On the Redundancy of the Implicit Welfarist Axiom in Bargaining Theory*

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Abstract

It has long been argued that there is a mismatch between the general motivation provided for Nash's (1950) axioms and their actual mathematical content because they are phrased in the space of joint (Bernoulli) utilities. Alternatively, it is easy to rephrase these axioms in an economic environment so as to match their intuitive meaning, but Nash's proof then applies only if one adds a cardinal welfarist axiom requiring that the solution of two problems that happen to have the same image in the space of joint utilities for some linear representation of von Neumann/Morgenstern preferences, must coincide in that space. Attemps so far at recovering Nash's uniqueness result without the cardinal welfarist axiom have not been successful, in that they all rely on the introduction of a non-straightforward axiom. The purpose of this paper is to show that the straightforward formulation of the arguments underlying Nash's axioms does in fact characterize his solution on a natural economic domain. A similar result holds for Kalai and Smorodinsky's (1975) characterization of their solution if and only if the domain contains multiple goods. The non-welfarist characterization of the Nash solution is shown to extend to a larger class of preferences that accommodate some forms of non-expected utility.

Keywords: Bargaining, Welfarism, Nash, Kalai-Smorodinsky, Non-Expected Utility

1. INTRODUCTION

Axioms in bargaining theory can be interpreted as systematic formalizations of reasonable arguments that one could plausibly hear in real-life situations.[1] Suppose that a couple just reached a compromise regarding where to spend their next vacation. Right before booking the ski trip they agreed upon, they learn that the resort in the Bahamas they considered earlier as a possible alternative, is in fact fully booked. Oftentimes this additional information will not change the couple's plans. Indeed, they decided not to go to the Bahamas anyway, and the ski

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[1]Axioms are sometimes interpreted differently. For instance, they could capture testable implications of a specific bargaining model one has in mind. I will not follow these possible alternative interpretations here.

trip they agreed upon remains feasible. The systematic formalization of this kind of argument leads to the classical property of independence of irrelevant alternatives (IIA). Other classical axioms, such as efficiency and anonymity when bargainers have equal bargaining abilities, can be justified on similar grounds. The best results in axiomatic bargaining theory then reach surprising conclusions, proving mathematically that there is a unique way of resolving the bargainers' conflictual preferences in a way that is consistent with the systematic application of a few basic principles.

Nash's (1950) result is often considered as the most famous example of such achievement. Using linear representations of von Neumann-Morgenstern preferences over lotteries, he showed that maximizing the product of the bargainers' Bernoulli utility functions is the only solution that is compatible with properties of IIA, symmetry, efficiency, and scale covariance. Did he really? The theorem of course is correct, but the way Nash phrases his axioms involves more than a simple systematization of reasonable arguments as discussed in the previous paragraph. Indeed his model is phrased in the space of joint utilities. For instance, the fact that the set of feasible utility profiles is smaller in a first problem compared to a second does not imply that the two problems are related through a reduction of the set of feasible agreements with fixed von Neumann/Morgenstern preferences, as should be in order for IIA to be consistent with the underlying argument it aims to capture. The reader is referred to Roemer (1986-88) or Rubinstein et al. (1992) for more thorough discussions of the shortcomings associated with phrasing axioms in the space of joint utilities. The question is then the following. Is there a way to recover Nash's surprising uniqueness result with the natural formulation of the informal arguments that usually motivate his axioms in an environment that involves explicit economic or social outcomes and von Neumann/Morgenstern preferences? Attempts so far tend to indicate that this is impossible, in the sense that an additional non-straightforward axiom is needed (see the discussion of the related literature below), thereby raising doubts as to whether the central role played by the Nash solution in bargaining theory is truly justified. The primary purpose of the present paper is to provide a natural economic framework and a new argument to answer the question positively.

The set of bargaining problems over which axioms apply is an intrinsic part of any characterization result. If, for instance, one were to consider a single problem in isolation, then axioms that relate the solution of different problems, such as IIA, would be vacuous. A bargaining problem for most of this paper will be any (compact) set of pairs of bundles (one bundle for each of the two bargainers), and a pair of selfish von Neumann-Morgenstern preferences over lotteries that select feasible bundles. Disagreement is assumed to result in the null bundle for both bargainers. This domain seems both natural in view of the existing literature in other subfields of economic theory (combining economic outcomes as modeled in general equilibrium with von Neumann/Morgenstern preferences that play a central role in non-cooperative games), and in line with Nash's (1950) own account of what underlies bargaining problems (cf. his description of an 'anticipation' and the example he presents at the end of his paper).

The natural meaning of Nash's axioms can easily be captured by properties phrased in our explicit economic environment, but his arguments would then require the addition of a cardinal[2] 'welfarist' axiom restricting the solution by imposing that two problems that happen to have the same image in the space of joint utilities for some linear representation of the bargainers' von Neumann/Morgenstern preferences, must have the same solution in that space. Unfortunately, this additional mathematical property hardly qualifies as an axiom because it is unclear what reasonable argument it systematizes. In fact it is even difficult to understand what it entails. Theorem 1 below establishes that natural analogues of Nash's original axioms do in fact characterize his solution in our economic environment with lotteries. In other words, the cardinal welfarist axiom turns out to be *redundant* on that domain.

This result comes as a surprise in perspective of the related literature where Nash's uniqueness result seems to be derivable only at the cost of an additional non-straightforward axiom that replaces or is a remnant of the cardinal welfarist axiom. Roemer (1988) reconstructs bargaining theory on economic environments without lotteries by showing that a property of "consistency with respect to additional dimensions" (CONRAD) is equivalent to his notion of welfarism. Valenciano and Zarzuelo (1997) elaborate on a feature of Rubinstein et al.'s (1997) symmetry axiom to characterize the cardinal welfarist axiom in terms of a property of invariance with respect to isomorphic transformations (see their ISO axiom, their Theorem 4, and the discussion in their Section 6). The use of this or Roemer's property constitute an improvement over the cardinal welfarist axiom in that they are phrased in terms of underlying preferences with no mention to specific utility representations. Unfortunately, it is not clear whether they are any easier to interpret or any more appealing as axioms. Roemer goes as far as to conclude that his reconstruction of bargaining theory via CONRAD demonstrates "*the lengths to which one must go to preserve the axiomatic characterization of the standard bargaining mechanisms on economic environments*" (Roemer (1988), page 30). Yet the complexity of one reconstruction does not necessarily imply that there are no more straightforward alternative routes, as my result shows. A superficial reading of Rubinstein et al. (1992) would make one think that their result already achieved the same objective as mine. After all, they prove in Proposition 2* that the Nash solution is the only one which satisfies axioms of Symmetry, EFF and IIA on another reasonable economic domain. Yet looking into what the axioms require beyond their labels, one realizes that their symmetry axiom does not follow from an idea of anonymity or from an assumption of equal bargaining abilities as usually understood.[3] It entails significantly more

[2]Welfarism broadly means that utility possibility sets should provide enough information to solve problems. Note though that there are as many notions of welfarism as there are interpretations of utilities. For instance, even though von Neumann-Morgenstern preferences can be represented by linear utility functions, there is no reason a priori to focus on such representations when formulating the welfarist axiom. Expected utility theory is indeed an ordinal theory of preference over lotteries. The Nash solution satisfy the welfarist property with respect to cardinal representations of von Neumann/Morgenstern preferences, but not with respect to the larger class of ordinal representations (see de Clippel (2009, Section 5) for more details). Hence the terminology of cardinal welfarism.

[3]Notice also that their axiom of IIA does not match its usual interpretation in terms of changes in the set of feasible agreements, but instead involves changes in preferences. This dimension has been discussed in subsequent results by Hanany (2007) and Hanany and Gal (2007).

than that.[4] As a first example, consider a problem where two bargainers can share \$1. The first bargainer is risk-averse, and evaluates lotteries with possible outcomes according to the expected utility criterion with a Bernoulli function $u_1(o_1) = \sqrt{o_1}$, for each $o_1 \in [0,1]$. The second bargainer, on the other hand, is risk-loving, and evaluates lotteries with possible outcomes according to the expected utility criterion with a Bernoulli function $u_2(o_2) = 1 - \sqrt{1-o_2}$, for each $o_2 \in [0,1]$. Rubinstein et al.'s symmetry axiom (in combination with EFF) prescribes to divide the dollar by giving a fourth to the first bargainer and three fourth to the second. Why? Because it is a fixed-point of the map $\phi : O \to O$, where $O = \{o \in [0,1]^2 | o_1 + o_2 \leq 1\}$ and $\phi(o) = (2 - o_2 - 2\sqrt{1-o_2}, 2\sqrt{o_1} - o_1)$, for each $o \in O$, which happens to have the properties that 1) $\phi(\phi(o)) = o$, for each $o \in O$, and 2) $\mu \succeq_i \nu$ is equivalent to $\phi(\mu) \succeq_j \phi(\nu)$, for both $i = 1,2$ and $j \neq i$, and for each pair μ, ν of lotteries defined on O, with $\phi(\mu)$ and $\phi(\nu)$ being the lotteries that deliver $\phi(o)$ with probability $\mu(o)$ and $\nu(o)$ respectively. Does this encompass a reasonable restriction? Unfortunately Rubinstein at al.'s informal justification for their symmetry axiom (see page 1175) isn't helpful, since they merely observe that the image of problems for which similar isomorphisms exist, appear to be symmetric in the space of joint utilities, for some linear representations of the underlying von Neumann-Morgenstern preferences, and then that their axiom amounts to require the solution to fall on the 45^0-line in that space. In other words, their justification – "*the above formulation of SYM is essentially the same as Nash's original symmetry axiom*" – is entirely welfarist.[5] As such, their result only establishes two facts as far as the discussion on welfarism is concerned:[6] 1) cardinal welfarism need not be imposed on all problems, but only on those whose image is symmetric in the space of joint utilities for some linear representations of the preferences, and 2) cardinal welfarism on those problems can be phrased entirely in terms of the underlying preferences with no reference to specific representations.[7] Of course, the absence of an explanation as to why Rubinstein et al.'s axiom captures an argument that is reasonable in bargaining, does not mean that there isn't one. Perhaps it is just a mathematical difficulty of truly understanding what an isomorphism means. There are reasons to believe otherwise. Consider an elementary problem where the

[4]Though not in relation to the discussion on welfarism, variants of Rubinstein et al.'s (1992) symmetry axioms have been proposed by Grant and Kajii (1995) and Hanany and Gal (2007). All the examples in the discussion to follow to their notions as well.

[5]Rubinstein et al. (1992) define their axioms on a larger class of preferences that allows for some forms of non-expected utility (also discussed later on in the present paper). Although the Nash solution satisfies their symmetry axiom on that larger class, they restrict it to apply only over von Neumann/Morgenstern preferences. A plausible explanation is that their informal justification for the axiom does not carry over to non-expected utility preferences, as they don't admit linear representations. This is another hint that they don't have an informal justification for their symmetry axiom that is independent of Nash's original cardinal welfarist formulation. It turns out that cardinal welfarism *can* in fact be defined on that larger class of preferences, because they admit representations that are linear on simple lotteries that involve only one outcome in addition to the disagreement point, as highlighted by Grant and Kajii's (1995) reformulation of Rubinstein et al. (1992). Grant and Kajii introduce a variant of Rubinstein et al.'s symmetry axiom, which is required on the whole domain, as the cardinal welfarist justification now goes through beyond von Neumann/Morgenstern preferences.

[6]Rubinstein et al. contains another contribution, which is to extend the Nash solution and its axiomatic characterization to larger domains of preference with possibly non-expected utility. A similar extension is also feasible on my economic domain - see Section 5.

[7]Valenciano and Zarzuelo's (1997) characterization of welfarism discussed earlier in this paragraph can be seen as an elaboration on that second point.

only feasible agreements are lotteries over three basic outcomes: 1) the null bundle for both bargainers, 2) a bundle x for the first bargainer and 0 for the second, and 3) a bundle y for the second bargainer and 0 for the first. Rubinstein et al. symmetry axiom (together with EFF) imposes that the solution must be the lottery that chooses $(x, 0)$ and $(0, y)$ with probability 1/2 each, independently of x and y.[8] Let's call this particular case of Rubinstein et al.'s symmetry axiom a property of "equal probability in elementary problems" (EPEP). Is it reasonable to require this as part of an axiom? Surprisingly Rubinstein et al. themselves argue earlier in their paper that it isn't, when discussing the axiom of invariance to positive affine transformations on domains comprising problems with different sets of alternatives (cf. page 1174, where $x = \$1$ and $y = \$1000$). They did not realize that their symmetry axiom has in fact the exact same immediate implication. There are more reasons to believe that EPEP does not qualify as a systematic formalization of reasonable arguments. Indeed, elementary bargaining problems have already been discussed independently by Roth (1979, pages 67-70) and Roemer (1996, Section 2.5). Roth reports experimental findings showing that subjects do not generally agree on the half-half lottery in experiments where the two prizes x and y are different. Roemer provides introspective/normative arguments against the half-half outcome. My contribution is to bring to light the fact that EPEP and more generally cardinal welfarism appear as unavoidable consequences of the straightforward reformulation of Nash's axioms in a natural economic environment with lotteries. Hence it is incorrect to understand arguments against EPEP or welfarism in general as new reasons to reject Nash's axiomatic system as justification for his solution. Alternatively, if one finds this set of axioms as a meaningful systematic application of arguments one may hear in real-life bargaining situations, then one must logically accept EPEP and welfarism. Violations of EPEP would then be a mistake, much in the same way that Savage considered a mistake his own violation of the independence axiom when subject to the Allais paradox. It remains an open question to know whether subjects might change their behavior in experimental studies as those reported by Roth (1979), if they were to fully understand the logical arguments developed in the present paper.

IIA is the axiom that has most often been criticized in Nash's model. Although it is undeniable that arguments along the lines of IIA are heard in real-life bargaining, it is not clear that they are systematically followed. Suppose for instance that the set of feasible outcomes O' is obtained from a larger set O by removing exclusively alternatives that are very favorable to the first bargainer. In such cases, the second bargainer may have a valid argument against IIA, because the reduction from O to O' seems to place the first bargainer in a weaker position. The main alternative cardinal welfarist solution that emerged from this criticism was proposed by Kalai and Smorodinsky (1975). They propose to replace IIA by a property of monotonicity

[8] In Rubinstein et al.'s notations, apply their symmetry axiom to the bargaining problem with $X = \Delta(O)$, where $O = \{(0, 0), (x, 0), (0, y)\}$. Any lottery in $\Delta(X)$ can be reduced into a lottery on $\Delta(O)$, in which case there is a unique way to rank marginals when preferences satisfy first-order stochastic dominance: more weight on the non-zero bundle is better. The symmetry function from X to X associates to any lottery $\mu \in \Delta(O)$ the lottery that picks $(x, 0)$ with probability $\mu(0, y)$, and $(0, y)$ with probability $\mu(x, 0)$. The efficient fixed-point of that "symmetry function" is the lottery in X that picks $(x, 0)$ and $(0, y)$ with probability 1/2 each.

that applies only when the bargainers' utopia points remain unchanged.[9] In Section 4, I investigate whether the straightforward economic reformulation of their axioms does characterize their solution. I show that it does if and only if the domain is rich enough in the sense that bargaining may involve two goods or more. I show in Section 5 that my non-welfarist characterization of the Nash solution extends to a larger class of preferences that allow for some forms of non-expected utility, that is similar to the class introduced by Grant and Kajii (1995). Again, the main innovative feature compared to their, or Rubinstein et al.'s (1992), result is the use of a symmetry axiom that matches its intuitive meaning. The related literature, more specifically Roemer (1988), is further discussed in Section 6.

2. BASIC MODEL

Let L be the set of goods. A *bargaining problem* will first be characterized by a compact subset[10] O of $\mathbb{R}^L_+ \times \mathbb{R}^L_+$ describing feasible outcomes. An *agreement* is a simple[11] lottery defined on O. Bargainers are assumed to be selfish, and hence the relevant lottery for i, when $\mu \in \Delta(O)$ is agreed upon, is its marginal μ_i, where a bundle $x \in \mathbb{R}^L_+$ occurs with probability $\sum_{o\in O|o_i=x} \mu(o)$. Let also O_i denote the projection of O on the i-component, i.e. the set of bundles $x \in \mathbb{R}^L_+$ for which there exists $o \in O$ such that $o_i = x$. The second component characterizing a bargaining problem is i's preference relation $\succ_i$ defined on $\Delta(O_i)$. The bargainers' preferences are assumed to be strictly increasing (i.e. $o \succ_i o'$ whenever $o \geq o'$ and $o \neq o'$), and of the *von Neumann-Morgenstern* (vN-M) type, meaning that they are complete, transitive, continuous, and satisfy the usual independence axiom (see e.g. Fishburn (1970, chapter 8) and references therein). The bargainers receive no good if they fail to reach an agreement, and I assume throughout the paper that they can agree to implement the disagreement outcome, i.e. $(0,0) \in O$. I also assume that there exists $\mu \in \Delta(O)$ such that $\mu_i \succ_i 0$, for both $i = 1,2$. Otherwise, the problem is trivial to solve by applying an argument of efficiency. A *solution* Σ associates to each bargaining problem $(O, \succeq_1, \succeq_2)$ a nonempty subset of $\Delta(O)$.

3. NASH

Nash's axioms can easily be rephrased in this economic context so as to match their usual intuitive interpretation, contrary to their classical formulation in the space of joint Bernoulli utilities. The following axioms are assumed to hold for each bargaining problem $(O, \succeq_1, \succeq_2)$.

Pareto Indifference (PI) *If μ and ν both belong to $\Sigma(O, \succeq_1, \succeq_2)$, then $\mu_1 \sim_1 \nu_1$ and $\mu_2 \sim_2 \nu_2$.*

Exhaustivity (EX) *Let $\nu \in \Delta(O)$ and $\mu \in \Sigma(O, \succeq_1, \succeq_2)$. If $\mu_1 \sim_1 \nu_1$ and $\mu_2 \sim_2 \nu_2$, then $\nu \in \Sigma(O, \succeq_1, \succeq_2)$.*

[9] Kalai and Smorodinsky's property of monotonicity implies a weak form of IIA that applies only when the bargainers' utopia points remain unchanged, thereby addressing to some extent the criticism formulated against IIA.

[10] O can be finite or infinite. All the results of the paper remain true if one restricts the domain to bargaining problems with a finite set O of outcomes.

[11] i.e. with finite support.

Efficiency (EFF) *If* $\mu \in \Sigma(O, \succeq_1, \succeq_2)$*, then there does not exist* $\nu \in \Delta(O)$ *such that* $\nu_i \succeq_i \mu_i$ *for both* $i \in \{1, 2\}$*, and* $\nu_i \succ_i \mu_i$ *for some* $i \in \{1, 2\}$*.*

Symmetry (SYM) *Let* $O^* = \{(x, y) \in \mathbb{R}^L_+ \times \mathbb{R}^L_+ | (y, x) \in O\}$*. For each* $\mu \in \Delta(O)$*, let* $\mu^* \in \Delta(O^*)$ *be the lottery defined as follows:* $\mu^*(x, y) = \mu(y, x)$*, for each* $(x, y) \in O^*$*. If* $O = O^*$ *and* $\succeq_1 = \succeq_2$*, then* $\mu \in \Sigma(O, \succeq_1, \succeq_2)$ *if and only if* $\mu^* \in \Sigma(O, \succeq_1, \succeq_2)$*.*

Independence of Irrelevant Alternatives (IIA) *If* $\mu \in \Sigma(O, \succeq_1, \succeq_2)$*,* $O' \subseteq O$*,* $\mu \in \Delta(O')$*, and* $\succeq'_i$ *coincides with* $\succeq_i$ *on* $\Delta(O')$*, for both* $i = 1, 2$*, then* $\mu \in \Sigma(O', \succeq'_1, \succeq'_2)$*.*

PI requires that the solution provides a unique answer to each bargaining problem, meaning formally that the solution is essentially single-valued. There can be multiple lotteries in the solution of a problem only if both bargainers are indifferent between all of them. Nash imposed this type of restriction in the very definition of a solution, by focusing on functions instead of correspondences. I prefer to formulate it as an explicit axiom, drawing attention to the fact that unicity in the space of utilities does not necessarily imply unicity in the underlying economic environment. EX is the dual property: if both bargainers are indifferent between a contract in the solution and an alternative feasible contract, then this alternative contract should also belong to the solution.[12] EFF guarantees that the solution makes the most out of the feasibility constraints faced by the two bargainers: there is no alternative contract that would make both of them at least as well off, and at least one of them strictly better off. By Symmetry, if a problem is symmetric, in the sense that the set of feasible outcomes is symmetric and the two bargainers' preferences coincides on the marginal distributions, then the symmetric image of any lottery in the solution should also be a solution. Anonymity, requiring that the solution does not depend on the identity of the bargainers, is perhaps more intuitive: $\Sigma(O^*, \succeq_2, \succeq_1) = \{\mu^* | \mu \in \Sigma(O, \succeq_1, \succeq_2)\}$, for each bargaining problem $(O, \succeq_1, \succeq_2)$. Anonymity sounds intuitively appealing if both bargainers have equal bargaining abilities, and immediately implies SYM. As for IIA, suppose that the bargainers recognize that the lottery $\mu \in \Delta(O)$ is a reasonable agreement for the problem $(O, \succeq_1, \succeq_2)$. Suppose now they learn that less alternatives are available, in that they must agree on a lottery over $O' \subseteq O$, but that μ is still feasible, i.e. $\mu \in \Delta(O')$. It is then assumed that the bargainers will recognize that μ' is a reasonable agreement for the problem $(O', \succeq'_1, \succeq'_2)$ as well, where $(\succeq'_1, \succeq'_2)$ is the restriction of $(\succeq_1, \succeq_2)$ to $\Delta(O')$.

Theorem 1 *There exists a unique solution* Σ *that satisfies PI, EX, EFF, SYM, and IIA. It is*

[12] EX is the only axiom which is not a direct non-welfarist analogue of one of Nash's explicit axioms or assumptions. It captures the commonly accepted idea that once a problem has been solved in Nash's welfarist setting, then any agreement that leads to the utilities in the welfarist solution is a reasonable compromise. Yet it could be termed an "hidden assumption" of Nash's construction, as cardinal welfarism has been called in the past. The difference though is that EX applies for a fixed problem, and what it entails is clear. Note that all the papers discussing welfarism in bargaining so far have assumed EX one way or another without much discussion. Roemer (1988) restricts his analysis to "full correspondences" (see page 6). Rubinstein et al. (1992) rule out the difficulty in their very definition of a bargaining problem by assuming that there is a one-to-one correspondence between outcomes and utility pairs (see their condition (v) on page 1173). Valenciano and Zarzuelo (1997) need to amend their ISO axiom in order to prove their main theorem by applying it to preference-based indifference classes in the outcome space.

computed as follows:

$$\Sigma(O, \succeq_1, \succeq_2) = \arg \max_{\mu \in \Delta(O)} (U_1(\mu) - U_1(0))(U_2(\mu) - U_2(0)),$$

where $U_i : \Delta(\mathbb{R}^L_+) \to \mathbb{R}$ *is any*[13] *linear representation of the vN-M preferences* $\succeq_i$ *(*$i = 1, 2$*).*

The solution derived from the axioms in Theorem 1 is simply the reformulation of the Nash bargaining solution in our economic environment. It will thus be denoted by Σ_N in the remainder of the paper. Observe that the use of linear representations of the preferences follows from the axioms, instead of being assumed in the model and the axioms themselves.[14] The proof of Theorem 1 can be found in the Appendix. The rest of this section is devoted to showing the independence of the axioms. The solution that picks all the Pareto efficient lotteries satisfies all the axioms except PI. It is also possible to construct non-welfarist solutions that satisfy all the axioms except PI. Consider for simplicity the case $L = 1$, and the solution that coincides with the Nash solution in all problems, except on elementary problems, in which case the solution contains the Nash solution plus the lottery that picks x with probablity $\frac{x}{x+y}$ and y with probability $\frac{y}{x+y}$, when $O = \{(0, 0), (x, 0), (0, y)\}$. The modified Nash solution that selects deterministic outcomes in Σ_N whenever possible, i.e. $\tilde{\Sigma}_N(O, \succeq_1, \succeq_2) = \Sigma_N(O, \succeq_1, \succeq_2) \cap O$, if this set is nonempty, and $\tilde{\Sigma}_N(O, \succeq_1, \succeq_2) = \Sigma_N(O, \succeq_1, \succeq_2)$, otherwise, satisfies all the axioms except EX. The solution that systematically picks $(0, 0)$ satisfies all the axioms except EFF. Weighted Nash solutions satisfy all the axioms except SYM. It is also possible to find a non-welfarist solution with this property. Consider for simplicity the case $L = 1$. For each set O of feasible outcomes, let $x^i(O)$ be the maximal feasible payment that i could receive. If $x^1(O)$ is larger or equal to $x^2(O)$, then the solution picks the outcome o such that $o_1 = x^1(O)$ and o_2 is maximal among all the outcomes with that property. Otherwise the solution picks the outcome o such that $o_2 = x^2(O)$ and o_1 is maximal among all the outcomes with that property. The solution satisfies all the axioms except SYM. The Kalai-Smorodinsky solution, studied in the next section, satisfies all the axioms except IIA. Of course, there are also numerous non-welfarist solutions with that property. For instance, for each set O of outcomes, let $\alpha_i(O) = \max_{o \in O} \sum_{l \in L} o^l_i$. Then the Nash solution weighted by $(\alpha_1(O), \alpha_2(O))$ satisfies all the axioms except IIA.

4. KALAI-SMORODINSKY

I start by redefining the Kalai-Smorodinsky solution and the property of conditional monotonicity in my economic framework:

$$\Sigma_{KS}(O, \succeq_1, \succeq_2) = \arg \max_{\mu \in \Delta(O)} \min_{i=1,2} \frac{U_i(\mu) - U_i(0)}{\max_{\nu \in \Delta(O)} U_i(\nu) - U_i(0)}, \tag{1}$$

[13] If two different sets of linear utility functions (U_1, U_2) and (V_1, V_2) represent $(\succeq_1, \succeq_2)$, then there exists $\alpha \in \mathbb{R}^2_+$ and $\beta \in \mathbb{R}^2$ such that $U_i = \alpha_i V_i + \beta_i$, for $i = 1, 2$. Hence $\arg\max_{\mu \in \Delta(O)} (U_1(\mu) - U_1(0))(U_2(\mu) - U_2(0)) = \arg\max_{\mu \in \Delta(O)} (V_1(\mu) - V_1(0))(V_2(\mu) - V_2(0))$, and the solution is thus well-defined.

[14] Obviously, "Scale Invariance" is not part of the axioms in Theorem 1, since this property made sense only in Nash's cardinal welfarist model.

where (U_1, U_2) is any[15] linear representation of the vN-M preferences $(\succeq_1, \succeq_2)$.

Conditional Monotonicity (C-MON) *Let* $(O, \succeq_1, \succeq_2)$ *be a bargaining problem, and* $(O', \succeq_1', \succeq_2')$ *be a larger bargaining problem* O*, in the sense that* $O \subseteq O'$*, and* $\succeq_i'$ *coincides with* $\succeq_i$ *on* $\Delta(O_i)$*, for both* $i = 1, 2$*. If there is no* $\mu' \in \Delta(O')$ *such that either* $\mu_1' \succ_1 \mu_1$*, for all* $\mu \in \Delta(O)$*, or* $\mu_2' \succ_2 \mu_2$*, for all* $\mu \in \Delta(O)$*, then for all* $\mu \in \Sigma(O, \succeq_1, \succeq_2)$ *there exists* $\mu' \in \Sigma(O', \succeq_1, \succeq_2)$ *such that* $\mu_1' \succeq_1 \mu_1$ *and* $\mu_2' \succeq_2 \mu_2$*.*[16]

Theorem 2 Σ_{KS} *is the only solution that satisfies PI, EX, EFF, SYM, and C-MON if and only if* $L \geq 2$*.*

The proof that Σ_{KS} is the only solution that satisfies PI, EX, EFF, SYM, and C-MON when $L \geq 2$ can be found in the Appendix. The next example shows that there exists non-welfarist solutions that satisfy the natural reformulation of Kalai-Smorodinsky's axioms in my economic environment when $L = 1$.

Example 1 *Consider the reformulation of the solutions characterized by Peters and Tijs (1985) in my non-welfarist model. Let* $\phi : [1, 2] \to conv\{(1, 0), (0, 1), (1, 1)\}$ *that is continuous, non-decreasing and such that* $\phi(1) = (1/2, 1/2)$ *and* $\phi(2) = (1, 1)$*. The function* ϕ *thus determines a monotonic curve in the subset* $conv\{(1, 0), (0, 1), (1, 1)\}$ *of the utility space. Let then* Σ^ϕ *be the solution that associates to each bargaining problem* $(O, \succeq_1, \succeq_2)$ *the set of lotteries* λ *that are Pareto optimal and such that* $(U_1^*(\lambda), U_2^*(\lambda)) = \phi(U_1^*(\lambda) + U_2^*(\lambda))$*, where* U_i^* *is the unique linear representation of* $\succeq_i$ *such that* $U_i^*(0) = 0$ *and* $\max_{o \in O} U_i^*(o_i) = 1$*. Suppose that* $L = 1$*, and let* x *and* y *be the two positive real numbers such that* $U_1^*(x) = U_2^*(y) = 1$*. Consider then the monotonic curve in the triangle* $conv\{(1, 0), (0, 1), (1, 1)\}$ *that starts at* $(1/2, 1/2)$*, and follows a direction parallel to the vector* $(\frac{x_1}{x_1+y_1}, \frac{y_1}{x_1+y_1})$ *until it reaches an edge of the triangle, in which case it continues until* $(1, 1)$ *on that edge. Let* ψ *be the functional description of that curve, i.e.* $\psi(t)$ *is the intersection of the curve with the line* $u_1 + u_2 = t$*, for each* $t \in [1, 2]$*. It is not difficult to check that* Σ^ψ *is well-defined, and satisfies the axioms listed in Theorem 2, but is different from* Σ_{KS}*.*

I now discuss the independence of the axioms appearing in Theorem 2. The solution that picks all the Pareto efficient lotteries satisfies all the axioms except PI. The modified Kalai-Smorodinsky solution that selects deterministic outcomes in Σ_{KS} whenever possible, i.e. $\tilde{\Sigma}_{KS}(O, \succeq_1, \succeq_2) = \Sigma_{KS}(O, \succeq_1, \succeq_2) \cap O$, if this set is nonempty, and $\tilde{\Sigma}_{KS}(O, \succeq_1, \succeq_2) = \Sigma_{KS}(O, \succeq_1, \succeq_2)$, otherwise, satisfies all the axioms except EX. The solution that systematically picks $(0, 0)$ satisfies all the axioms except EFF. Weighted Kalai-Smorodinsky solutions satisfy all the axioms except SYM. The Nash solution, studied in the previous section, satisfies all the axioms except IIA. Of course, there are also numerous non-welfarist solutions with that property, including for instance the one introduced when discussing the independence of IIA in Theorem 1.

[15]See Footnote 13.

[16]The Kalai-Smorodinsky solution also satisfies a stronger monotonicity property requiring that $\mu' \succeq_1 \mu$ and $\mu' \succeq_2 \mu$, for all $\mu \in \Sigma(O, \succeq_1, \succeq_2)$ and all $\mu' \in \Sigma(O', \succeq_1, \succeq_2)$, and Theorem 2 is a fortiori true with this stronger version of C-MON. Notice also that this stronger monotonicity property implies PI.

5. NON-EXPECTED UTILITY

I now consider a larger class of bargaining problems to accommodate some forms of non-expected utility, as initiated by Rubinstein et al. (1992), and further extended and clarified by Grant and Kajii (1995). As before, a bargaining problem is a triplet $(O, \succeq_1, \succeq_2)$, where O is a compact subset of $\mathbb{R}^L_+ \times \mathbb{R}^L_+$ describing feasible outcomes, and $\succeq_i$ describes bargainer i's preferences over $\Delta(O_i)$. In order to present the assumptions made on preferences, I introduce some notation. Given a lottery $\mu \in \Delta(O)$, and a number $p \in [0,1]$, $p\mu$ will denote the compound lottery where μ is implemented with probability p and $(0,0)$ prevails with probability $1-p$. Preferences are still assumed to be strictly increasing, in the sense $o_i \succ_i o'_i$ whenever $o_i \geq o'_i$ and $o_i \neq o'_i$. The disagreement point $(0,0)$ is also assumed to be feasible, i.e. $(0,0) \in O$, and there exists $\mu \in \Delta(O)$ such that $\mu_i \succ_i 0$, for both $i = 1, 2$. The new feature is that preferences, though still complete, transitive, and continuous, are not required to be of the von Neumann-Morgenstern type. Instead, I will assume that they satisfy first-order stochastic dominance, and that there exists $u_i : \Delta(O_i) \to \mathbb{R}$ such that $p\lambda \succeq_i q\mu$ if and only if $u_i(\lambda)p \geq u_i(\mu)q$, for each $p, q \in [0,1]$, and each $\lambda, \mu \in \Delta(O_i)$. Some form of convexity will also be required: for both $i \in \{1,2\}$, there exists $o^i \in O_i$ that is a best outcome for i (i.e. $o^i \succeq x$, for each $x \in O_i$), and such that

$$(\forall \mu, \nu \in \Delta(O))(\forall p_1, p_2, q_1, q_2 \in [0,1])(\exists \lambda \in \Delta(O)) :$$

$$\mu_1 \sim_1 p_1 o^1, \mu_2 \sim_2 p_2 o^2, \nu_1 \sim_1 q_1 o^1 \text{ and } \nu_2 \sim_2 q_2 o^2 \Rightarrow \lambda_1 \succeq_1 \frac{p_1 + q_1}{2} o^1 \text{ and } \lambda_2 \succeq_2 \frac{p_2 + q_2}{2} o^2.$$

With these assumptions, for any lottery $\mu \in \Delta(O)$ and either $i \in \{1,2\}$, there exists a unique real number, call it $U_i(\mu_i)$, such that $\mu_i \sim_i U_i(\mu_i) o^i$. Obviously U_i represents bargainer i's preferences, in the sense that $\mu_i \succeq \nu_i$ if and only if $U_i(\mu_i) \geq U_i(\nu_i)$, for each $\mu, \nu \in \Delta(O)$.[17] With our assumptions on preferences, U_i is linear in probabilities on elementary compound lotteries that pick either a lottery in $\Delta(O_i)$ or 0, but not necessarily on all lotteries in $\Delta(O_i)$. Let then

$$U(O, \succeq_1, \succeq_2) = \{v \in \mathbb{R}^2 | \exists \mu \in \Delta(O) : v_1 \leq U_1(\mu_1) \text{ and } v_2 \leq U_2(\mu_2)\}.$$

The convexity assumption imposed on bargaining problems guarantee that $U(O, \succeq_1, \succeq_2)$ is a convex subset of $\mathbb{R}^2_+$.

The notion of a solution, and the axioms presented in Section 2, remain unchanged, except that they are now defined on our larger class of bargaining problems.

Theorem 1' *Let Σ be a bargaining on the larger class of bargaining problems considered in this section. Then Σ satisfies PI, EX, EFF, SYM, and IIA on that larger class of problems if and only if*

$$\Sigma(O, \succeq_1, \succeq_2) = \{\mu \in \Delta(O) | (U_1(\mu_1), U_2(\mu_2)) \in \arg \max_{v \in U(O, \succeq_1, \succeq_2)} v_1 v_2\}. \tag{2}$$

The proof of Theorem 1' is available in the Appendix. Whether Theorem 2 also extends to the larger domain of this section remains an open question.

[17] It is easy to check that U_i does not depend on the choice of the optimal o^i, if multiple such choices exist.

The conditions defining a bargaining problem are direct analogues of those proposed by Grant and Kajii (1995) as a weakening of the original conditions proposed by Rubinstein et al. (1992). A relevant difference, though, is that Grant and Kajii apply these conditions to a (large) set of deterministic outcomes, while I apply them to lotteries over a (possibly small) set of deterministic outcomes. The next example shows that the bargaining problem they considered as an illustration of the richness of their expanded domain also belongs to the domain I consider in this section.

Example 2 *Consider the case $L = 1$, and a bargaining problem with the set of outcomes $O = \{x \in \mathbb{R}^2_+ | x_1 + x_2 \leq 1\}$. For any lottery μ with finite support in $[0,1]$, let $(x_k(\mu))_{k=1}^K$ be the increasing sequence where $x_1(\mu) = 0$ and $\{x_k(\mu) | 2 \leq k \leq K\}$ is the set of non-zero outcomes that come with positive probability under μ. Then define i's utility for μ as follows:*

$$V_i(\mu) = \sum_{k=1}^{K-1} (x_{k+1}(\mu) - x_k(\mu)) (\sum_{j=k+1}^{K} \mu(x_j))^{\alpha_i},$$

for each $i \in \{1,2\}$, where α_i is any given parameter strictly larger than 1. Let then $\succeq_i$ be the preference over defined $\Delta(O_i)$ derived from V_i. I now prove that any such $(O, \succeq_1, \succeq_2)$ qualify as a bargaining problem, as defined in this section. First-order stochastic dominance is obviously verified. Next notice that $V_i(p\mu) = p^{\alpha_i} V_i(\mu)$, for any $p \in [0,1]$ and any $\mu \in \Delta(O_i)$. Hence $p\mu \succeq_i q\nu$ if and only if $p(V_i(\mu))^{1/\alpha_i} \geq q(V_i(\nu))^{1/\alpha_i}$, and i's preference indeed admits a linear representation on elementary compound lotteries that involve 0 and any lottery in $\Delta(O_i)$ (taking $u_i = V_i^{1/\alpha_i}$). Finally, consider $\mu, \nu \in \Delta(O)$ and p_1, p_2, q_1, q_2 such that $\mu_1 \sim_1 p_1 o^1$, $\mu_2 \sim_2 p_2 o^2$, $\nu_1 \sim_1 q_1 o^1$ and $\nu_2 \sim_2 q_2 o^2$. Requiring the parameters α_1, α_2 to be larger than 1 is equivalent to assume that the two bargainers are risk-averse (see Chew et al. (1987, Theorem 1, p. 374)). If (x_1, x_2) and (y_1, y_2) denote the expected values of μ and ν respectively, then $V_i(x_i) \geq V_i(\mu_i)$ and $V_i(x_i) \geq V_i(\nu^i)$, or $x_i \geq p_i^{\alpha_i} o^i$ and $y_i \geq q_i^{\alpha_i} o^i$. Let then λ be the lottery that picks $(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2})$ for sure. We have:

$$V_i(\lambda_i) = \frac{1}{2}(x_i + y_i) \geq \frac{1}{2}(p_i^{\alpha_i} + q_i^{\alpha_i}) o^i \geq (\frac{1}{2}(p_i + q_i))^{\alpha_i} o^i = V_i(\frac{1}{2}(p_i + q_i) o^i),$$

thereby showing that the convexity assumption is satisfied as well, and that $(O, \succeq_1, \succeq_2)$ does indeed qualify as a bargaining problem.

6. FURTHER COMPARISON WITH THE RELATED LITERATURE

This section is devoted to a more detailed comparison of Theorems 1 and 2 with Roemer's (1988) results. A first difference between our two approaches is that I do not aim at characterizing welfarism, but instead observe that natural reformulations of Nash's and Kalai and Smorodinsky's axioms imply it in my model (when $L \geq 2$ in the latter case). My 'reconstruction' is thus more successful, but also less ambitious, since the fact that the cardinal welfarist property is redundant for these two axiomatic systems does not imply that it will be in others (and indeed we already observed, for instance, that it is not redundant in the case of Kalai and Smorodinsky's axioms when $L = 1$). It is also important to recognize the role played by the domain of definition of solutions when formulating axiomatic results. Particularly, the Nash bargaining solution would not be uniquely characterized in Roemer's paper if CONRAD was dropped. Here are the main differences between our two papers regarding the definition of a bargaining problem. First, instead of restricting attention to economic problems that result from all the possible reallocations of some collective endowment to be shared, my domain includes bargaining problems build on any compact set of bundles. A stark consequence of Roemer's assumption is that every solution satisfies IIA in his framework, since an efficient allocation cannot remain feasible if the total endowment to distribute diminishes. Second, bargainers can use lotteries to reach an agreement in my model, and these lotteries are evaluated via von Neumann-Morgenstern ordinal preferences. Roemer, instead, endows the bargainers with a concave utility function defined over a set of deterministic contracts. Thus his theory is still rooted in a notion of utility that is not ordinally invariant (as an increasing transformation of a concave function is not necessarily concave). Third, my reasoning works for any fixed number of goods (including the interesting case of only one good), while Roemer's argument depends crucially on the possibility of adding goods, thereby considering a framework with a variable (possibly infinite) number of goods. Fourth, Roemer include the bargainers' preferences over *unfeasible* outcomes in the description of a bargaining problem. By analogy, this would mean in my framework that $\succeq_i$ is defined over $\Delta(\mathbb{R}^L_+)$ instead of $\Delta(O)$. My reformulation of Nash's axioms do not characterize uniquely his solution in that alternative model. Here is indeed an alternative class of solutions satisfying them, inspired by Pazner and Schmeidler's (1978) concept of egalitarian equivalence.

Example 3 *Let $(O, \succeq_1, \succeq_2)$ be a bargaining problem with $\succeq_1$ and $\succeq_2$ defined over $\Delta(\mathbb{R}^L_+)$ instead of $\Delta(O)$, and let $d \in \mathbb{R}^L_{++}$. For each $\mu \in \Delta(O)$, let $\alpha^d_i(\mu)$ be the unique real number such that i is indifferent between the lottery μ and receiving $\alpha^d_i(\mu)d_l$ units of each good l, for sure. Let $\hat{\alpha}^d(\mu)$ be the vector in $\mathbb{R}^2_+$ obtained by rearranging the components of $\alpha^d(\mu)$ increasingly. The egalitarian equivalent solution Σ^d_{EE} is obtained by maximizing $\hat{\alpha}^d$ according to the lexicographic order. It is not difficult to check that Σ^d_{EE} satisfies PI, EX, AN, EFF, and IIA. Notice that multiplying d by a scalar does not change the solution. There is thus a unique egalitarian equivalent solution when $L = 1$, the vector $\alpha(\mu)$ determining the certainty equivalent of μ for both bargainers. The solution varies with the direction d when $L \geq 2$.*

Restricting preferences to be defined over feasible outcomes is standard in economic theory. The concept of (pure or mixed-strategy) Nash equilibrium, for instance, does not depend on preferences over unfeasible outcomes. The assumption is standard in bargaining theory as well, cf. Rubinstein et al. (1992) and Valenciano and Zarzuelo (1997), for instance. Also, it is satisfied by any solution that is justified by the Nash program, in the sense of coinciding with the subgame-perfect Nash equilibrium outcome of some non-cooperative bargaining procedure defined over feasible agreements. The interested reader is referred to de Clippel (2009) for a more thorough discussion of the property of independence with respect to preferences over unfeasible alternatives.

Taking into account these four main differences, I believe that my framework is closer to Nash's (1950) original construction of a bargaining problem, starting with explicit economic bundles instead of his more abstract notion of 'anticipation.'

I conclude the paper with yet one more illustration, in complement to those given in the Introduction, that Rubinstein et al.'s (1992) symmetry axiom entails more than an argument of anonymity, or of equal outcome when bargainers have equal bargaining ability. Indeed, notice that replacing their symmetry axiom by SYM or an anonymity property does not characterize the Nash solution in most problems in their framework. Consider for instance the case $L = 1$, and consider a set $O = \{o \in \mathbb{R}^2_+ | \lambda_1 o_1 + \lambda_2 o_2 = 1\}$, for some $\lambda \in \mathbb{R}^2_{++}$. If $\lambda_1 \neq \lambda_2$, then $O^* \neq O$, and SYM remains silent. The solution that picks o^* independently of the bargainers' preferences thus satisfies SYM, as well as Rubinstein et al.'s axioms of efficiency and independence over irrelevant alternatives, for any $o^* \in \mathbb{R}^2_+$ such that $\lambda_1 o^*_1 + \lambda_2 o^*_2 = 1$. Their characterization result does not hold even when $\lambda_1 = \lambda_2$, and thus $O = O^*$, if one uses SYM instead of their stronger welfarist-related version of it. Fix $o^* \in \mathbb{R}^2_{++}$ such that $\lambda_1 o^*_1 + \lambda_2 o^*_2 = 1$. The solution that picks o^* for any pair $(\succeq_1, \succeq_2)$ such that $\succeq_1 \neq \succeq_2$ on $\Delta([0, \min\{o_1, o_2\})$, and $(1/2, 1/2)$ otherwise, provides an example.

References

Chew, S. H., E. Karni, and Z. Safra, 1987. Risk Aversion in the Theory of Expected Utility with Rank Dependent Probabilities. *Journal of Economic Theory* **42**, 304-318.

de Clippel, G., 2009. Axiomatic Bargaining on Economic Enviornments with Lotteries. Brown University Economics Department Working Paper 2009-5.

Fishburn, P. C., 1970. Utility Theory for Decision Making. New York: John Wiley and Sons.

Grant, S., and A. Kajii, 1995. A Cardinal Characterization of the Rubinstein-Safra-Thomson Axiomatic Bargaining Theory. *Econometrica* **63**, 1241-1249.

Hanany, E., 2007. Appeals Immune Bargaining Solution with Variable Alternative Sets. *Games and Economic Behavior* **59**, 72-84.

Hanany, E., and R. Gal, 2007. Asymmetric Nash Bargaining with Surprised Players. *The B.E. Journal of Theoretical Economics* **7** (Topics), Article 29.

Husseinov, F., 2010. Monotonic Extension. Bilkent University Economics Discussion Paper 10-4.

Kalai, E., and M. Smorodinsky, 1975. Other Solutions to Nash's Bargaining Problem. *Econometrica* **43**, 513-518.

Nash J., 1950. The bargaining problem. *Econometrica* **18**, 155-162.

Pazner, E., and D. Schmeidler, 1978. Egalitarian Equivalent Allocations: a New Concept of Economic Equity. *Quarterly Journal of Economics* **92**, 671-687.

Peters, H., and S. Tijs, 1985. Characterization of all Individually Monotonic Bargaining Solutions. *International Journal of Game Theory* **14**, 219-228.

Roemer, J. E., 1986. The mismarriage of bargaining theory and distributive justice. *Ethics* **97**, 88-110.

Roemer, J. E., 1988. Axiomatic bargaining theory on economic environments. *Journal of Economic Theory* **45**, 1-31.

Roemer, J. E., 1996. Theories of Distributive Justice. Cambridge: Harvard University Press.

Roth, A. E., 1979. Axiomatic Models of Bargaining. Lecture Notes in Economics and Mathematical Systems #170, Springer Verlag. Also available at
http://kuznets.fas.harvard.edu/~aroth/Axiomatic_Models_of_Bargaining.pdf.

Rubinstein, A., Z. Safra, and W. Thomson, 1992. On the interpretation of the Nash bargaining solution and its extension to non-expected utility preferences. *Econometrica* **60**, 1171-1186.

Valenciano, F., and J. M. Zarzuelo, 1997. On Nash's Hidden Assumption. *Games and Economic Behavior* **21**, 266-281.

APPENDIX

Proof of Theorem 1

The fact that Σ_N satisfies the axioms follows from the usual properties of the Nash bargaining solution defined in the space of joint Bernoulli utilities. I will thus focus on proving uniqueness. Let Σ be a solution that satisfies the axioms, and let $(O, \succeq_1, \succeq_2)$ be a bargaining problem. We have to prove that $\Sigma(O, \succeq_1, \succeq_2) = \Sigma_N(O, \succeq_1, \succeq_2)$. Given that Σ satisfies PI, it is sufficient to show that $\Sigma_N(O, \succeq_1, \succeq_2) \subseteq \Sigma(O, \succeq_1, \succeq_2)$. Let thus $\mu^* \in \Sigma_N(O, \succeq_1, \succeq_2)$. The next few steps show that $\mu^* \in \Sigma(O, \succeq_1, \succeq_2)$, as desired.

Step 1 *Let o^1 be an element of O that is maximal for 1 ($o^1_1 \succeq o'_1$, for each $o' \in O$), let o^2 be an element of O that is maximal for 2 ($o^2_2 \succeq o'_2$, for each $o' \in O$), let (U_1, U_2) be two linear representations of $(\succeq_1, \succeq_2)$ such that $U_1(0) = U_2(0) = 0$ and $U_1(o^1_1) = U_2(o^2_2) = 1$, let $(\sigma^*_1, \sigma^*_2) = (U_1(\mu^*_1), U_2(\mu^*_2))$, and let $U(O, \succeq_1, \succeq_2) = \{(U_1(\mu_1), U_2(\mu_2)) | \mu \in \Delta(O)\}$. Then $U(O, \succeq_1, \succeq_2)$ is included in the triangle with extreme points $(0,0)$, $(2\sigma^*_1, 0)$, and $(0, 2\sigma^*_2)$.*

<u>Proof</u>: Let

$$\mathcal{V} = \{x \in \mathbb{R}^2 | x \geq (0,0) \text{ and } x_1 x_2 \geq \sigma^*_1 \sigma^*_2\}.$$

The sets $U(O, \succeq_1, \succeq_2)$ and $\mathcal{V}$ are both convex, and $U(O, \succeq_1, \succeq_2) \cap \mathcal{V} = \{\sigma^*\}$. The separating hyperplane theorem implies that[18]

$$U(O, \succeq_1, \succeq_2) \subseteq \{x \in \mathbb{R}^2 | \frac{x_1}{\sigma^*_1} + \frac{x_2}{\sigma^*_2} \leq 2\},$$

because the gradient of the function $x_1 x_2$ at σ^* is proportional to (σ^*_2, σ^*_1). $\square$

Step 2 *Notice that $\sigma^*_1 \geq 1/2$, by convexity of $U(O, \succeq_1, \succeq_2)$ (which follows from the linearity of U_1 and U_2). If $\sigma^*_1 = 1/2$, then read the rest of the proof with $x = o^1$. Otherwise, pick a bundle x that is larger and different from o^1, and define $\succeq'_1$ as the following preference ordering on $\Delta(O_1 \cup \{x\})$:*

$$\mu_1 \succeq'_1 \nu_1 \text{ if and only if } U'_1(\mu_1) \geq U'_1(\nu_1)$$

for each $\mu_1, \nu_1 \in \Delta(O_1 \cup \{x\})$, where

$$U'_1(\mu_1) = 2\mu_1(x)\sigma^*_1 + (1 - \mu_1(x))U_1(\mu_1 | \neg x),$$

for each $\mu_1 \in \Delta(O_1 \cup \{x\})$, where $\mu_1 | \neg x$ is the lottery on O_1 derived by Bayesian updating from μ_1 when conditioning on the fact that the outcome is different from x. Similarly, $\sigma^*_2 \geq 1/2$, by convexity of $U(O, \succeq_1, \succeq_2)$. If $\sigma^*_2 = 1/2$, then read the rest of the proof with $y = o^2$. Otherwise,

[18] Notice that both σ^*_1 and σ^*_2 must be strictly positive since there exists $\mu \in \Delta(O)$ such that $\mu_1 \succ_1 0$ and $\mu_2 \succ_2 0$, by definition of a bargaining problem.

pick a bundle y that is larger and different from o^2, and define $\succeq_2'$ as the following preference ordering defined on $\Delta(O_2 \cup \{y\})$:

$$\mu_2 \succeq_2' \nu_2 \textit{ if and only if } U_2'(\mu_2) \geq U_2'(\nu_2),$$

for each $\mu_2, \nu_2 \in \Delta(O_2 \cup \{y\})$, where

$$U_2'(\mu_2) = 2\mu_2(y)\sigma_2^* + (1 - \mu_2(y))U_2(\mu_2|\neg y),$$

for each $\mu_2 \in \Delta(O_2 \cup \{y\})$, where $\mu_2|\neg y$ is the lottery on O_2 derived by Bayesian updating from μ_2 when conditioning on the fact that the outcome is different from y. Let $O' = O \cup \{(x,0),(0,y)\}$. Then $(O', \succeq_1', \succeq_2')$ is a bargaining problem.

Proof: It is straightforward to check that O' inherits from O the property of compactness, that $\succeq_i'$ inherits from $\succeq_i$ the quality of being a von Neumann/Morgenstern preference, and that there exists $\mu' \in \Delta(O')$ such that $\mu_1' \succ_1' 0$ and $\mu_2' \succ_2' 0$, since there exists $\mu \in \Delta(O)$ such that $\mu_1 \succ_1 0$ and $\mu_2 \succ_2 0$. □

Step 3 *Let $\mu \in \Sigma(O', \succeq_1', \succeq_2')$. Then $U_1'(\mu_1) = U_1'(\mu_1')$ and $U_2'(\mu_2) = U_2'(\mu_2')$, for all $\mu' \in \Sigma(\{(0,0),(x,0),(0,y)\}, \cdot, \cdot)$.*[19]

Proof: Following Step 1, and the definitions of U_1' and U_2', notice that

$$\mathcal{U}(O', \succeq_1', \succeq_2') = \{(U_1'(\nu), U_2'(\nu)) | \nu \in \Delta(O')\}$$

coincides with the triangle whose extreme points are $(0,0)$, $(2\sigma_1^*, 0)$, and $(0, 2\sigma_2^*)$. By EFF, there exists $\alpha \in [0,1]$ such that $U_1'(\mu_1) = 2\alpha\sigma_1^*$ and $U_2'(\mu_2) = 2(1-\alpha)\sigma_2^*$. EX implies that the lottery that gives $(x,0)$ with probability α, and $(0,y)$ with probability $1-\alpha$ belongs to $\Sigma(O', \succeq_1', \succeq_2')$. IIA implies that this lottery also belongs to $\Sigma(\{(0,0),(x,0),(0,y)\}, \cdot, \cdot)$. We have thus found one lottery $\mu' \in \Sigma(\{(0,0),(x,0),(0,y)\}, \cdot, \cdot)$ such that $U_1'(\mu_1) = U_1'(\mu_1')$ and $U_2'(\mu_2) = U_2'(\mu_2')$. The result then follows by PI. □

Step 4 $\Sigma(\{(0,0),(x,0),(0,y)\}, \cdot, \cdot) = \{\alpha(x,0) \oplus (1-\alpha)(0,y)\}$, *for some $\alpha \in]0,1[$.*[20]

Proof: By EFF, any lottery in $\Sigma(\{(0,0),(x,0),(0,y)\}, \cdot, \cdot)$ must give $(x,0)$ with some probability α, and $(0,y)$ with probability $1-\alpha$. The difficult part is to show that the solution is strictly individually rational, i.e. α is different from both 0 and 1. Suppose, to the contrary of what we want to prove, that $\Sigma(\{(0,0),(x,0),(0,y)\}, \cdot, \cdot) = \{x\}$ (a similar argument applies

[19] There is only one preference satisfying first-order stochastic dominance, and a fortiori of the von Neumann/Morgenstern type, when defined over 0 and a non-zero bundle. To save on notations, I omit to write the unique pair of preferences when looking at the bargaining problem whose feasible outcomes are $(0,0)$, $(x,0)$, or $(0,y)$.

[20] For each pair o, o' in O, and each $\alpha \in [0,1]$, $\alpha o \oplus (1-\alpha)o'$ stands for the lottery that picks o with probability α and o' with probability $1-\alpha$.

if $\Sigma(\{(0,0),(x,0),(0,y)\}) = \{y\})$. Let's show that this leads to a contradiction by considering four different cases:[21]

Case 1: $y = x$

In that case, PI and SYM imply that $\alpha = 1/2$, and we are done.

Case 2: $y > x$

Let for instance $\bar{U}_2$ be the utility function defined as follows on $\{0, x, y\}$:

$$\bar{U}_2(0) = 0, \bar{U}_2(x) = 1/2, \bar{U}_2(y) = 1,$$

and let $\bar{\succsim}_2$ be the associated von Neumann/Morgenstern preference defined over $\Delta(\{0, x, y\})$. Efficiency implies that any lottery ν in $\Sigma(\{(0,0),(x,0),(0,x),(0,y)\}, \cdot, \bar{\succsim}_2)$ must place positive weights on $(x,0)$ and $(0,y)$ only. By IIA, this lottery will also belong to $\Sigma(\{(0,0),(x,0),(0,y)\}, \cdot, \cdot)$, which is equal to $\{(x,0)\}$. Hence ν must pick $(x,0)$ for sure, but then applying IIA again implies that $\Sigma(\{(0,0),(x,0),(0,x)\}, \cdot, \cdot) = \{(x,0)\}$, which contradicts the combination of PI and SYM.

Case 3: $x > y$

Let $x' << x$, let $y' << y$, let $\bar{U}_1$ be the utility function defined as follows on $\{0, x', x\}$:

$$\bar{U}_1(0) = 0, \bar{U}_1(x') = 1/4, \bar{U}_1(x) = 1,$$

let $\bar{\succsim}_1$ be the associated von Neumann/Morgenstern preference defined over $\Delta(\{0, x', x\})$, let $\bar{U}_2$ be the utility function defined as follows on $\{0, y', y\}$:

$$\bar{U}_2(0) = 0, \bar{U}_2(y') = 3/4, \bar{U}_2(y) = 1,$$

and let $\bar{\succsim}_2$ be the associated von Neumann/Morgenstern preference defined over $\Delta(\{0, y', y\})$. Let ν be a lottery in $\Sigma(\{(0,0),(x,0),(0,y),(x',y')\}, \bar{\succsim}_1, \bar{\succsim}_2)$. Since the lottery $\frac{1}{4}(x,0) \oplus \frac{3}{4}(0,y)$ leads to the same expected utilities (under $(\bar{U}_1, \bar{U}_2)$) for both players than the outcome (x', y'), there must exists a lottery ν' in $\Delta(\{(x,0),(0,y)\})$ such that $\bar{U}_i(\nu') = \bar{U}_i(\nu)$, for both $i \in \{1,2\}$. EX implies that $\nu' \in \Sigma(\{(0,0),(x,0),(0,y),(x',y')\}, \bar{\succsim}_1, \bar{\succsim}_2)$. IIA implies that $\nu' \in \Sigma(\{(0,0),(x,0),(0,y)\}, \cdot, \cdot) = \{(x,0)\}$. Hence $\bar{U}_1(\nu) = \bar{U}_1(\nu') = 1$ and $\bar{U}_2(\nu) = \bar{U}_2(\nu') = 0$, or ν must be the lottery that picks $(x,0)$ for sure. IIA implies that $\Sigma(\{(0,0),(x,0),(x',y')\}, \bar{\succsim}_1, \cdot) = \{(x,0)\}$.

Consider now $\bar{U}'_2$, the utility function defined as follows on $\{0, y', x\}$:

$$\bar{U}'_2(0) = 0, \bar{U}'_2(y') = 3/4, \bar{U}'_2(x) = 1,$$

let $\bar{\succsim}'_2$ be associated von Neumann/Morgenstern preference defined over $\Delta(\{0, y', x\})$. Let ρ be a lottery in $\Sigma(\{(0,0),(x,0),(0,x),(x',y')\}, \bar{\succsim}_1, \bar{\succsim}'_2)$. Since the lottery $\frac{1}{4}(x,0) \oplus \frac{3}{4}(0,x)$ leads to the same expected utilities (under $(\bar{U}_1, \bar{U}'_2)$) for both players than the outcome (x', y'),

[21]It is perhaps a good place to emphasize that utility functions are defined in all the proofs so as to lead to valid preferences, One has to be careful in particular to avoid decreasing functions.

there must exists a lottery ρ' in $\Delta(\{(x,0),(0,x)\})$ such that $\bar{U}_i(\rho') = \bar{U}_i(\rho)$, for both $i \in \{1,2\}$. EX implies that $\rho' \in \Sigma(\{(0,0),(x,0),(0,x),(x',y')\},\bar{\succeq}_1,\bar{\succeq}'_2)$. IIA implies that $\rho' \in \Sigma(\{(0,0),(x,0),(0,x)\},\cdot,\cdot) = \{\frac{1}{2}(x,0)\oplus\frac{1}{2}(0,x)\}$ by PI, SYM and EFF. Notice that $\frac{1}{3}(x,0)\oplus\frac{2}{3}(x',y')$ gives the same expected utility (under $(\bar{U}_1,\bar{U}'_2)$) to both players than this last lottery, and hence $\frac{1}{3}(x,0)\oplus\frac{2}{3}(x',y') \in \Sigma(\{(0,0),(x,0),(0,x),(x',y')\},\bar{\succeq}_1,\bar{\succeq}'_2)$, by EX. IIA implies that $\frac{1}{3}(x,0)\oplus\frac{2}{3}(x',y') \in \Sigma(\{(0,0),(x,0),(x',y')\},\bar{\succeq}_1,\cdot)$. This, combined with the conclusion from the previous paragraph, contradicts PI.

Case 4: x and y are not comparable

Let $\bar{U}_2$ be the utility function defined as follows on $\{0,x,y\}$:

$$\bar{U}_2(0) = 0, \bar{U}_2(x) = \bar{U}_2(y) = 1,$$

and let $\bar{\succeq}_2$ be the associated von Neumann/Morgenstern preference defined over $\Delta(\{0,x,y\})$. Efficiency implies that any lottery ν in $\Sigma(\{(0,0),(x,0),(0,x),(0,y)\},\cdot,\bar{\succeq}_2)$ must place positive weights on $(x,0)$, $(0,x)$, and $(0,y)$ only. For any such lottery, there exists another lottery ν' in $\Delta(\{(x,0),(0,y)\})$ such that both bargainers are indifferent between ν and ν'. EX implies that $\nu' \in \Sigma(\{(0,0),(x,0),(0,x),(0,y)\},\cdot,\bar{\succeq}_2)$. By IIA, this lottery will also belong to $\Sigma(\{(0,0),(x,0),(0,y)\},\cdot,\cdot)$, which is equal to $\{(x,0)\}$. Hence ν must pick $(x,0)$ for sure, but then applying IIA again implies that $\Sigma(\{(0,0),(x,0),(0,x)\},\cdot,\cdot) = \{(x,0)\}$, which contradicts the combination of PI and SYM. $\square$

Step 5 $\Sigma(\{(0,0),(x,0),(0,y)\}) = \{\frac{1}{2}(x,0)\oplus\frac{1}{2}(0,y)\}$.

Proof: By Step 4, we know that $\Sigma(\{(0,0),(x,0),(0,y)\}) = \{\alpha(x,0)\oplus(1-\alpha)(0,y)\}$, for some $\alpha \in]0,1[$. We have to show that $\alpha = 1/2$. This follows trivially from PI and SYM if $y = x$. Suppose thus that $y \neq x$. Hence there exists $\pi \in \mathbb{R}^L_{++}$ such that $\pi\cdot x > \pi\cdot y$ or $\pi\cdot y > \pi\cdot x$.[22] I will assume that the former inequality holds - a similar argument applies in the other case. Let $\beta \in]0,\min\{\alpha,1-\alpha\}[$, let $a \in \mathbb{R}^L_{++}$ be such that $\pi\cdot a = (1-\beta)\pi\cdot y$, and let $b \in \mathbb{R}^L_{++}$ be such that $\pi\cdot b = \beta\pi\cdot y$. Let also $U''_1 : \{0,a,b,x\} \to \mathbb{R}$ and $U''_2 : \{0,a,b,y\} \to \mathbb{R}$ be defined as follows: $U''_1(0) = U''_2(0) = 0$, $U''_1(a) = U''_2(a) = \pi\cdot a$, $U''_1(b) = U''_2(b) = \pi\cdot b$, $U''_1(x) = U''_2(y) = \pi\cdot y$.

Let $\succeq''_1$ be the von Neumann/Morgenstern preference defined on $\Delta(\{0,a,b,x\})$ associated with the Bernoulli function U''_1, and $\succeq''_2$ be the von Neumann/Morgenstern preference defined on $\Delta(\{0,a,b,y\})$ associated with the Bernoulli function U''_2.[23] Consider now a lottery ν in $\Sigma(\{(0,0),(x,0),(0,y),(a,b),(b,a)\},\succeq''_1,\succeq''_2)$. EFF implies that $\nu((0,0)) = 0$. For any $\gamma \in [0,1]$, both agents are indifferent given $(\succeq''_1,\succeq''_2)$ between their marginal of the lottery that gives (a,b) with probability γ and (b,a) with probability $1-\gamma$, and their marginal of the lottery that gives $(x,0)$ with probability $\beta+\gamma-2\beta\gamma$ and $(0,y)$ with probability $1-\beta-\gamma+2\beta\gamma$. Hence both bargainers must be indifferent between their marginal of ν and their marginal of some lottery in $\Delta(\{(x,0),(0,y)\})$. This lottery must belong to $\Sigma(\{(0,0),(x,0),(0,y),(a,b),(b,a)\},\succeq''_1$

[22] The proof might be easier to understand at first when $L = 1$, in which case π can be normalized to 1.

[23] It is easy to check that $\succeq''_1$ and $\succeq''_2$ are necessarily strictly increasing given our definition of U''_1, U''_2, and the fact that $\pi\cdot x > \pi\cdot y$.

$,\succeq_2''$), by EX, and IIA implies that it also belong $\Sigma(\{(0,0),(x,0),(0,y)\},\cdot,\cdot)$. PI thus implies that the lottery that gives $(0,y)$ with probability α and $(x,0)$ with probability $1-\alpha$ belongs to $\Sigma(\{(0,0),(x,0),(0,y),(a,b),(b,a)\},\succeq_1'',\succeq_2'')$. Both bargainers are indifferent given $(\succeq_1'',\succeq_2'')$ between their marginal of this last lottery and their marginal of the lottery that gives (b,a) with probability $\frac{\alpha-\beta}{1-2\beta}$ and (a,b) with probability $\frac{1-\alpha-\beta}{1-2\beta}$ (these are well-defined probabilities because $\beta<\min\{\alpha,1-\alpha\}<1/2$). EX implies that this new lottery belongs to $\Sigma(\{(0,0),(x,0),(0,y),(a,b),(b,a)\},\succeq_1'',\succeq_2'')$, and hence also to $\Sigma(\{(0,0),(a,b),(b,a)\},\succeq_1''',\succeq_2''')$, by IIA, where $\succeq_i'''$ is the restriction of $\succeq_i''$ to $\Delta(\{0,a,b\})$. Notice that $\succeq_1'''$ coincides with $\succeq_2'''$ on $\Delta(\{0,a,b\})$. SYM and PI imply that the only element of $\Sigma(\{(0,0),(a,b),(b,a)\},\succeq_1''',\succeq_2''')$ is the lottery that puts an equal weight on (a,b) and on (b,a). Hence $\frac{\alpha-\beta}{1-2\beta}=1/2$, or $\alpha=1/2$. □

Step 6 $\mu^*\in\Sigma(O,\succeq_1,\succeq_2)$.

Proof: Steps 3 and 5 imply that $U_1'(\mu)=\sigma_1^*=U_1'(\mu^*)$ and $U_2'(\mu)=\sigma_2^*=U_2'(\mu^*)$, and hence $\mu^*\in\Sigma(O',\succeq_1',\succeq_2')$, by PI. IIA implies that $\mu^*\in\Sigma(O,\succeq_1,\succeq_2)$, as desired. ■

Proof of Theorem 2

The fact that Σ_{KS} is not the only solution satisfying the axioms when $L=1$ has already been shown in the main text (cf. Example 1). The fact that Σ_{KS} satisfies the axioms follows from the usual properties of the Kalai-Smorodinsky solution defined in the space of joint Bernoulli utilities. I will thus focus on proving uniqueness. Let Σ be a solution that satisfies the axioms, and let $(O,\succeq_1,\succeq_2)$ be a bargaining problem. We have to prove that $\Sigma(O,\succeq_1,\succeq_2)=\Sigma_{KS}(O,\succeq_1,\succeq_2)$. Given that Σ satisfies PI, it is sufficient to show that $\Sigma_{KS}(O,\succeq_1,\succeq_2)\subseteq\Sigma(O,\succeq_1,\succeq_2)$. Let $\lambda^*\in\Sigma_{KS}(O,\succeq_1,\succeq_2)$. The next few steps show that $\lambda^*\in\Sigma(O,\succeq_1,\succeq_2)$, as desired.

Step 1 *Let (U_1,U_2) be two linear representations of $(\succeq_1,\succeq_2)$. Then*

$$\frac{U_1(\lambda_1^*)-U_1(0)}{\max_{\mu\in\Delta(O)}U_1(\mu_1)-U_1(0)}=\frac{U_2(\lambda_2^*)-U_2(0)}{\max_{\mu\in\Delta(O)}U_2(\mu_2)-U_2(0)}. \tag{3}$$

Proof: Suppose on the contrary that one of the two ratios, let's say the one on the left-hand side, is strictly smaller than the other one. Let x be an element of O such that $U_1(x)=\max_{\mu\in\Delta(O)}U_1(\mu_1)$. Then the lottery that picks x with probability ϵ, and λ^* with probability $1-\epsilon$, guarantees a larger minimal ratio if ϵ is small enough, thereby contradicting the fact that $\lambda^*\in\Sigma_{KS}(O,\succeq_1,\succeq_2)$. This establishes equation (3). □

Step 2 *Let ρ denote the common number defined in (3). If $\rho=1$, then $\Sigma_{KS}(O,\succeq_1,\succeq_2)$ coincides with the Pareto frontier, and coincides with $\Sigma(O,\succeq_1,\succeq_2)$, since Σ satisfies EFF. It will thus be assumed throughout the rest of the proof that $\rho<1$. There exist $x,y,\bar{x},\bar{y}$ in $\mathbb{R}^L_{++}$ such that $\bar{x}<<x$, $\bar{y}<<y$, and $(\bar{x},\bar{y})\in\Sigma(O',\succeq_1',\succeq_2')$ implies that $\lambda^*\in\Sigma(O,\succeq_1,\succeq_2)$, where*

$O' = \{(0,0),(\bar{x},\bar{y}),(x,0),(0,y)\}$ *and* $\succeq_1'$ *(resp.* $\succeq_2'$*) is the preference ordering on* $\Delta(O_1')$ *(resp.* $\Delta(O_2')$*) derived via expected utility from the following utility function* U_1' *(resp.* U_2'*):*

$$U_1'(0) = U_2'(0) = 0,\ U_1'(\bar{x}) = U_2'(\bar{y}) = \rho,\ U_1'(x) = U_2'(y) = 1.$$

Proof: The classical Tietze theorem guarantees that any continuous function on a compact subset of $\mathbb{R}_+^L$ can be extended into a continuous function defined on $\mathbb{R}_+^L$. Strict increasingness can be preserved too (see Husseinov (2010), Corollary 2). Let thus $\bar{U}_1 : \mathbb{R}_+^L \to \mathbb{R}$ and $\bar{U}_1 : \mathbb{R}_+^L \to \mathbb{R}$ be two continuous strictly increasing functions such that $\bar{U}_1$ coincides with U_1 on O_1 and $\bar{U}_2$ coincides with U_2 on O_2. Let $o_1 \in O_1$ be such that $U_1(o_1) = \max_{\mu\in\Delta(O)} U_1(\mu_1)$. Let x' be an element on the diagonal that falls above o_1. Monotonicity implies that $x' \succeq_1 \mu_1$, for all $\mu \in \Delta(O)$. Continuity of $\bar{U}_1$ implies that there exist a convex combinations between 0 (the worst element of O) and x' whose associated (extended) utility is equal to $U_1(o_1)$. Let's call x this new bundle (x might belong to O_1, or not - it does not matter). Since $U_1(\lambda^*) < U_1(x)$, there exists a convex combination between x and 0 whose associated (extended) utility is equal to $U_1(\lambda^*)$. Let's call $\bar{x}$ this new bundle (again, $\bar{x}$ might belong to O_1, or not - it does not matter). A similar construction leads to y and $\bar{y}$. Clearly, $x << \bar{x}$ and $y << \bar{y}$, $\bar{U}_1$ restricted to $\{0,\bar{x},x\}$ is another linear representation of $\succeq_1'$, and $\bar{U}_1$ restricted to $\{0,\bar{x},x\}$ is another linear representation of $\succeq_1'$. Suppose now that $(\bar{x},\bar{y}) \in \Sigma(O',\succeq_1',\succeq_2')$. Let $\bar{\succeq}_1$ be the von Neumann/Morgenstern preference on $\Delta(O_1 \cup \{\bar{x},x\})$ derived via expected utility from $\bar{U}_1$ restricted to $O_1 \cup \{\bar{x},x\}$. Let $\bar{\succeq}_2$ be the von Neumann/Morgenstern preference on $\Delta(O_2 \cup \{\bar{y},y\})$ derived via expected utility from $\bar{U}_2$ restricted to $O_2 \cup \{\bar{y},y\}$. C-MON applies. Notice that there is no lottery in $\Delta(O \cup \{(\bar{x},\bar{y}),(x,0),(0,y)\})$ that strictly Pareto dominates $(\bar{x},\bar{y})$. PI implies that $(\bar{x},\bar{y}) \in \Sigma(O \cup \{(\bar{x},\bar{y}),(x,0),(0,y)\},\bar{\succeq}_1,\bar{\succeq}_2)$. C-MON also applies when moving from O to $O \cup \{(\bar{x},\bar{y}),(x,0),(0,y)\}$. Notice that the set of utilities remain the same when adding $(\bar{x},\bar{y})$, $(x,0)$, and $(0,y)$ to O. So any lottery in $\Sigma(O,\succeq_1,\succeq_2)$ must generate the same utilities as $(\bar{x},\bar{y})$ under $(\bar{U}_1,\bar{U}_2)$. Since λ^* generates the same utilities as well, EX implies that $\lambda^* \in \Sigma(O,\succeq_1,\succeq_2)$, as desired. □

Step 3 *There exist* $\xi,\bar{\xi}$ *in* $\mathbb{R}_{++}^L$ *such that* $\bar{\xi} << \xi$, *and* $(\bar{\xi},\bar{\xi}) \in \Sigma(O'',\succeq'',\succeq'')$ *implies that* $\lambda^* \in \Sigma(O,\succeq_1,\succeq_2)$, *where* $O'' = \{(0,0),(\bar{\xi},\bar{\xi}),(\xi,0),(0,\xi)\}$ *and* $\succeq''$ *is preference ordering on* $\Delta(\{0,\bar{\xi},\xi\}$ *derived via expected utility from the following utility function* U''*:*

$$U''(0) = 0,\ U''(\bar{\xi}) = \rho,\ U''(\xi) = 1.$$

Proof: Let $x,\bar{x},y,\bar{y}$ as in Step 2. Let ϵ be a strictly positive number, and let ξ be the vector in $\mathbb{R}^L$ defined as follows:

$$(\forall l \geq 3) : \xi_l = \min\{x_l, y_l\},$$

$$\xi_1 = \frac{1}{1-\epsilon^2}[y_1 - \epsilon^2 x_1 + \epsilon(y_2 - x_2) + \epsilon \sum_{i=3}^{L}(y_i - \xi_i) - \epsilon^2 \sum_{i=3}^{L}(x_i - \xi_i)],$$

$$\xi_2 = \frac{1}{1-\epsilon^2}[\epsilon(x_1 - y_1) + x_2 - \epsilon^2 y_2 + \epsilon \sum_{i=3}^{L}(x_i - \xi_i) - \epsilon^2 \sum_{i=3}^{L}(y_i - \xi_i)].$$

Let $\bar{\xi}$ be the vector derived by applying the same equations to $(\bar{x}, \bar{y})$. It is easy to check that all the components of both ξ and $\bar{\xi}$ are strictly positive and that $\xi >> \bar{\xi}$, if ϵ is chosen small enough. Indeed, the limit of $\bar{\xi}_1$ (resp. $\bar{\xi}_2$) when ϵ tends to zero is $\bar{y}_1 > 0$ (resp. $\bar{x}_2 > 0$), the limit of $\xi_1 - \bar{\xi}_1$ (resp. $\xi_2 - \bar{\xi}_2$) when ϵ tends to zero is $y_1 - \bar{y}_1 > 0$ (resp. $x_2 - \bar{x}_2 > 0$), and the inequalities regarding the other components are obvious. Straightforward algebra also allows to show that

$$\xi_1 + \epsilon \sum_{l=2}^{L} \xi_l = y_1 + \epsilon \sum_{l=2}^{L} y_l \text{ and } \bar{\xi}_1 + \epsilon \sum_{l=2}^{L} \bar{\xi}_l = \bar{y}_1 + \epsilon \sum_{l=2}^{L} \bar{y}_l$$

$$\xi_2 + \epsilon \sum_{l=1, l\neq 2}^{L} \xi_l = x_2 + \epsilon \sum_{l=1, l\neq 2}^{L} x_l \text{ and } \bar{\xi}_2 + \epsilon \sum_{l=1, l\neq 2}^{L} \bar{\xi}_l = \bar{x}_2 + \epsilon \sum_{l=1, l\neq 2}^{L} \bar{x}_l.$$

In other words, ξ (resp. $\bar{\xi}$) has been chosen in the intersection of the hyperplane of normal $(1, \epsilon, \ldots, \epsilon)$ that goes through y (resp. $\bar{y}$) and the hyperplane of normal $(\epsilon, 1, \epsilon, \ldots, \epsilon)$ that goes through x (resp. $\bar{x}$). Hence ξ is not comparable to either x or y, and $\bar{\xi}$ is not comparable to either $\bar{x}$ or $\bar{y}$. Consider then the utility functions $U_1''' : O_1' \cup O_1'' \to \mathbb{R}$ and $U_2''' : O_2' \cup O_2'' \to \mathbb{R}$ defined as follows:

$$U_1'''(0) = U_2'''(0) = 0, U_1'''(\bar{x}) = U_1'''(\bar{\xi}) = U_2'''(\bar{y}) = U_2'''(\bar{\xi}) = \rho,$$

$$U_1'''(x) = U_1'''(\xi) = U_2'''(y) = U_2'''(\xi) = 1.$$

Let $\succeq_1'''$ and $\succeq_2'''$ be the associated von Neumann/Morgenstern preferences defined on $\Delta(O_1'')$ and $\Delta(O_2'')$ respectively. Notice that if $\mu_1 \succeq_1''' \bar{\xi}$ and $\mu_2 \succeq_2''' \bar{\xi}$, then $\mu_1 \sim_1''' \bar{\xi}$ and $\mu_2 \sim_2''' \bar{\xi}$, for each $\mu \in \Delta(O' \cup O'')$. We also have that $\bar{x} \sim_1''' \bar{\xi}$ and $\bar{y} \sim_2''' \bar{\xi}$. C-MON and EX thus imply that $(\bar{x}, \bar{y}) \in \Sigma(O' \cup O'', \succeq_1''', \succeq_2''')$ if $(\bar{\xi}, \bar{\xi}) \in \Sigma(O'', \succeq'', \succeq'')$. Similarly, C-MON implies that any lottery in $\Sigma(O', \succeq_1', \succeq_2')$ must also belong to $\Sigma(\{(0,0), (O' \cup O'', \succeq_1''', \succeq_2''')$. Hence $(\bar{x}, \bar{y}) \in \Sigma(O', \succeq_1', \succeq_2')$, and Step 2 allows us to conclude that $\lambda^* \in \Sigma(O, \succeq_1, \succeq_2)$. □

Step 4 $\lambda^* \in \Sigma(O, \succeq_1, \succeq_2)$.

Proof: Let $\mu \in \Sigma(O'', \succeq'', \succeq'')$. EFF implies that $\mu((0,0)) = 0$. AN implies that $\mu^* \in \Sigma(O'', \succeq'' , \succeq'')$. PI implies that $\mu((\xi, 0)) = \mu((0, \xi))$. Notice that $\rho \geq 1/2$. If $\rho = 1/2$, then then both bargainers are indifferent between $(\bar{\xi}, \bar{\xi})$ and the lottery that gives $(\xi, 0)$ and $(0, \xi)$ with equal probabilities. There are multiple lotteries in $\Sigma(O'', \succeq'', \succeq'')$, and EX implies that $(\bar{\xi}, \bar{\xi}) \in \Sigma(O'', \succeq'', \succeq'')$. If $\rho > 1/2$, then EFF implies that $\mu((\xi, 0)) = \mu((0, \xi)) = 0$, and one concludes again that $(\bar{\xi}, \bar{\xi}) \in \Sigma(O', \succeq'', \succeq'')$. Step 3 thus implies that $\lambda^* \in \Sigma(O, \succeq_1, \succeq_2)$, as desired. ■

Proof of Theorem 1'

Given the usual properties of the Nash solution in the space of joint utilities, it is easy to check that the extended Nash solution defined in (2) satisfies the axioms listed in Theorem 1' (PI follows from the fact that $U(O, \succeq_1, \succeq_2)$ is convex). I will thus focus on proving uniqueness. Let Σ be a solution that satisfies the axioms, and let $(O, \succeq_1, \succeq_2)$ be a bargaining problem. As for the proof of Theorem 1, it is sufficient to prove that the extended Nash solution to the problem $(O, \succeq_1, \succeq_2)$ is included in $\Sigma(O, \succeq_1, \succeq_2)$. All the steps in the proof of that Theorem and their proofs, excepts for Step 2, apply immediately to our extended domain as well, using the representations (U_1, U_2), as defined in Section 5, instead of the linear representations of the two von Neumann/Morgenstern preferences previously. Notice particularly how Steps 3 to 5 apply whenever the domain simply contains von Neumann/Morgenstern preferences. I will thus focus on establishing Step 2, whose statement remains unchanged, except again for using the utility representations defined in Section 5, which accommodate some forms of non-expected utility.

It is easy to check that $\succeq_1'$ (resp. $\succeq_2'$) inherits from $\succeq_1$ (resp. $\succeq_2$) the properties of first-order stochastic dominance, and the existence of a linear representation when comparing simple compound lotteries that involve a lottery in $\Delta(O_1 \cup \{x\})$ (resp. $\Delta(O_2 \cup \{y\})$) and 0. Let $O' = O \cup \{(x,0),(0,y)\}$. It remains to check the convexity condition for $(O', \succeq_1', \succeq_2')$ to qualify as a bargaining problem. Notice that x and y are best outcomes for 1 and 2 respectively in $(O', \succeq_1', \succeq_2')$. Let thus $\mu, \nu \in \Delta(O')$ and $p_1, p_2, q_1, q_2 \in [0,1]$ be such that $\mu_1 \sim_1' p_1 x$, $\nu_1 \sim_1' q_1 x$, $\mu_2 \sim_2' p_2 y$, $\nu_2 \sim_2' q_2 y$. Notice that

$$U_1'(\mu_1) = 2\mu(x,0)\sigma_1^* + (1-\mu(x,0))U_1((\mu|\neg(x,0))_1) = 2\mu(x,0)\sigma_1^* + (1-\mu(x,0)-\mu(0,y))U_1((\mu|O)_1),$$

where $\mu|O$ is the lottery derived from μ by Bayesian updating if one knows that the outcome will belong to O (the first equality follows from the definition of U_1', while the second follows from the fact that U_1 is linear on simple compound lotteries that involve 0 and a lottery in $\Delta(O)$). Similarly, we have:

$$U_1'(\nu_1) = 2\nu(x,0)\sigma_1^* + (1-\nu(x,0)-\nu(0,y))U_1((\nu|O)_1),$$

$$U_2'(\mu_2) = 2\mu(0,y)\sigma_2^* + (1-\mu(x,0)-\mu(0,y))U_2((\mu|O)_2),$$

$$U_2'(\nu_2) = 2\nu(0,y)\sigma_2^* + (1-\nu(x,0)-\nu(0,y))U_2((\nu|O)_2).$$

Let

$$\alpha = \frac{1}{2\sigma_1^*}\frac{U_1'(\mu_1)+U_1'(\nu_1)}{2}$$

$$\beta = \frac{1}{2\sigma_2^*}\frac{U_2'(\mu_2)+U_1'(\nu_2)}{2}.$$

Hence $\alpha + \beta$ is equal to

$$\frac{\mu(x,0)+\mu(0,y)+\nu(x,0)+\nu(0,y)}{2} + \frac{1-\mu(x,0)-\mu(0,y)}{2}[\frac{U_1((\mu|O)_1)}{2\sigma_1^*} + \frac{U_2((\mu|O)_2)}{2\sigma_2^*}]$$

$$+\frac{1-\nu(x,0)-\nu(0,y)}{2}[\frac{U_1((\nu|O)_1)}{2\sigma_1^*} + \frac{U_2((\nu|O)_2)}{2\sigma_2^*}],$$

which is no larger than 1, since both $\frac{U_1((\mu|O)_1)}{2\sigma_1^*} + \frac{U_2((\mu|O)_2)}{2\sigma_2^*}$ and $\frac{U_1((\mu|O)_1)}{2\sigma_1^*} + \frac{U_2((\mu|O)_2)}{2\sigma_2^*}$ are no larger than 1, given that $U(O, \succeq_1, \succeq_2) \subseteq \{x \in \mathbb{R}^2 | \frac{x_1}{\sigma_1^*} + \frac{x_2}{\sigma_2^*} \leq 2\}$ (see Step 1 in the proof of Theorem 1). Consider then the lottery λ that gives $(x,0)$ with probability α, $(0,y)$ with probability β, and $(0,0)$ with probability $1-\alpha-\beta$. We have: $\lambda_1 \sim_1' \alpha x = \frac{p_1+q_1}{2}x$ and $\lambda_2 \sim_2' \beta y = \frac{p_2+q_2}{2}y$, since $U_1'(\mu_1) = 2\sigma_1^* p_1$, $U_1'(\nu_1) = 2\sigma_1^* q_1$, $U_2'(\mu_2) = 2\sigma_2^* p_2$, and $U_2'(\nu_2) = 2\sigma_2^* q_2$. This establishes the convexity condition for $(O', \succeq_1', \succeq_2')$, and concludes this proof. ■