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No profitable decompositions in quasi-linear allocation problems ☆

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Abstract

We study the problem of allocating a bundle of perfectly divisible private goods from an axiomatic point of view, in situations where compensations can be made through monetary transfers. The key property we impose on the allocation rule requires that no agent should be able to gain by decomposing the problem into sequences of subproblems. Combined with additional standard properties, it leads to a characterization of the rule that shares the total surplus equally. Hence a traditional welfarist rule emerges as the unique consequence of our axioms phrased in a natural economic environment.
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1. Introduction

We consider situations where a group of people have to share a bundle of perfectly divisible private goods. We assume that compensations can be achieved through monetary transfers (quasi-linear framework). As often, instead of solving each specific problem in isolation, we study allocation rules that may be applied in many different instances. For most allocation problems

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and most rules, some participants can gain by decomposing the stakes in some way, requesting for instance to allocate good l before l', or to share a proportion of the total amount of goods available before allocating what remains. Of course, such decompositions often lead to an efficiency loss, which is not desirable. Even when there is no efficiency loss, a gain for one participant must result in a loss for another one when the allocation rule selects efficient outcomes. Hence the normative appeal of a rule may be lost if stakes are decomposed when implementing it. Finally, one advantage of agreeing on an allocation rule is to reduce conflict when it comes to solving particular problems. This advantage may be limited when implementing rules that are subject to such profitable decompositions, as participants will have conflicting preferences when it comes to setting the agenda. For all these reasons, we are interested in studying rules that satisfy a property of "No Profitable Decompositions" (NPD), requiring that no individual can gain by decomposing the problem into sequences of subproblems.

The main result of the paper establishes that NPD, once combined with other standard axioms, characterizes the allocation rule that corresponds to an equal split of the maximal total surplus among the participants. Equal surplus sharing being probably the simplest notion of microeconomic justice, one would think that there exist numerous axiomatic characterizations of this solution in bargaining and social choice theory. In reality there are only relatively few such results. The reason is that most contributions in axiomatic bargaining and social choice are phrased while taking utility possibility sets as primitive. Equal surplus sharing follows trivially from the properties of anonymity and efficiency in quasi-linear environments under this welfarist assumption. Most of the literature focuses instead on finding extensions of the equal surplus sharing solution to environments that are not quasi-linear. Unfortunately, the welfarist assumption lacks a clear normative and/or positive content, and is thus hard to accept as an axiom or postulate (see [35,36]). The existence of appealing contextual solutions (e.g. egalitarian equivalence, or competitive equilibrium with equal income) also shows that the welfarist assumption is far from being innocuous. To be precise, we are not arguing that a solution is unappealing because it is welfarist. Instead, we suggest that the axiomatic approach should be applied more systematically to explicit economic and social environments. Some properties that were incompatible in the utility space may lead to the characterization of new (necessarily contextual) solutions. In other cases, welfarism will come as a consequence of axioms, hence giving us a deeper understanding of classical solutions. Our main result belongs to this second category. It is worth noting that NPD cannot even be phrased under the welfarist assumption, since the set of utilities that are feasible in the subsequent step of a decomposition depends on the economic description of the problem. This set may be strictly smaller than, and unrelated to, the set of utilities that are achievable when solving the problem in its entirety.

Beyond usual properties of anonymity, efficiency, and continuity, the result requires an axiom of independence with respect to preferences over non-feasible allocations (IND). As hinted by its name, IND requires that the solution of two allocation problems that differ only in the participants' preferences over outcomes that are not feasible coincide. As far as we can tell, this type of property was first mentioned explicitly by Karni and Schmeidler in [19].[1] It has been invoked on various occasions since then.[2] Though IND may appear completely innocuous at first sight, we must point out that it rules out solutions such as Pazner and Schmeidler's [29] egalitarian equivalence.

[1] Karni and Schmeidler themselves refer to a 1969 *mimeo* written by A. Gibbard.

[2] Here are a few references: [32,17,3,9–11,4,12,7,8]. The list is not exhaustive, but it illustrates well the various contexts where a property in the spirit of IND has been used, and the various formulations that have been proposed.

We can now provide some intuition for our main characterization result. Consider various countries that have an equal claim over a newly-discovered field of natural gas. A total quantity Q is available to share. Let v_i be the function that measures the net social surplus for country i, as a function of the share it receives.[3] These functions are most likely to vary across countries because of different transportation costs and different needs (e.g. existence of alternative sources, and use of different technologies that make the resource more or less productive). NPD is more restrictive when it applies to many decompositions of the original problem. Consider for instance the case where the division of Q is tested against the iteration cubic meter by cubic meter of the solution. Suppose that $Q' < Q$ cubic meters have already been shared (combined with some monetary transfers). Given the possibility of monetary compensations, the efficient allocation of Q' prescribed by the solution must equalize the marginal social surplus across countries (assuming for simplicity that we have an interior solution). When considering the additional cubic meter to be shared in the next iteration of the decomposition, all the countries look identical, because a cubic meter is essentially an infinitesimal quantity when compared to Q, and the countries' social surplus functions over quantities that are larger than this infinitesimal amount must be irrelevant under IND. In order to be anonymous (a minimal requirement for equitability), the solution should give an equal share to each country of the additional total surplus generated by the additional cubic meter to allocate. Iterating the process, it follows that the total surplus associated to Q should be shared equally across countries. The formal reasoning is more general (e.g. allowing for multiple goods, and without restricting attention to functions v_i that guarantee interior solutions), but also requires to focus on solutions that are regular (formalized in an axiom of continuity) in order to make the argument at the margin complete.

The paper unfolds as follows. Section 2 presents the model. The axioms and the main result are included in Section 3, while its proof is postponed to Section 5. Section 4 offers a review of the related literature.

2. Model

A set $\mathcal{I}$ of $I \geqslant 2$ individuals have to allocate a bundle ω of L perfectly divisible goods ($\omega \in \mathbb{R}^L_+$). Some compensation can be achieved through monetary transfers. An *allocation* is a couple $(x, t) \in \mathbb{R}^{IL}_+ \times \mathbb{R}^I$ where, for each $i \in \mathcal{I}$, t_i (resp. x_i) represents the net amount of money (resp. bundle of goods) that individual i receives. It is *feasible* if $\sum_{i \in I} x_i \leqslant \omega$ and $\sum_{i \in I} t_i \leqslant 0$.[4] The set of feasible allocations will be denoted by $\mathcal{F}(\omega)$.

Utilities are quasi-linear. The *utility function* $u_i : \mathbb{R}^L_+ \to \mathbb{R}_+$ determines the maximal amount of money $u_i(x)$ that individual i is ready to pay to consume each bundle $x \in \mathbb{R}^L_+$ ($u_i(x)$ will be called i's willingness to pay for x). The utility functions are assumed to be non-decreasing,

[3] The story is of course rather stylized, the objective being to emphasize the argument behind the main result of our paper. Still, the model is more general than it may seem at first sight. For instance, the costs of extraction seem to be overlooked, but they can possibly be expressed in terms of the energy required to extract the gas, which itself can be obtained from a fraction of the natural gas extracted. Q can then be interpreted as the *net* quantity available in the field. Also, our story does not incorporate time explicitly, but the functions v_i can be reinterpreted as the net present value of streams of resources to be extracted.

[4] As usual in the standard quasi-linear model (see the definition of utility functions in the next paragraph), monetary endowments are not modeled explicitly. This is without loss of generality provided that the participants' endowments are large enough and/or that there is enough money to share in addition to the collective endowment ω of other goods. How to adapt our results in the absence of this standard implicit assumption remains an open problem.

continuous and such that $u(0) = 0$.[5] The set of all such functions is denoted by $\mathcal{U}$. Agent i's total utility associated to the allocation (x,t) is $u_i(x_i) + t_i$. A *utility profile* is a vector $\mathfrak{u}$ in $\mathbb{R}^I$. It is *feasible* if there exists a feasible allocation (x,t) such that $\mathfrak{u}_i = u_i(x_i) + t_i$, for each $i \in \mathcal{I}$.

An *allocation problem* P is a couple (ω, u), where ω is the bundle of L goods to share, and $u = (u_i)_{i\in\mathcal{I}} \in \mathcal{U}^I$ is the list of utility functions. The set of all allocation problems is denoted by $\mathcal{P}$.

An *allocation rule* (or simply a *rule*) is a correspondence $\mathcal{R}: \mathcal{P} \to \mathbb{R}_+^{IL} \times \mathbb{R}^I$, which associates to each allocation problem a nonempty set of feasible allocations. We will assume throughout the paper that the allocation rules determine a single utility profile:

$$\big\{(x,t) \in \mathcal{R}(P) \text{ and } \big(x',t'\big) \in \mathcal{R}(P)\big\} \quad \Rightarrow \quad \big\{u_i(x_i) + t_i = u_i\big(x_i'\big) + t_i', \ \forall i \in \mathcal{I}\big\}, \tag{1}$$

for each $P \in \mathcal{P}$, and each pair of allocations $((x,t),(x',t'))$.

A *solution* is a function $\sigma : \mathcal{P} \to \mathbb{R}^I$ that associates a utility profile to each allocation problem. Condition (1) makes it meaningful to consider the solution associated to a rule $\mathcal{R}$ that is defined as follows:

$$\sigma_i^{\mathcal{R}}(P) = u_i(x_i) + t_i, \quad \forall i,$$

for some (or each, by (1)) $(x,t) \in \mathcal{R}(P)$, and each $P \in \mathcal{P}$.

For each allocation problem $P = (\omega, u)$,

$$s(P) = \max_{x\in\mathbb{R}_+^{IL}} \left\{ \sum_{i\in\mathcal{I}} u_i(x_i) \;\Big|\; \sum_{i\in\mathcal{I}} x_i \leqslant \omega \right\}$$

denotes the maximal total surplus achievable. The *equal surplus sharing solution*[6] σ^{ESS} is then given by:

$$\sigma_i^{ESS}(P) = \frac{s(P)}{I},$$

for each $i \in \mathcal{I}$, and each $P = (\omega, u) \in \mathcal{P}$. The *equal surplus sharing allocation rule* $\mathcal{R}^{ESS}$ is then naturally defined as follows:

$$\mathcal{R}^{ESS}(P) = \big\{(x,t) \in \mathcal{F}(\omega) \;\big|\; u_i(x_i) + t_i = \sigma_i^{ESS}(P), \ \forall i \in \mathcal{I}\big\},$$

for each $P = (\omega, u) \in \mathcal{P}$.

Finally, an allocation rule $\mathcal{R}$ is *welfarist* if $\sigma^{\mathcal{R}}(P) = \sigma^{\mathcal{R}}(P')$, for each pair (P, P') of allocation problems with $s(P) = s(P')$. This definition should make precise the discussion we had in the Introduction and that we will pursue in Section 4.

[5] It is natural to assume that an individual's willingness to pay for consuming nothing is zero. Dropping this assumption would require to change some notations, but not the substance of our argument.

[6] One could argue that σ^{ESS} is actually the egalitarian solution. We refrain from using this terminology, because it also coincides with many other solutions such as the Nash or the Kalai–Smorodinsky solutions applied to the bargaining problem $(U(P), d(P))$, where $d(P) = 0$ and $U(P) = \{\mathfrak{u} \in \mathbb{R}^I \mid \sum_{i\in\mathcal{I}} \mathfrak{u}_i \leqslant s(P)\}$, for each $P \in \mathcal{P}$. The problems being quasi-linear, σ^{ESS} actually coincides with any solution that is welfarist, and satisfies the properties of "Efficiency" and "Equal Treatment of Equals" (cf. definitions below in the main text).

3. Main result

Here are the axioms that we will impose on the allocation rule.

Efficiency (EFF). $\sum_{i\in\mathcal{I}}\sigma_i^{\mathcal{R}}(P)=s(P)$, for each $P\in\mathcal{P}$.

Equal Treatment of Equals (ETE). $\sigma_i^{\mathcal{R}}(P)=\sigma_j^{\mathcal{R}}(P)$, for each $P=(\omega,u)\in\mathcal{P}$, and each $i,j\in\mathcal{I}$ such that $u_i=u_j$.

No Profitable Decompositions (NPD). Let $\tilde{P}=(\widetilde{\omega},u)\in\mathcal{P}$, let $\omega\in\mathbb{R}_+^L$ be such that $\omega\leqslant\widetilde{\omega}$, let $P=(\omega,u)$, let $i\in\mathcal{I}$, and let $(\tilde{x},\tilde{t})\in\mathcal{R}(\tilde{P})$. Then, there exist $(x,t)\in\mathcal{R}(P)$ and $(y,r)\in\mathcal{R}(P_x)$ such that

$$u_i(x_i+y_i)+t_i+r_i\leqslant u_i(\tilde{x}_i)+\tilde{t}_i,$$

where $P_x=(\widetilde{\omega}-\omega,u^x)$ is the "residual problem" obtained after distributing (x,t), i.e. with $u_i^x(y_i)=u_i(x_i+y_i)-u_i(x_i)$, for each $y_i\in\mathbb{R}_+^L$ and each $i\in\mathcal{I}$.

Independence of Preferences over Non-Feasible Allocations (IND). Let $P=(\omega,u)\in\mathcal{P}$ and $\tilde{P}=(\omega,\tilde{u})\in\mathcal{P}$ be such that $u_i(x)=\tilde{u}_i(x)$, for each $i\in\mathcal{I}$ and each $x\in\mathbb{R}_+^L$ with $x\leqslant\omega$. Then $\sigma_i^{\mathcal{R}}(P)=\sigma_i^{\mathcal{R}}(\tilde{P})$.

Continuity (CONT).

(a) Let $\omega\in\mathbb{R}_+^L$ and let $(\omega_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+^L$ that converges to ω. Then the sequence $(\sigma^{\mathcal{R}}(\omega_k,u))_{k\in\mathbb{N}}$ converges to $\sigma^{\mathcal{R}}(\omega,u)$, for each $u\in\mathcal{U}^I$.

(b) For every compact set $K\subseteq\mathbb{R}_+^L$, there exists $M>0$ such that[7]

$$\left\|\sigma^{\mathcal{R}}(\omega,u)-\sigma^{\mathcal{R}}(\omega,\tilde{u})\right\|\leqslant Md(u,\tilde{u}),$$

for every $\omega\in K$ and $u,\tilde{u}\in\mathcal{U}^I$.

EFF simply imposes on the rule to specify allocations that are Pareto efficient. It should not be possible to find another feasible allocation that would make all the individuals happier. ETE guarantees some minimal form of equity, in that two individuals with the same utility functions are treated identically. NPD guarantees that no participant can have an interest in manipulating the allocation rule through some decomposition of the stakes.[8] As explained in the Introduction, a violation of that property may lead to conflict and inefficiency when it comes to implementing the rule, as well as a violation of the equity principles that motivated the solution in the first place. Different people may have different opinions regarding what is the right way of formalizing NPD depending on the agents relative optimism/pessimism when decomposing the stakes (given that allocation rules can be multi-valued). Our formulation presumes that the agents are most pessimistic, making a rule robust to profitable decompositions as soon as the combination of some element $(x,t)\in\mathcal{R}(P)$ and $(y,r)\in\mathcal{R}(P_x)$ makes them no better than the solution of

[7] $d(u,\tilde{u})=\max_{i\in\mathcal{I}}\sup_{x\in\mathbb{R}_+^L}|u_i(x)-\tilde{u}_i(x)|$.

[8] As clear from the statement of the axiom, we consider situations where previous agreements cannot be renegotiated when stakes are decomposed. We will come back to this point when discussing the link between NPD and Kalai's [18], Step-by-Step Decomposition axiom in the next section.

the original problem $\tilde{P}$. The property is thus the weakest version one can think of, making the uniqueness result in the next theorem only more interesting. On the other hand, observe that Σ^{ESS} does satisfy the stronger version of NPD, in which agents are most optimistic.[9]

Strong NPD. Let $\tilde{P} = (\widetilde{\omega}, u) \in \mathcal{P}$, let $\omega \in \mathbb{R}^L_+$ be such that $\omega \leqslant \widetilde{\omega}$, let $P = (\omega, u)$, let $i \in \mathcal{I}$, and let $(\tilde{x}, \tilde{t}) \in \mathcal{R}(\tilde{P})$. Then, for each $(x, t) \in \mathcal{R}(P)$, and each $(y, r) \in \mathcal{R}(P_x)$, we have:

$$u_i(x_i + y_i) + t_i + r_i \leqslant u_i(\tilde{x}_i) + \tilde{t}_i,$$

where $P_x = (\widetilde{\omega} - \omega, u^x)$ is the "residual problem" obtained after distributing (x, t), i.e. with $u^x_i(y_i) = u_i(x_i + y_i) - u_i(x_i)$, for each $y_i \in \mathbb{R}^L_+$ and each $i \in \mathcal{I}$.

While discussing the independence of the axioms after stating the Theorem, we will encounter a simple solution that satisfies NPD, but not its stronger version. Before that, let us motivate the last two axioms. A rule must specify feasible allocations, and hence no individual can ever receive more than the amounts that are available for division. It is then natural to assume that the individuals' willingness to pay for bundles that are not feasible should be irrelevant in the determination of the final allocation, as required by IND (see references in the introduction). It is also meaningful to require some form of continuity on the allocation rule. CONT formalizes the idea that small measurement mistakes should not trigger a major difference when computing the solution. Part (a) applies this principle to the total resources available, while part (b) requires the stronger property of Lipschitz continuity with respect to the utility functions.

Theorem. *$\mathcal{R}^{ESS}$ satisfies EFF, ETE, NPD, IND and CONT. Conversely, any allocation rule that satisfies the axioms must be such that $\sigma^{\mathcal{R}} = \sigma^{ESS}$.*

We already gave some intuition for this theorem in the Introduction, and we defer the complete proof to Section 5. We now discuss the independence of the axioms. The equal split allocation rule, $\mathcal{R}^{ES}$, which shares ω equally among all the individuals without making any monetary compensation, satisfies all the axioms except EFF. A rule that selects those feasible allocations at which the total surplus is split in some fixed (but not equal) proportions across individuals (as in Kalai's [18] proportional solutions) clearly satisfies all our axioms, except ETE. Consider next the solution proposed by Moulin in [24]. For each $P = (\omega, u) \in \mathcal{P}$, let

$$\mathcal{R}^M(P) = \bigl\{(x, t) \in \mathcal{F}(\omega) \bigm| u_i(x_i) + t_i = Sh_i\bigl(v^{(\omega,u)}\bigr),\ \forall i\bigr\},$$

where Sh denotes the Shapley value, and $v^{(\omega,u)}$ is the characteristic function defined as follows:

$$v^{(\omega,u)}(\mathcal{S}) = \max_{x \in \mathbb{R}^{IS}_+} \left\{ \sum_{i \in \mathcal{S}} u_i(x_i) \,\middle|\, \sum_{i \in \mathcal{S}} x_i \leqslant \omega \right\},$$

for each coalition $\mathcal{S} \subseteq \mathcal{I}$ (i.e. the maximal surplus that members of $\mathcal{S}$ could share if they were free to distribute ω among themselves). $\mathcal{R}^M$ satisfies EFF (resp. ETE; resp. CONT (a); resp. (b)) because the Shapley value is efficient (resp. symmetric; resp. continuous; resp.[10] linear). It obviously satisfies IND, given the way $v^{(\omega,u)}$ is defined. The Theorem thereby implies that it violates

[9] Observe that the strong version of NPD actually implies condition (1).

[10] To show that $\mathcal{R}^M$ satisfies CONT (b), one also needs to observe that $|v^{(\omega,u)}(\mathcal{S}) - v^{(\omega,\tilde{u})}(\mathcal{S})| \leqslant Sd(u, \tilde{u})$, which is shown explicitly in Section 5 for the special case $\mathcal{S} = \mathcal{I}$ (when checking that σ^{ESS} satisfies CONT).

NPD. More explicitly, consider for instance the allocation problem $\tilde{P}$ with $L = 1$, $\mathcal{I} = \{1, 2\}$, $\widetilde{\omega} = 2$, $u_1(x) = 2x$ if $x \leqslant 1$ (resp. $1 + x$ if $x \geqslant 1$), and $u_2(x) = \min\{x, 1\}$, for each $x \in \mathbb{R}_+$. Any element of $\mathcal{R}^M$ gives a utility of 2.5 to the first agent and 0.5 to the second agent. Even a pessimistic agent 2 (as in NPD) would want to decompose the stakes, starting for instance by allocating a single unit of the good. Indeed, the Moulin solution of that problem contains a unique allocation, with the first agent receiving the good and paying a half dollar to agent 2. Solving the residual problem, we conclude that the second agent can guarantee himself a utility of at least \$1 via this decomposition. To conclude, we have unfortunately not been able to prove separately the independence of IND and CONT from the rest of the axioms. While we clearly use both axioms in the proof in Section 5, it remains a possibility (and would make the Theorem only more interesting) that one of them might be dropped, or at least weakened. We will therefore show only that they cannot both be dropped and, for notational simplicity, we will do so for $L = 1$. Let $\hat{\mathcal{P}}$ be the set of problems (ω, u) for which there exist $\alpha \in \mathbb{R}^I_{++}$ and $x^* \geqslant 0$ such that u_i is differentiable and $u_i'(x) = \alpha_i$, for each $x \geqslant x^*$ and each $i \in \mathcal{I}$. Consider then the following allocation rule:

$$\mathcal{R}(P) = \mathcal{R}^{ESS}(P), \quad \text{for each } P \in \mathcal{P} \setminus \hat{\mathcal{P}}, \quad \text{and}$$
$$\mathcal{R}(P) = \left\{(x, t) \in \mathcal{F}(\omega) \;\middle|\; u_i(x_i) + t_i = Sh_i\left(v^u\right)s(\omega, u),\ \forall i \in \mathcal{I}\right\},$$

for each $P = (\omega, u) \in \hat{\mathcal{P}}$, where v^u is the characteristic function defined as follows:

$$v^u(\mathcal{S}) = \frac{\max_{i \in \mathcal{S}} \alpha_i}{\max_{i \in \mathcal{I}} \alpha_i},$$

for each coalition $\mathcal{S} \subseteq \mathcal{I}$. It is not difficult to check that $\mathcal{R}$ satisfies EFF, ETE, and NPD (because $\hat{\mathcal{P}}$ is closed under decompositions, and $s(\omega, u)$ is additive, as shown in Section 5 when checking that σ^{ESS} satisfies NPD), but violates both CONT and IND.

We conclude this section with three remarks regarding our main theorem.

Remark 1. Following Roemer's [36] terminology, a rule $\mathcal{R}$ is said to be a *full correspondence* if, for each $P \in \mathcal{P}$, an allocation $(x', t') \in \mathcal{R}(P)$ whenever it is feasible and it generates the same utility profile as an allocation $(x, t) \in \sigma(P)$. Observe that our characterization of the equal surplus sharing allocation rule does not require the rule to be a full correspondence, while many papers that characterize classical welfarist solutions in non-welfarist environments do make such an assumption. While $\mathcal{R}^{ESS}$ is a full correspondence, observe for instance that $\mathcal{R}^{ES}$ is not. Of course, it can be extended into the following full correspondence:

$$\bar{\mathcal{R}}^{ES}(P) = \left\{(x, t) \in \mathcal{F}(\omega) \;\middle|\; u_i(x_i) + t_i = u_i\left(\frac{\omega}{I}\right),\ \forall i\right\},$$

for each $P \in \mathcal{P}$. While $\mathcal{R}^{ES}$ satisfies also the stronger version of NPD, $\bar{\mathcal{R}}^{ES}$ satisfies only NPD. To see that, consider for instance the allocation problem $\tilde{P}$ with $L = 1$, $\mathcal{I} = \{1, 2\}$, $\widetilde{\omega} = 4$, $u_1(x) = x$, and $u_2(x) = \min\{x, 1\}$, for each $x \in \mathbb{R}_+$. Any element of $\bar{\mathcal{R}}^{ES}$ gives a utility of 2 to the first agent and 1 to the second agent. An optimistic agent 2 may hope to be better off by first receiving nothing of the good plus a compensation of one dollar when allocating the first two units, and then getting 1 unit of the good with no compensation in the residual problem. A pessimistic agent 2, on the other hand, would have no strict incentive to decompose the stakes when $\bar{\mathcal{R}}^{ES}$ is used (any rule that contains a rule which satisfies strong NPD – in this case $\mathcal{R}^{ES} \subseteq \bar{\mathcal{R}}^{ES}$ – must necessarily satisfy NPD). We conclude that restricting the analysis to full correspondences may thus be restrictive in that it eliminates reasonable allocation rules when used in conjunction with other axioms (strong NPD in this example).

Remark 2. Our Theorem remains true on the restricted domain of concave utility functions. Formally, any rule $\mathcal{R}$ that satisfies EFF, ETE, NPD, IND and CONT on $\mathcal{P}_{\mathcal{C}}$ must be such that $\sigma^{\mathcal{R}} = \sigma^{ESS}$, where $\mathcal{C} = \{u \in \mathcal{U} \mid u \text{ is concave}\}$ and $\mathcal{P}_{\mathcal{C}} = \{(\omega, u) \mid u \in \mathcal{C}\}$. The proof of this variant of our Theorem is deferred to the end of Section 5.

Remark 3. We conclude this section by arguing that the natural analogues of EFF, ETE, NPD, and IND are likely to be incompatible when monetary compensations are not available. When there is a single good to be allocated, the equal split solution is the only solution that satisfies the axioms, at least if preferences are strictly increasing. This is a direct consequence of ETE, since there is only one possible such ordinal preference – the more the better. Moving to two goods or more leads to an impossibility. This follows from Moulin and Thomson's [25, Theorem 1] impossibility result. Indeed, the natural extension of NPD in a framework without monetary compensations will imply their property of "Resource Monotonicity." At the same time, IND and NPD will imply their "Individual Rationality" axiom, which requires that each individual prefers the final outcome to an equal split of the total endowment. Notice that applying the natural extension of NPD good by good will imply that property, since IND imply that the solution of each smaller problem (focus on one good) depends only on the individuals' preferences for that good, and as before, there is only one such preference (restricting attention to preferences that are strictly monotonic). Moulin and Thomson's [25] two-good two-individual counter-examples therefore apply, and it is not difficult to extend them to counter-examples with any number of goods and individuals. It remains an interesting question to find restricted domains that are different from the quasi-linear case, and where the axioms would be compatible again (see [25, comment (D), Section 4]).

4. Related literature

Graham et al. [14, Section II] characterized the equilibrium allocation rule that prevails in single-unit second-price auctions in the presence of nested buyer rings. Its computation is reminiscent of the principle of serial cost sharing (see [21]) and each resulting allocation happens to coincide with the Shapley value of some characteristic function derived from the buyers' willingness to pay. Indeed, the payoffs have a strong normative appeal as well (see [24, Section 5]). There seems to be a natural procedure to adapt this allocation rule to problems that involve a quantity Q of a *divisible* good: decompose the problem into a sequence of allocation problems with infinitesimal quantities, solve each infinitesimal problem via the previous solution (treating each infinitesimal quantity as indivisible), and integrate in order to obtain a solution for the original problem. Of course, the procedure works well only for problems with decreasing marginal utilities, as otherwise the resulting allocations are not necessarily efficient. Suppose also that the utility functions are regular, that is differentiable and such that the efficient allocation of any positive Q gives a positive amount of the good to each participant (interior solutions). It turns out that the resulting solution then coincides with equal surplus sharing. This is true not only when applying the constructive procedure to the Graham et al. [14] or Moulin [24] allocation rule, but also to any solution that guarantees to each agent a payoff that is larger than or equal to his valuation for the indivisible good to be allocated divided by the number of participants,[11] a rather weak equity property first introduced by Moulin and Thomson [25], which plays a central role

[11] The competitive equilibrium with equal income, $\mathcal{R}^{CE}$, satisfies this property, for instance, since each individual gets a final payoff that is larger or equal to his willingness to pay for ω/I, which in turn is larger or equal to his willingness to pay for ω divided by I (the utility functions being concave). Since the competitive equilibrium with equal income is

in [24]. The proof of this new result is very similar to Step 1 in the proof of our Theorem (see Section 5). The general idea is that, at every step of the continuous summation, the lower bound on the participant's final utility is binding, and equal to the common marginal utility (which is also equal to the derivative of the total surplus, by the envelope theorem) divided by the number of participants.[12] The details for the full proof are left to the dedicated reader. It is interesting to note that the Graham et al./Moulin allocation rule, as well as many of the rules that meet Moulin and Thomson's [25] lower-bound requirement, are not welfarist. Yet, once iterated to obtain a solution for the divisible case, they all result in the same welfarist solution.

At first sight, NPD may seem very similar to Kalai's [18] axiom of *step-by-step negotiations* (see also [26], and Young's [38] composition principle in taxation problems). In reality, the two axioms are rather different. Indeed, NPD cannot even be phrased in Kalai's [18] welfarist framework, because the set of utility profiles that are feasible when sharing the bundle $\omega - \omega'$ after having solved for $\omega' < \omega$ may be strictly smaller than the set of utility profiles that are feasible when sharing the bundle ω. Kalai [18] assumes instead that the solution for the problem of dividing ω' is a partial agreement that serves as a disagreement point in a new bargaining problem where any division of the bundle ω can still be agreed upon. NPD, on the contrary, assumes that any partial agreement is final and non-renegotiable.[13] Kalai's arguments in support of the egalitarian solution are not very informative for the quasi-linear case that we focus on. Indeed, equal sharing of the surplus follows immediately from the properties of efficiency and anonymity when one is ready to work in the space of utilities. The purpose of Kalai's argument instead is to characterize proportional solutions in a welfarist framework when utilities are non-transferable. It may be interesting to test the robustness of Kalai's result, by trying to rephrase it in explicit economic environments. As shown on different occasions, and most forcefully by Roemer [36], axioms that characterize a solution in the space of utilities are usually satisfied by other non-welfarist solutions as well.

The additivity/super-additivity property[14] that plays a key role in various axiomatic results of social choice and cooperative game theory is often motivated by referring to multiple issues (see e.g., [37,31,33]).[15] The story behind the axiom is that the participants' payoffs when bar-

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different from $\mathcal{R}^{ESS}$, it must be that it violates NPD. Indeed, here is a simple direct example. Consider a two-consumer, one-good, quasi-linear economy in which consumers' utilities over bundles $(x,t) \in \mathbb{R}_+ \times \mathbb{R}$ are $\ln(1+x)+t$ and $\sqrt{x}+t$. It is easy to check that $\mathcal{R}^{CE}(14) = \{((5, \frac{1}{3}), (9, -\frac{1}{3}))\}$ and $\mathcal{R}^{CE}(7) = \{((3, \frac{1}{8}), (4, -\frac{1}{8}))\}$. In the residual problem (after allocating first half of the endowment), the equilibrium allocation is $((2, \frac{1}{4}), (5, -\frac{1}{4}))$. Thus, consumer 1 gets a utility of $\ln 6 + \frac{3}{8}$ by decomposing the stakes, which is strictly higher than $\sigma^{CE}(14)$.

[12] Notice that requiring the efficient allocations to be interior is important. If the first participant's utility function equals the quantity he consumes, while the second participant's utility function equals twice the quantity she consumes, then the solution obtained by iterating the Graham et al./Moulin allocation rule does not coincide with an equal split of the total surplus.

[13] Kalai himself [18, p. 1627] offers a very clear discussion of his axiom of step-by-step negotiations, emphasizing that, although a property in the spirit of NPD would be a natural formulation of the general principle, his axiom must have an alternative interpretation in terms of partial agreements because NPD cannot be phrased in the space of utilities. It is thus surprising that, to the best of our knowledge, the property of NPD has not been studied sooner in non-welfarist environments.

[14] Quasi-linear problems lead to utility possibility sets that are half-spaces, and super-additivity is then equivalent to additivity.

[15] The additivity/super-additivity property is sometimes given an alternative interpretation in terms of a preference to agree before the resolution of some uncertain events, see e.g. [27,30]. This kind of argument is unrelated to our analysis, since there is no uncertainty in our framework, and utility functions do not contain any information regarding risk attitudes.

gaining over all the issues at once should be larger than or equal to the sum of their payoffs when bargaining over the different issues separately. A difficulty though is that all the papers in that vein are written in welfarist frameworks. Yet it is usually impossible to derive the utility possibility set when bargaining over two issues simultaneously, from the two sets of the utilities that are feasible when bargaining over each issue separately. The usual motivation behind the additivity/superadditivity property is thus meaningful only when utility functions are assumed to be additively separable across issues, in which case the former set is indeed the sum of the other two.[16] So, while applying NPD to decompositions good by good is reminiscent of these ideas on multi-issue bargaining, we believe that NPD is a more appropriate formulation. Arguing in a non-welfarist framework, we are indeed able to treat problems with no underlying restriction on utility functions. NPD also highlights another class of multi-issue problems that arise from alternative decompositions. Indeed, a participant may insist, for instance, on sharing first a fraction of the total endowment, before sharing what remains. The two issues that this decomposition generates are inter-dependent, even if the utility functions are additively separable (or even if $L = 1$), and therefore cannot be phrased in any welfarist model. Note also that the proof of our result has no analogue in the literature on multi-issue bargaining, since the equal surplus sharing solution follows trivially from the axioms of efficiency and anonymity when working exclusively in the space of utilities.

O'Neill et al. [28] introduce a new welfarist model of bargaining, where the set of feasible utility profiles expands over time according to a differentiable function. Our two papers thus share a common line of argument, in that a solution is ultimately characterized by integrating its local behavior, which can be determined by imposing rather weak axioms. A first obvious difference is that there is no given bargaining agenda in our model. The integration step follows from the NPD property instead. More importantly, the arguments bear on different objects in our two papers. Working in the space of utilities, equal surplus sharing is not derived by O'Neill et al. [28], but instead assumed by their symmetry property. The key ingredient in their result is that the efficient frontier of the expanded set of feasible utilities at time $t + \Delta t$ that lies above the agreement reached at time t is essentially linear when Δt is infinitesimal. Scale covariance then leads to a problem in the space of utilities that can be solved by direct application of the symmetry axiom. The key ingredient in our result is that the participants' preferences are essentially identical when an infinitesimal quantity $\Delta\omega$ has to be divided after a strictly positive quantity ω has already been distributed (assuming that we have an interior solution). Notice how the set of feasible utilities at time t does not depend on previous agreements in O'Neill et al.'s [28] model. Rephrased in an economic environment like ours, this implies that the whole quantity of all the goods that have been bargained in the past must be renegotiated at every t, as in Kalai's [18] interpretation of the property of step-by-step decomposition. In our case, to integrate the solution of local problems that follow a path from 0 to ω often leads to an inefficient solution because past agreements are

[16] Green [15] has taken a first step away from welfarism in quasi-linear problems, by dissociating monetary compensations from the set of utilities that are achievable in the absence of transfers (see also [16,5,6] for more recent results). The additivity/superadditivity property is subject to the same limitation as far as its interpretation is concerned, but it is worth noting that these authors do obtain interesting solutions that are both anonymous and efficient, while different from the equal surplus sharing rule. These solutions would trivially satisfy IND if they were rephrased in our explicit economic environments, because the utility possibility set obtained in the absence of monetary transfers does not change when one modifies the utility function of any participant over bundles that involve more goods than available in the total endowment. Those solutions must therefore violate NPD and/or CONT.

assumed to be non-renegotiable (except when $L = 1$ and marginal utilities are decreasing, as in the first paragraph of the present section).

NPD is related to the CONRAD property that Roemer [36] introduced to recover most classical results in bargaining theory with axioms phrased in economic environments. Though weaker, the CONRAD property is far more cumbersome than NPD, because it restricts in a rather ad-hoc way the set of decompositions over which it applies (adding goods in which at most one agent is interested, provided the set of feasible utility profiles remains the same). If a person likes Roemer's idea of consistency in CONRAD, then we think that he or she will prefer to go all the way to NPD. Notice that Roemer's proof cannot be adapted to our framework because he makes crucial use of preferences that are not quasi-linear. Our result has also the advantage of holding for any fixed number of goods, while Roemer works with a variable and potentially infinite number of goods.

The present paper studies the exact same problem as Moulin [24], but from a different perspective. Moulin [24] introduces four new properties: resource monotonicity, population solidarity, (weak and strong) individual rationality, and the stand-alone test. He then shows that these four properties, as well as most possible combinations of two or three properties out of the list, are incompatible both on the general domains and when restricting attention to concave utility functions. On the other hand, there exists a solution that satisfies the four axioms simultaneously (using the weaker version of individual rationality) on the restricted domain where goods are substitutes, that is when restricting attention to utility functions that are concave in each good, as well as submodular. We have already discussed Moulin's solution when checking the independence of our axioms at the end of the previous section. There we noted that it satisfies all our axioms, except NPD. It is thus subject to strategic manipulations of the agenda, leading, as we argued earlier, to possible conflict, inefficiency, and violation of the equity principles that motivated the solution in the first place.

It is not difficult to check, on the other hand, that the equal surplus sharing solution that we characterized, satisfies all of Moulin's axioms except the stand-alone test (not only on the restricted domain where goods are substitute, but over the entire domain). Let us thus explain briefly the content of that test and why, though interesting, we do not see it as an uncontroversial principle of equity. A solution passes the stand-alone test if no coalition of agents receives a higher aggregated payoff than the maximal surplus that its members could achieve if they were free to share the whole total endowment, giving nothing to non-members. The solution must thus belong to the anti-core of the fictitious characteristic function used to compute the Moulin solution. It implies for instance that an agent on his own cannot get a payoff that is larger than his willingness to pay for consuming the total endowment. The equal surplus sharing solution takes a different standpoint on equity. Even in the limit case where an agent does not care for the goods being shared, we think that he should not be treated as irrelevant because he is a member of the group that collectively owns the total endowment. More generally, it is true that efficiency requires that an agent should not consume much of the goods being shared when others have higher marginal utilities, but this does not mean that there should be no or little monetary compensations in order to reach an equitable outcome. It remains a fact that consuming less is a favor to other agents, insofar as it lets them consume more, and it seems fair to reward agents on the basis of that criterion as well.

We close this literature review by briefly discussing two alternative axiomatic characterizations of the equal surplus sharing solution in non-welfarist environments. Moulin [23, Theorem 2] provides one such result in the context of adopting a public decision, together with monetary compensations, when there are at least three participants. Interestingly, his key axiom, No Advantageous Reallocations (NAR), is another property of robustness against some class of potential manipulations of the solution to be implemented. Indeed it requires that no coalition of individuals can be better off by publicly changing their utility functions via contingent monetary transfers. NPD on the other hand operates through decompositions of the total endowment, while the participants' utility functions are fixed. Ginés and Marhuenda [13] study economies where money is used to produce multiple public goods. They succeed in characterizing the equal surplus sharing solution by giving some economic content to Kalai's [18] monotonicity property. The axiom restricts the behavior of the solution when the individuals' satisfaction from consuming the public goods increase. This kind of principle has nothing to do with the axioms we discussed in Section 3. Ginés and Marhuenda also show that their result does not extend to the production of private goods. This confirms that there is no connection between our result and theirs.

5. Proofs

Proof of the Theorem. It is clear that $\mathcal{R}^{ESS}$ satisfies EFF, ETE and IND. Part (a) of CONT is an immediate consequence of Berge's [2] maximum theorem. As for part (b) of CONT, let $x \in \mathbb{R}^{IL}_+$ be such that $\sum_{i\in\mathcal{I}} x_i \leqslant \omega$ and $\sum_{i\in\mathcal{I}} u_i(x_i) = s(\omega, u)$. Then $\sum_{i\in\mathcal{I}} \tilde{u}_i(x_i) \leqslant s(\omega, \tilde{u})$, and hence $s(\omega, u) - s(\omega, \tilde{u}) \leqslant Id(u, \tilde{u})$. A similar argument also implies that $s(\omega, \tilde{u}) - s(\omega, u) \leqslant Id(u, \tilde{u})$. Hence $|\sigma_i^{ESS}(\omega, u) - \sigma_i^{ESS}(\omega, \tilde{u})| \leqslant d(u, \tilde{u})$, for every $u, \widetilde{u} \in \mathcal{U}^I$ (independently of ω), and thus $\mathcal{R}^{ESS}$ satisfies CONT. Finally, to check that it satisfies NPD (or even its strong version), it is enough to observe that

$$s(\omega', u) \geqslant \max\left\{ \sum_{i\in\mathcal{I}} u_i(x_i^* + y_i) \;\middle|\; y \in \mathbb{R}^{IL}_+ \text{ and } \sum_{i\in\mathcal{I}} y_i \leqslant \omega' - \omega \right\},$$

for any x^* that maximizes $\sum_{i\in\mathcal{I}} u_i(x_i)$ over the set of vectors $x \in \mathbb{R}^{IL}_+$ such that $\sum_{i\in\mathcal{I}} x_i \leqslant \omega$.

We prove next the second part of the theorem. Let thus $\mathcal{R}$ be a rule that satisfies the five axioms, and let $(\bar{\omega}, u)$ be an allocation problem. We have to prove that $\sigma^{\mathcal{R}}(\bar{\omega}, u) = \sigma^{ESS}(\bar{\omega}, u)$. For each $\bar{\omega} \in \mathbb{R}^L_{++}$, let $X(\bar{\omega}) := \{x \in \mathbb{R}^L_+ \mid x \leqslant \bar{\omega}\}$, and let $\mathcal{V}(\bar{\omega})$ be the set of functions $u \in \mathcal{U}$ that are twice continuously differentiable on the interior of $X(\bar{\omega})$ and satisfy the following Inada condition:

$$\forall d \in \partial\mathbb{R}^L_+ \cap X(\bar{\omega}), \quad \forall l \in \{1, \dots, L\}, \qquad \text{if } d_l = 0 \quad \text{then } \lim_{x\to d} \frac{\partial u}{\partial x_l}(x) = +\infty, \tag{2}$$

where the limit is taken with respect to $x \in int(X(\bar{\omega}))$.

We are now ready to proceed with the proof in four steps.

Step 1. *Suppose that $\bar{\omega} \in \mathbb{R}^L_{++}$. If $u \in \mathcal{V}(\bar{\omega})^I$, then $\sigma_i^{\mathcal{R}}(\cdot, u)$ admits right partial derivatives at any point ω in the interior of $X(\bar{\omega})$. In addition, for every $\omega \in int(X(\bar{\omega}))$ and every $l \in \{1, \dots, L\}$,*

$$\lim_{\varepsilon\to 0^+} \frac{\sigma_i^{\mathcal{R}}(\omega + \varepsilon\mathbf{1}_l, u) - \sigma_i^{\mathcal{R}}(\omega, u)}{\varepsilon} = \frac{1}{I}\frac{\partial}{\partial\omega_l} s(\omega, u)$$

where $\mathbf{1}_l \in \mathbb{R}^L$ denotes the l-th vector of the canonical base of $\mathbb{R}^L$.

Proof. Let ω, l as above, let $\varepsilon \in (0, 1]$, and let $i \in \mathcal{I}$. NPD applied with $\widetilde{\omega} = \omega + \varepsilon\mathbf{1}_l$ implies that there exists $(x^i(\varepsilon), t^i(\varepsilon)) \in \mathcal{R}(\omega, u)$ such that

$$\sigma_i^{\mathcal{R}}(\omega + \varepsilon\mathbf{1}_l, u) - \sigma_i^{\mathcal{R}}(\omega, u) \geqslant \sigma_i^{\mathcal{R}}(\varepsilon\mathbf{1}_l, u^{x^i(\varepsilon)}). \tag{3}$$

For each $j \in \mathcal{I}$, let

$$\alpha^i_j(\varepsilon) = \max_{0 \leqslant y \leqslant \varepsilon \mathbf{1}_l} \left| \nabla u_j\big(x^i_j(\varepsilon)\big) \cdot y - u_j^{x^i(\varepsilon)}(y) \right|,$$

and let $u_j^{i,\varepsilon}$ be the utility function defined as follows:

$$u_j^{i,\varepsilon}(y) = \begin{cases} \nabla u_j(x^i_j(\varepsilon)) \cdot y & \text{if } |\nabla u_j(x^i_j(\varepsilon)) \cdot y - u_j^{x^i(\varepsilon)}(y)| \leqslant \alpha^i_j(\varepsilon), \\ u_j^{x^i(\varepsilon)}(y) + \alpha^i_j(\varepsilon) & \text{if } \nabla u_j(x^i_j(\varepsilon)) \cdot y - u_j^{x^i(\epsilon)}(y) > \alpha^i_j(\varepsilon), \\ u_j^{x^i(\varepsilon)}(y) - \alpha^i_j(\varepsilon) & \text{if } u_j^{x^i(\varepsilon)}(y) - \nabla u_j(x^i_j(\varepsilon)) \cdot y > \alpha^i_j(\varepsilon), \end{cases}$$

for each $y \in \mathbb{R}^L_+$. It is easy to check that $u_j^{i,\epsilon} \in \mathcal{U}$, for each $j \in \mathcal{I}$.

Let K be the compact set $K = \{y \in \mathbb{R}^L_+ \mid y \leqslant \mathbf{1}_l\}$. Part (b) of CONT implies that there exists $M > 0$ such that

$$\sigma_i^{\mathcal{R}}\big(\varepsilon \mathbf{1}_l, u^{x^i(\varepsilon)}\big) \geqslant \sigma_i^{\mathcal{R}}\big(\varepsilon \mathbf{1}_l, u^{i,\varepsilon}\big) - M d\big(u^{x^i(\varepsilon)}, u^{i,\varepsilon}\big) \tag{4}$$

for each $\varepsilon \in (0, 1]$. Since $u \in \mathcal{V}(\bar{\omega})^I$ and $x^i(\varepsilon)$ is an efficient split of ω, it must be interior, and thus $\nabla u_j(x^i_j(\varepsilon)) = \nabla u_k(x^i_k(\varepsilon))$, for each $j \neq k$. Then IND, ETE, and EFF imply $\sigma_i(\varepsilon \mathbf{1}_l, u^{i,\varepsilon}) = \frac{\varepsilon}{I} \frac{\partial u_i}{\partial x_l}(x^i_i(\varepsilon))$. This in turn equals $\frac{\varepsilon}{I} \frac{\partial}{\partial \omega_l} s(\omega, u)$, by the envelope theorem.[17] Observe also that the uniform distance between $u^{x^i(\varepsilon)}$ and $u^{i,\varepsilon}$ is equal to $\alpha^i(\varepsilon) = \max_{j \in \mathcal{I}} \alpha^i_j(\varepsilon)$. Hence (3) and (4) imply that

$$\frac{\sigma_i^{\mathcal{R}}(\omega + \varepsilon \mathbf{1}_l, u) - \sigma_i^{\mathcal{R}}(\omega, u)}{\varepsilon} \geqslant \frac{1}{I} \frac{\partial}{\partial \omega_l} s(\omega, u) - M \frac{\alpha^i(\varepsilon)}{\varepsilon}. \tag{5}$$

We are now ready to prove, by contradiction, that the ratio on the left-hand side of (5) converges to $\frac{1}{I} \frac{\partial}{\partial \omega_l} s(\omega, u)$ when ε converges to 0, for each $i \in \mathcal{I}$. For simplicity, let's refer to this ratio as $r_i(\varepsilon)$. If the property is not true, then we can find $j \in \mathcal{I}$, $\beta > 0$, and a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ of strictly positive numbers that converges to 0 such that

$$\left| r_j(\varepsilon_k) - \frac{1}{I} \frac{\partial}{\partial \omega_l} s(\omega, u) \right| \geqslant \beta, \tag{6}$$

for each k. Taylor's theorem implies[18] that $M \frac{\alpha^j(\varepsilon_k)}{\varepsilon_k}$ converges to 0 when k goes to infinity, and hence there exists $k_0 \in \mathbb{N}$ such that $M \frac{\alpha^j(\varepsilon_k)}{\varepsilon_k} < \beta$, for each $k \geqslant k_0$. Combining this with (5) and (6), we must have $r_j(\varepsilon_k) - \frac{1}{I} \frac{\partial}{\partial \omega_l} s(\omega, u) \geqslant \beta$, for all those k's. Summing up the last inequality over $j \in \mathcal{I}$, we obtain:

[17] Note that $s(\cdot, u)$ is differentiable on the interior of $X(\bar{\omega})$ if $u \in \mathcal{V}(\bar{\omega})^I$. This follows from the implicit function theorem, using the fact that the maximum in the definition of s is attained at an interior point.

[18] Taylor's theorem implies indeed that, for each $\varepsilon > 0$, $\alpha^j(\varepsilon)$ is equal to the absolute value of the remainder term, which is smaller than ε^2 times the supremum of the absolute value of $\frac{\partial^2 u_j}{\partial x_l{}^2}(x^j_j(\varepsilon) + \alpha \mathbf{1}_l)$, over all $\alpha \in [0, \varepsilon]$, and all vectors $x^j_j(\varepsilon)$ that are part of an element in $\mathcal{R}(\omega, u)$. EFF and the definition of $\mathcal{V}(\bar{\omega})$ guarantee the set of all such $x^j_j(\varepsilon)$ is contained in a compact subset of $(int\, X(\bar{\omega}))^I$ (closedness of the set of efficient vectors of bundles for u follows from Berge's maximum theorem [2]), and hence the supremum is finite.

$$\frac{s(\omega+\varepsilon_k \mathbf{1}_l, u) - s(\omega, u)}{\varepsilon_k} = \sum_{i\in\mathcal{I}} r_i(\varepsilon_k) \geqslant \frac{\partial}{\partial\omega_l} s(\omega, u) + \beta \cdot I,$$

for each $k \geqslant k_0$. Taking the limit when k tends to infinity, we get a contradiction: $\frac{\partial}{\partial\omega_l} s(\omega, u) \geqslant \frac{\partial}{\partial\omega_l} s(\omega, u) + \beta \cdot I$. □

Step 2. *Suppose that $\bar{\omega} \in \mathbb{R}^L_{++}$. Let $u \in (\mathcal{V}(\bar{\omega}))^I$. Then $\sigma_i^{\mathcal{R}}(\bar{\omega}, u) = \sigma_i^{ESS}(\bar{\omega}, u)$, for each $i \in \mathcal{I}$.*

Proof. Fix $i \in \mathcal{I}$. We start by proving that $\sigma_i^{\mathcal{R}}(\omega, u) - \frac{1}{I}s(\omega, u)$ is constant on the interior of $X(\bar{\omega})$. For this, consider ω and ω^* in the interior of $X(\bar{\omega})$. Define the function $f : [0, 1] \to \mathbb{R}$ by $f(t) = \sigma_i^{\mathcal{R}}(t\omega_1, \omega_2, \ldots, \omega_I, u) - \frac{1}{I}s(t\omega_1, \omega_2, \ldots, \omega_I, u)$. Part (a) of CONT implies that f is continuous and, according to Step 1, f also has a right derivative with $f'_+(t) = 0$ for each $t \in (0, 1)$. Then f must be a constant function (for a proof, see for example [20]) and thus, $\sigma_i^{\mathcal{R}}(\omega, u) - \frac{1}{I}s(\omega, u) = \sigma_i^{\mathcal{R}}((\omega_1^*, \omega_{-1}), u) - \frac{1}{I}s((\omega_1^*, \omega_{-1}), u)$. One can iterate the argument by rescaling now the second component in the function on the right-hand side of the equality, to conclude that $\sigma_i^{\mathcal{R}}(\omega, u) - \frac{1}{I}s(\omega, u) = \sigma_i^{\mathcal{R}}((\omega_1^*, \omega_2^*, \omega_{-12}), u) - \frac{1}{I}s((\omega_1^*, \omega_2^*, \omega_{-12}), u)$. Further iterating the argument delivers the desired conclusion that $\sigma_i^{\mathcal{R}}(\omega, u) - \frac{1}{I}s(\omega, u)$ is constant on the interior of $X(\bar{\omega})$. CONT(a) implies that the function is constant on $X(\bar{\omega})$. Step 2 follows after observing that this constant is zero, which follows from the fact that the function evaluated at $\omega = 0$ is equal to zero. Indeed, IND implies that $\sigma_i^{\mathcal{R}}(0, u) = \sigma_i^{\mathcal{R}}(0, v)$ for any utility profile v. In particular, one can take a utility profile in which all agents are identical. Then ETE together with $s(0, u) = 0$ implies that $\sigma_i^{\mathcal{R}}(0, u) = 0 = s(0, u)/I$. □

Step 3. *For each $i \in \mathcal{I}$, there exists a sequence $(u_i^n)_{n\in\mathbb{N}}$ of functions in $\mathcal{V}(\bar{\omega})$ that converges uniformly to u_i.*

Proof. Let $(Q_i^n)_{n\in\mathbb{N}}$ be the sequence of multivariate Bernstein polynomials derived from u_i on $X(\bar{\omega})$ (a definition can be found in [22], for instance). It is well known that it converges uniformly to u_i on $X(\bar{\omega})$. Also, the elements of the sequence are smooth and non-decreasing on $X(\bar{\omega})$ (because u_i is non-decreasing). However, they may be decreasing on some regions out of $X(\bar{\omega})$. Let then $\tilde{Q}_i^n : \mathbb{R}^L_+ \to \mathbb{R}$ be the function obtained by projecting bundles in $\mathbb{R}^L_+$ on $X(\bar{\omega})$ before applying the polynomial Q_i^n, i.e., $\tilde{Q}_i^n(x) := Q_i^n((\min\{x^l, \bar{\omega}^l\})_{l\in L})$, for each $x \in \mathbb{R}^L_+$. These functions are continuous and non-decreasing on the whole domain, by construction. They coincide with the underlying polynomials on $X(\bar{\omega})$, and hence are smooth on the interior of that domain. Yet, they do not belong to $\mathcal{V}(\bar{\omega})$, because they do not satisfy the Inada condition (2). For each $n \in \mathbb{N}$, let then $v_i^n : \mathbb{R}^L_+ \to \mathbb{R}$ be the function defined as follows:

$$v_i^n(x) = \left(1 - \frac{1}{n}\right)\left(\tilde{Q}_i^n(x) - \tilde{Q}_i^n(0)\right) + \frac{1}{n}\left(e^{\sum_{l=1}^L \sqrt{x_l}} - 1\right),$$

for each $x \in \mathbb{R}^L_+$. It is now easy to check that $v_i^n \in \mathcal{V}(\bar{\omega})$. Observe also that

$$\max_{x\in X(\bar{\omega})} \left|u_i(x) - v_i^n(x)\right| \leqslant \left(1 - \frac{1}{n}\right)\left|\tilde{Q}_i^n(0)\right| + \max_{x\in X(\bar{\omega})}\left|u_i(x) - \tilde{Q}_i^n(x)\right| + \frac{1}{n}\max_{x\in X(\bar{\omega})}\left|\tilde{Q}_i^n(x) - e^{\sum_{l=1}^L \sqrt{x_l}} + 1\right|.$$

Each of the three terms on the right-hand side converges to 0 when n tends to infinity. Indeed, $\lim_{n\to\infty} \tilde{Q}_i^n(0) = u_i(0) = 0$, and the sequence $(\tilde{Q}_i^n)_n$ is uniformly bounded on $X(\bar{\omega})$, since it is uniformly convergent. This proves that $(v_i^n)_{n\in\mathbb{N}}$ is uniformly convergent to u on $X(\bar{\omega})$, but not necessarily on the whole domain. Hence we propose one last transformation of the sequence. For each $n \in \mathbb{N}$, let

$$\gamma(n) = \max_{x\in X(\bar{\omega})} \bigl|u_i(x) - v_i^n(x)\bigr|,$$

and let u_i^n be the utility function defined as follows:

$$u_i^n(x) = \begin{cases} v_i^n(x) & \text{if } |u_i(x) - v_i^n(x)| \leqslant \gamma_i(n), \\ u_i(x) - \gamma_i(n) & \text{if } u_i(x) - v_i^n(x) > \gamma_i(n), \\ u_i(x) + \gamma_i(n) & \text{if } v_i^n(x) - u_i(x) > \gamma_i(n) \end{cases}$$

for each $x \in \mathbb{R}_+^L$. It is easy to check that $u_i^n \in \mathcal{V}(\bar{\omega})$, for each $n \in \mathbb{N}$, and that the sequence converges uniformly to u_i on $\mathbb{R}_+^L$, as desired. $\square$

Step 4. $\sigma^{\mathcal{R}}(\bar{\omega}, u) = \sigma^{ESS}(\bar{\omega}, u)$.

Proof. Suppose first that $\bar{\omega} \in \mathbb{R}_{++}^L$. For each $i \in \mathcal{I}$, construct a sequence $(u_i^n)_{n\in\mathbb{N}}$ of functions in $\mathcal{V}(\bar{\omega})$ that converges uniformly to u_i, as in Step 3, and let $u^n = (u_1^n, \dots, u_I^n)$. We have:

$$\sigma_i^{\mathcal{R}}(\bar{\omega}, u) = \lim_{n\to\infty} \sigma_i^{\mathcal{R}}\bigl(\bar{\omega}, u^n\bigr) = \lim_{n\to\infty} \frac{s(\bar{\omega}, u^n)}{I} = \frac{s(\bar{\omega}, u)}{I},$$

for each $i \in \mathcal{I}$, where the first equality follows from part (b) of CONT, the second equality follows from Step 2, and the third equality follows from the fact that $\mathcal{R}^{ESS}$ satisfies part (b) of CONT.

Suppose finally that $\bar{\omega} \in \mathbb{R}_+^L$. We can construct a sequence $(\omega^n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{++}^L$ that converges to $\bar{\omega}$. We have:

$$\sigma_i^{\mathcal{R}}(\bar{\omega}, u) = \lim_{n\to\infty} \sigma_i^{\mathcal{R}}\bigl(\omega^n, u\bigr) = \lim_{n\to\infty} \frac{s(\omega^n, u)}{I} = \frac{s(\bar{\omega}, u)}{I},$$

for each $i \in \mathcal{I}$, where the first equality follows from part (a) of CONT, the second equality follows from the previous paragraph, and the third equality follows from the fact that $\mathcal{R}^{ESS}$ satisfies part (a) of CONT. $\square$

Proof of Remark 2. Only Steps 1 and 3 need to be amended for the proof of our Theorem to be applicable to the restricted domain $\mathcal{P}_{\mathcal{C}}$. The class of utility functions used in Step 1 needs to be chosen so that (i) it is rich enough to guarantee that any element of $\mathcal{P}_{\mathcal{C}}$ can be approximated by functions in that class (as in Step 3), and (ii) the modified functions $u_j^{i,\epsilon}$ appearing in Step 1 belong to $\mathcal{P}_{\mathcal{C}}$. This can be done by considering, instead of $\mathcal{V}(\bar{\omega})$, the following subset of $\mathcal{P}_{\mathcal{C}}$:

$$\mathcal{D} = \bigl\{u \in \mathcal{C}^0 \cap \mathcal{D}\bigl(\mathbb{R}_{++}^L\bigr) \bigm| u \text{ satisfies (2)}\bigr\},$$

where $\mathcal{C}^0$ is the set of *strictly* concave functions in $\mathcal{C}$, and $\mathcal{D}(\mathbb{R}_{++}^L)$ denotes the set of differentiable functions on the interior of $\mathbb{R}_+^L$. Note that if $u \in (\mathcal{C}^0)^I$ and $(x, t), (x', t') \in \mathcal{R}(\omega, u)$, since $\mathcal{R}$ satisfies EFF, we must have $x = x'$. Thus, the allocations $x^i(\varepsilon)$ defined in Step 1 of the main proof do not depend on ε. Observe also that for $u \in \mathcal{D}^I$, $\alpha_j^i(\varepsilon) = \varepsilon \frac{\partial}{\partial x_l} u_j(x_j^i(\varepsilon)) - u_j^{x^i(\varepsilon)}(\varepsilon \mathbf{1}_l) \geqslant 0$.

These two observations imply that $\lim_{\varepsilon\to 0^+}\frac{\alpha^i_j(\varepsilon)}{\varepsilon}=0$ (no need to resort to Taylor's theorem anymore, and hence no need to require utility functions to be twice continuously differentiable). Moreover, the definition of utility functions $u^{i,\varepsilon}_j$ reduces to

$$u^{i,\varepsilon}_j(y)=\min\bigl\{\nabla u_j\bigl(x^i_j(\varepsilon)\bigr)\cdot y, u^{x^i(\varepsilon)}_j(y)+\alpha^i_j(\varepsilon)\bigr\}$$

for $u_j\in\mathcal{D}$ and thus $u^{i,\varepsilon}_j\in\mathcal{C}$, since the minimum between two concave functions is concave.

The arguments proving Step 1 can then be reproduced after showing that $s(\cdot,u)$ is differentiable, for each $u\in\mathcal{D}^I$, so as to guarantee that the envelope theorem applies.[19] Observe that $s(\cdot,u)$ is concave, since utility functions are concave. Differentiability will be established via the following lemma:

Lemma 4. *(See Benveniste and Scheinkman [1].) Let V be a real-valued concave function defined on a convex set $D\subseteq\mathbb{R}^n$. If W is a concave differentiable function in a neighborhood N of x_0 in D with the property that $W(x_0)=V(x_0)$ and $W(x)\leqslant V(x)$ for each $x\in N$, then V is differentiable at x_0.*

Fix some $\bar\omega\in\mathbb{R}^L_{++}$ and let $x(\bar\omega)\in\mathbb{R}^{LI}_{++}$ be such that $s(\bar\omega,u)=\sum_{i=1}^I u_i(x_i(\bar\omega))$. Let N be an open neighborhood of $\bar\omega$ such that for each $\omega\in N$, $x_i(\bar\omega)+\frac{\omega-\bar\omega}{I}\in\mathbb{R}^L_{++}$, for every $i\in\mathcal{I}$. Define the function $W:N\to\mathbb{R}$ by

$$W(\omega)=\sum_{i=1}^I u_i\left(x_i(\bar\omega)+\frac{\omega-\bar\omega}{I}\right).$$

Then by definition $W(\bar\omega)=s(\bar\omega,u)$. Moreover, W is concave and differentiable on N, and $W(\omega)\leqslant s(\omega,u)$ for every $\omega\in N$. Thus, according to Lemma 4, $s(\cdot,u)$ is differentiable at $\bar\omega$.

Finally, Step 3 has to be modified as follows.

Step 3′. *For every $u\in\mathcal{C}$ there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of functions in $\mathcal{D}$ that converges uniformly to u.*

Proof. Let $u\in\mathcal{C}$ and define its monotone conjugate $u^-:\mathbb{R}^L_+\to\mathbb{R}\cup\{-\infty\}$ as $u^-(y)=\inf_{x\in\mathbb{R}^L_+}\{xy-u(x)\}$. Then u^- is non-decreasing, upper semi-continuous and concave (see [34, Theorem 12.4, p. 111]).

For every $n\in\mathbb{N}$, define the function $v_n:\mathbb{R}^L_+\to\mathbb{R}\cup\{-\infty\}$ by

$$v_n(y)=u^-(y)-\frac{1}{n}\bigl(e^{-\sum_{l=1}^L\sqrt{y_l}}-1\bigr).$$

Each function v_n is increasing, upper semi-continuous and *strictly* concave. Moreover, for every $y\in\mathbb{R}^L_+$,

$$v_n(y)-\frac{1}{n}\leqslant u^-(y)\leqslant v_n(y).\tag{7}$$

[19] Differentiability of $s(\cdot,u)$ was obtained as a consequence of the implicit function theorem in Step 1 of the main proof, under the assumption that utility functions were twice continuously differentiable. This need not be the case here since utility functions belong to $\mathcal{D}$.

Passing to monotone conjugates in (7) and using that $(u^-)^- = u$ we obtain

$$v_n^-(x) + \frac{1}{n} \geqslant u(x) \geqslant v_n^-(x), \tag{8}$$

for every $x \in \mathbb{R}_+^L$. Thus $v_n^-(x)$ is finite for every $x \in \mathbb{R}_+^L$ and $\lim_{n\to\infty} v_n^-(0) \to 0$. Moreover, since v_n is strictly concave, v_n^- is differentiable (see [34, Theorem 26.3, p. 253]).

Finally, for each $n \in \mathbb{N}$, define the function $u_n : \mathbb{R}_+^L \to \mathbb{R}$ by

$$u_n(x) = \big(v_n^-(x) - v_n^-(0)\big) - \frac{1}{n}\big(e^{-\sum_{l=1}^L \sqrt{x_l}} - 1\big).$$

Then $u_n \in \mathcal{D}$ and $\|u_n - u\| \leqslant \|v_n^- - u\| + \|v_n^-(0)\| + \frac{1}{n}$. The three terms on the right-hand side converge to 0 when n tends to infinity and thus u_n is uniformly convergent to u. □

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