile, then it is strictly positive.

Raiffa argues as follows. The most selfish demand that some player i could make is

$$a^i(V, d_{S-i}) := \arg \max_{v\in V \text{ s.t. } v_{S-i}=d_{S-i}} v_i.$$

A natural way to achieve fairness is to take the simple average of these vectors:

$$F_V^1(d) := \sum_{i \in S} \frac{a^i(V, d_{S-i})}{s}.$$

It represents a reasonable compromise, to be renegotiated if not Pareto optimal. *Raiffa's discrete bargaining solution* is obtained by iterating the procedure up to the Pareto frontier:

$$F_V^R(d) := \sum_{i \in S} \frac{a^i(V, (F_V^{R-1}(d))_{S-i})}{s}$$

for each integer $R \geq 2$, and

$$\mathcal{R}(V, d) := \lim_{R \to \infty} F_V^R(d).$$

The limit is well-defined because V is compact and the sequence of vectors $(F_V^R(d))_{R \in \mathbb{N}}$ is non-decreasing.

Bargaining problems are not suitable for studying the influence of partial cooperation. A *game in characteristic function form V* specifies the set of utility profiles that each coalition can guarantee to its members: $V(S) \subseteq \mathbb{R}^S$ for each subset S of N. I assume that $V(S)$ is compact, convex and non-level for each coalition S. In this framework, the *disagreement point* summarizes what the players can obtain on their own: $d_i = \max_{v \in V(\{i\})} v_i$ for each $i \in N$. The game is assumed to be *monotonic*. If some utility profile v is feasible for some coalition S, then it remains feasible when an additional player i joins the group while receiving his disagreement payoff: $\{v \in \mathbb{R}^{S+i} | v_i = d_i \wedge v_S \in V(S)\} \subseteq V(S+i)$.

3. The bargaining procedure

The objective of the section is to formally define the procedure described in the introduction. Let R be the maximal number of rounds that can be played within each coalition. The extensive form game $\Gamma(R)$ is made up of stage games indexed by couples (S, r) meaning that coalition S is active for the rth time ($r \in \{1,\ldots, R\}$). The stage game (S, r) is played as follows. A member of S (let's say i) is chosen at random according to a uniform probability distribution. He proposes a utility profile (let's say v) that is feasible for S. The other members of S sequentially (in some pre-specified order) choose whether to accept the proposal. If everybody accepts, then the game stops with the utility profile v being enforced. If somebody rejects and $r \leq R-1$, then the stage game $(S, r+1)$ is played. If somebody rejects and $r=R$, then player i is excluded from the active coalition. He receives his disagreement payoff d_i and the stage game $(S-i, 1)$ is played. Notice that if S is a singleton, then there is no other member and the 'proposal' v is assumed to be automatically realized. Utilities are not discounted. The model may be accommodated though as in Sjöström (1991) where the time period between two offers get shorter and shorter.

4. Unique SPE outcome

Proposition 1. *Let S be a coalition with at least two members and let r be an integer between* 1 *and R. Each subgame of* $\Gamma(R)$ *that starts with the stage game* (S, r) *admits a unique* SPE *outcome. Let* SPE(R, S, r) *denote its projection on* $\mathbb{R}^S$. *The finite sequence obtained by varying S and r is characterized by the following recursion*:

$$\begin{cases} (\forall S \in P^*(N)) : \mathrm{SPE}(R,S,R) = \frac{1}{s}\sum_{i\in S} a^i(V(S), \mathrm{SPE}(R, S-i, 1)) \\ (\forall S \in P^*(N))(\forall r \in \{1,\ldots,R-1\}) : \mathrm{SPE}(R,S,r) = \frac{1}{s}\sum_{i\in S} a^i(V(S), \mathrm{SPE}_{S-i}(R,S,r+1)) \end{cases}$$

where SPE(R, $\{i\}$, 1)$=d_{\{i\}}$ *for each* $i\in N$. *In addition*, SPE(R, S, r)$\in V(S)$ *and* SPE(R, S, r)$\geq d_S$.

Proof. The unique equilibrium outcome of the last stage game is the disagreement point. This serves as a first step for a recursive argument. I consider a subgame of $\Gamma(R)$ that starts with the stage game (S, r) and I assume that the result is satisfied for each subgame that starts with a stage game that follows (S, r). I focus, in addition, on the case where player $i\in S$ has been selected by nature. Let $v\in V(S)$ be the proposal he makes in some SPE of $\Gamma(R)$, and let u be the expected payoff of the members of S, as specified by the inductive hypothesis, if player i's proposal is rejected. Observe that $u\in V(S)$, using the monotonicity of V when $r=R$. Hence a^i ($V(S)$, u_{S-i}) is well-defined. I prove that the SPE outcome $o\in V(S)$ of the subgame starting at the specified node is $a^i(V(S)$, $u_{S-i})$. Notice that $o\geq_S u$ as each player $j\in S$ can guarantee himself u_j. So the result is proved if $u\in\partial_S V(S)$. Suppose now that $u\notin\partial_S V(S)$. Any proposal $v'\in V(S)$ such that $v'_{S-i}>>_{S-i} u_{S-i}$ is accepted in equilibrium. This implies that player i's proposal is accepted in equilibrium. Indeed, if somebody rejects it, then player i receives u_i and he has a profitable deviation to some $v'\in V(S)$ that strictly Pareto S-dominates u. So, $o=v$. Hence $v\geq_S u$. Remember that $V(S)$ is non-level. If $v_{S-i}>_{S-i} u_{S-i}$ or if $v_i<a_i^i$ ($V(S)$, u_{S-i}), then player i has a profitable deviation to some $v'\in V(S)$ such that $v'_{S-i}>>_{S-i} u_{S-i}$ and $v'_i>v_i$. Hence $o=v=a^i$ ($V(S)$, u_{S-i}). □

5. The procedural value

Given R, I focus on the SPE outcome associated to the subgames of $\Gamma(R)$ that start with a coalition being active for the first time. This defines a payoff configuration: SPE(R):=(SPE(R, S, 1))$_{S\in P(N)}\in\times_{S\in P(N)}V(S)$. The *procedural value* is the limit of this payoff configuration when R tends to infinity: $\mathcal{PV} := \lim_{R\to\infty}\mathrm{SPE}(R)$. It is characterized recursively in terms of the original Raiffa bargaining solution, as the following proposition shows.

Proposition 2. $\mathcal{PV}(S) = \mathcal{R}\left(V(S), \sum_{i\in S}\frac{a^i(V(S),\mathcal{PV}(S-i))}{s}\right)$, *for each* $S\in P^*(N)$, *and* $\mathcal{PV}(\{i\}) = d_{\{i\}}$, *for each* $i\in N$.

Proof. The result is obvious when S is a singleton. Let us apply an argument by induction on the cardinality of the coalitions. Let S be a coalition with at least two members. The second equation appearing in proposition 1 may be rewritten as follows:

$$(\forall S\in P^*(N))(\forall r\in\{1,\ldots,R-1\}) : \mathrm{SPE}(R,S,r) := F_{V(S)}^{R-r}(x(R))$$

where $x(R):=\sum_{i\in S}a^i$ ($V(S)$, SPE(R,$S-i$, 1))$/s$. The result is then easy to prove when S has only two members, as $x(R)=F^1_{V(S)}(d_S)$ for each positive integer R. This summarizes the main argument of Sjöström (1991) if one makes abstraction of the discount factor. With more than two players, the 'personalized threats' SPE(R,S$-i$, 1) used to start the procedure varies with R and the convergence result is slightly less trivial. Let $\epsilon\in\,]0, 1]$, let $y:=\sum_{i\in S}a^i(V(S), \mathcal{PV}\,(S-i))/s$ and let

$$A_\epsilon := \{v\in\mathbb{R}^S | (\forall i\in S) : v_i > \mathcal{R}_i(V(S), y) - \epsilon\}.$$

As $\mathcal{R}_i(V(S),y)=\lim_{R\to\infty}F^R_{V(S)}(y)$, there exists $\hat{\mathrm{R}}$ such that $F^{\hat{R}-1}_{V(S)}(y)\in A_\epsilon$. As $F^{\hat{R}-1}_{V(S)}$ is continuous and $\lim_{R\to\infty}x(R)=y$, there exist $\bar{R}$ such that $F^{\hat{R}-1}_{V(S)}(x(R))\in A_\epsilon$ for each $R\geq\bar{R}$. I may assume for simplicity that $\bar{R}\geq\hat{\mathrm{R}}$. Hence $F^{\hat{R}-1}_{V(S)}(x(R))\in A_\epsilon$ for each $R\geq\bar{R}$. I conclude that $\mathcal{PV}(S)\in\cap_{\epsilon\in]0,1]}A_\epsilon\cap V(S)=\{\mathcal{R}(V(S),y)\}$. The last equality follows from the fact that $V(S)$ is non-level. □

I interpret the procedural value as follows. It specifies a reasonable compromise for each coalition. Even if the grand coalition is forming, the fairness of any utility profile depends on what would happen should sub-coalitions form. Applying an argument by induction, I may assume that $\mathcal{PV}(S)$ represents a reasonable compromise for each $S\subsetneq N$. The most selfish demand that some player i could make is a^i_i $(V(N), \mathcal{PV}(\mathrm{N}-i))$. Indeed, player i fears to be excluded from cooperation in which case the outcome is $\mathcal{PV}(N-i)$. Alternatively, $a^i_i(V(N), \mathcal{PV}\ (N-i))$ may be considered as a natural extension of the concept of marginal contribution to games with non-transferable utility. Notice that it is usually impossible to recover by projection the various threats $\mathcal{PV}(N-i)$ from some common vector such as the disagreement point. This constitutes an important difference with respect to bargaining problems. Fairness is achieved by taking a simple average: $y=\sum_{i\in N}a^i\ (V(N), \mathcal{PV}\ (N-i))/n$. It represents a natural compromise, to be renegotiated if not Pareto optimal. After this first step, threats are derived by projection from a unique vector and iterating the argument amounts to apply the Raiffa solution to the partial agreement y.

It follows from its very definition that the procedural value is single-valued, scale covariant and specifies utility profiles that are Pareto optimal. It coincides with the Raiffa solution on the class of bargaining problems. As a consequence of proposition 1, it specifies utility profiles that are individually rational: $\mathcal{PV}\ (S)\geq d_S$ for each $S\in P(N)$. A game is hyperplane if the feasible set of each coalition coincides with a half-space on the set of utility profiles that are individually rational. Hyperplane games are such that the vector $\sum_{i\in S}a^i(V(S), \mathcal{PV}(S-i))/s$ is optimal in $V(S)$ for each S. It follows from Proposition 2 that the procedural value coincides with the consistent Shapley value for these games (see Maschler and Owen, 1989, Lemma 1). In particular, it coincides with the Shapley value on the class of TU-games.

Making R tend to infinity allows to reach efficiency. Notice though that efficiency is already obtained with $R=1$ when V is a hyperplane game. The above procedure then coincides with the procedure of Hart and Mas-Colell (1996), taking the probability of ejection after refusal equal to one.

6. Related literature

The link with Hart and Mas-Colell (1996) has already been discussed in the introduction. I make some further comments. Their extensive form games can potentially last infinitely many periods, although this happens with probability zero. They thus have to focus on SPE that are stationary, as folk-like theorems hold otherwise. Moreover, they obtain the weaker result that every sequence of SSPE outcomes indexed by ϵ (see the introduction) converges to some element of the consistent Shapley value, as ϵ tends to zero. They don't establish the converse. This is important since the consistent Shapley value is not necessarily single-valued. Our approach also shows the limits of the robustness of their result: a slight variation (different from those suggested in their Section 6) of their bargaining procedure may yield the Shapley value in the TU-case and something different from the consistent Shapley value in the NTU-case.

Gomes et al. (1999) study a finite-length version of the Hart/Mas-Colell procedure. The players have up to R rounds to reach an agreement. Otherwise, there is full disagreement. Each (ϵ, R)-parametrized extensive form game admits a unique SPE outcome as in my case. I denote it

SPE(ϵ, R). They show how the limits of these SPE outcomes depend on the relative speed of convergence of R to infinity and of to ϵ zero. Their procedure is closest to mine when ϵ equals zero or at least converges to zero more rapidly than R converges to infinity, so that the probability of the grand coalition remaining active at the last round is close to one. In such cases, SPE(ϵ, R) converges to the Raiffa solution applied to the n-person pure bargaining problem ($V(N)$, d). This is indeed the result of Sjöström (1991) with small risks ϵ of disagreement instead of discounting. As already argued, it is not related at all to the procedural value: the possibility of partial cooperation is not taken into account. More generally, Gomes et al. show that the limits of the SPE outcomes are intimately related to the solution of a dynamic system characterized by the 'consistent vector field'. It is connected with the consistent Shapley value in that it indicates the direction in which to move in order to decrease the inconsistency of every efficient payoff configuration.

I conclude by comparing the procedural value with the main existing values. The procedural value coincides with the Raiffa solution on the class of bargaining problems while most of the values defined so far coincide with the Nash solution. Remember also that the procedural value coincides with the consistent Shapley value on the class of hyperplane games. This shows again that it differs from both the Harsanyi (1963) and the Shapley (1969) NTU values (see Hart, 2004). Here is a three-player example where it is possible to appreciate the differences between the Harsanyi, the Shapley, the consistent and the procedural values: $V(\{i\}) := \{v \in \mathbb{R}^{\{i\}} | v_i \leq 0\}$ for each $i \in \{1, 2, 3\}$; $V(\{1, 3\}) = V(\{2,3\}) := \{(0, 0, 0)\}$; $V(\{1, 2\})$ is the convex hull of the vectors (0, 0, 0), (100, 0, 0), (50, 25, 0) and (0, 30, 0); $V(\{1, 2, 3\})$ is the convex hull of the vectors (0, 0, 0), (100, 0, 0), (0, 100, 0) and (0, 0, 100). Notice that utilities are transferable in the grand coalition. The Harsanyi value is obtained by computing the dividends. Coalition {1, 2} pays 300/11 units to its members while the grand coalition pays 500/33 units to its members. The Harsanyi value for the grand coalition is thus (1400/33, 1400/33, 500/33) which is approximately equal to (42.4, 42.4, 15.2). The Shapley NTU value linearizes the feasible set of coalition {1, 2} and is therefore (50, 50, 0). The consistent Shapley value involves the computation of the Nash bargaining solution for coalition {1, 2} (= (50, 25, 0)) which is used as a threat to compute the solution for the grand coalition. It is therefore equal to (50, 125/3, 25/3) or approximately (50, 41.7, 8.3). Finally, the procedural value involves the computation of the Raiffa bargaining solution for coalition {1, 2} (= (60, 20, 0)) which is used as a threat to compute the solution for the grand coalition. It is therefore equal to (160/3, 120/3, 20/3) or approximately (53.5, 40, 6.5).

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