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An axiomatization of the inner core using appropriate reduced games

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Abstract

I adapt a reduction process introduced by Serrano and Volij [Serrano, R., Volij, O., 1998. Axiomatization of neoclassical concepts for Economies. Journal of Mathematical Economics 30, 87–108] so that the reduced games of convex-valued games are convex-valued. I use the corresponding consistency property and its converse to axiomatize the inner core for games that are convex-valued, non-level and smooth.

Keywords: Consistency; Reduced games; Inner core

1. Introduction

The consistency property can be used to axiomatize most solution concepts in game theory (see Thomson, 1998, for a survey). It requires that the restriction of a payoff vector in the solution of a game to a subset of players belongs to the solution of the reduced game. The key is to adequately define the reduced games.

Davis and Maschler (1965) introduce the first reduction process for cooperative games with transferable utility (TU). It involves some re-evaluation of the coalitional bargaining power. Let N be the set of players, let $u \in \mathbb{R}^N$ be a potential agreement and let $S \subseteq N$ be a coalition. Then, the reduced game defined on S is obtained by considering that each strict subset of S *may* buy up the cooperation of any passive player (i.e. not in S), while S itself *has to* buy up the cooperation of all the passive players, the utility 'price' of these players being specified by u.

Peleg (1986) observes that the Davis–Maschler consistency property can be used to axiomatize the core on the class of balanced TU-games. Peleg (1985) extends the reduction process to games with non-transferable utility (NTU) and uses the corresponding consistency property (and its converse) to axiomatize the core on a large class of NTU-games.

One may question the asymmetric treatment of the coalitions in the Davis–Maschler-Peleg reduction process. Serrano and Volij (1998, definition 3) indeed suggest that the grand coalition in the reduced games also considers the possibility of choosing a subset of passive players with whom to cooperate. They apply the related consistency property and axiomatize the core for a large class of production economies. I re-state their result on the class of NTU-games in Section 2.

It is natural in some contexts to restrict attention to NTU-games that are convex-valued. For instance, the players may agree on lotteries, and evaluate them according to the expected utility criterion. Unfortunately, the results discussed so far do not apply to this class of games, as the reduced game of a convex-valued game is not necessarily convex-valued.

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The study of the reduced game property for convex-valued games was left as an open question by Peleg (1985, Section 7.6).

I propose to "convexify" the Serrano–Volij reduction process. A natural interpretation is that coalitions may use lotteries to determine the set of passive players with whom to cooperate. It appears, perhaps as a surprise, that the analogue of the axioms introduced by Serrano and Volij (1998) then characterize the inner core for games that are convex-valued, non-level and smooth (see Proposition 2).

As its name suggests, the inner core is a refinement of the core. It is obtained by applying Shapley's (1969) fictitious-transfer procedure to the core defined for TU-games. A first mention to the inner core can be found in Shapley and Shubik (1975). Though perhaps cryptic in its original constructive definition, the inner core admits a natural interpretation in terms of random blocking plans (Myerson, 1991, Section 9.8; Qin, 1993; de Clippel and Minelli, 2005). Consider for instance an allocation x in the core of an exchange economy. By definition, no coalition S can improve the satisfaction of all its members by reallocating their initial endowments. It is also impossible for S to improve the satisfaction of all its members by reallocating $(x_i)_{i \in S}$, since this would contradict the Pareto efficiency of x. Yet, it may happen that S can improve the satisfaction of all its members by randomizing between the two types of objection. It turns out that the inner core is precisely the set of feasible allocations that are immune to these random objections if the utility functions are smooth and concave (see de Clippel and Minelli, 2005). Lemma 1 proposes a variant of this result for NTU-games, and will play a key role in the proof of the main result. It is worth noting that competitive equilibria always belong to the inner core in exchange economies with concave utility functions. The inner core being a subset of the core, the converge and equivalence results established for the core also apply to the inner core (Qin, 1994; de Clippel and Minelli, 2005).

2. Preliminaries

If N is a finite set, then $P(N)$ denotes the set of non-empty subsets of N. Let $S \in P(N)$ and let (u, u') be a couple of vectors in $\mathbb{R}^S$. Then $u \leq u'$ if $u_i \leq u'_i$ for each $i \in S$, $u < u'$ if $u \leq u'$ and $u \neq u'$, while $u \ll u'$ if $u_i < u'_i$ for each $i \in S$. If $u \in \mathbb{R}^N$, then u_S denotes the projection of u on $\mathbb{R}^S$. Let X be a subset of $\mathbb{R}^S$. Then u is *efficient* in X if $u \in X$ and there does not exists $u' \in X$ such that $u < u'$.

A *game* is a couple (N, V) where N is the finite set of players and V is a correspondence that associates to every coalition $S \in P(N)$ a non-empty compact subset $V(S)$ of $\mathbb{R}^S$. The class of all games is denoted by $\mathcal{G}$.

A *solution* Σ associates to every game $(N, V) \in \mathcal{G}$ a subset $\Sigma(N, V)$ of $V(N)$. The *core* for instance specifies the set of feasible utility profiles that no coalition can improve upon:

$$C(N, V) = \{\sigma \in V(N) | \neg[(\exists S \in P(N))(\exists u \in V(S)) : \sigma_S < u]\}$$

for each $(N, V) \in \mathcal{G}$.

Serrano and Volij (1998) axiomatize the core in the context of production economies. I adapt their argument to my framework. Here are properties that one could impose on a solution Σ defined on $\mathcal{G}$.

Axiom 1. (One-person rationality, OPR) $\Sigma(N, V) = arg \max_{v \in V(N)} v$, for each $(N, V) \in \mathcal{G}$ such that $\#N = 1$.

Axiom 2. (Consistency, CONS) Let $(N, V) \in \mathcal{G}$, let $S \in P(N)$ and let $\sigma \in \Sigma(N, V)$. Then $(S, V_{S,\sigma}) \in \mathcal{G}$ and $\sigma_S \in \Sigma(S, V_{S,\sigma})$, where $(S, V_{S,\sigma})$ is the reduced game with respect to S and σ defined as follows:

$$V_{S,\sigma}(T) := \bigcup_{Q \in P(N \setminus S) \cup \{\emptyset\}} \{u \in \mathbb{R}^{\mathrm{T}} | (\exists v \in V(T \cup Q)) : v_T = u \text{ and } v_Q \geq \sigma_Q\}$$

for each $T \in P(S)$.

Axiom 3. (Converse consistency, CO-CONS) Let $(N, V) \in \mathcal{G}$ be a game with at least two players and let $\sigma \in V(N)$. If $(S, V_{S,\sigma}) \in \mathcal{G}$ and $\sigma_S \in \Sigma(S, V_{S,\sigma})$ for each $S \in P(N) \setminus \{N\}$, then $\sigma \in \Sigma(N, V)$.

OPR requires that the solution is compatible with the maximization of individual utilities when there is only one player. Axiom 2 is the usual consistency property: the restriction of a payoff vector in the solution of a game to a subset of players must belong to the solution of the reduced game. As already discussed in Section 1, the reduction process differs from Peleg's (1985) definition only as far as the feasible set associated to the grand coalition in the reduced games is concerned: *every* coalition in the reduced game is now free to choose the set of passive players with whom to cooperate. It is indeed more natural to treat all the coalitions symmetrically. CO-CONS is a dual version of CONS. If the relevant projections of a feasible allocation σ belong to the solution of the reduced games $(S, V_{S,\sigma})$, then it belongs to the solution of the game (N, V). Peleg (1985) also uses a property of converse consistency in his axiomatization of the core. There are two main differences. First, as already discussed, his reduced games differ from those defined in CONS. Second, his axiom is slightly stronger as he imposes that $\sigma \in \Sigma(N, V)$ whenever $\sigma_S \in \Sigma(S, V_{S,\sigma})$ for each coalition S with exactly two members.

Proposition 1 is similar to Theorem 4 in Serrano and Volij (1998), dealing with NTU-games instead of production economies. The proof is included for the sake of completeness.

Proposition 1. *The core is the maximal solution to satisfy OPR and CONS on $\mathcal{G}$. It is also the minimal solution to satisfy OPR and CO-CONS on $\mathcal{G}$. Hence it is the only solution to satisfy OPR, CONS and CO-CONS on $\mathcal{G}$.*

Proof.

1. The core obviously satisfies OPR. It also satisfies CONS and CO-CONS as the following arguments show.

(1.a) CONS: Let $(N, V) \in \mathcal{G}$, let $S \in P(N)$ and let $\sigma \in V(N)$. If $\sigma \in C(N, V)$, then $(S, V_{S,\sigma}) \in \mathcal{G}$, as the finite union of compact sets is a compact set. On the other hand, if $\sigma_S \notin C(S, V_{S,\sigma})$, then there exist $T \in P(S)$ and $u \in V_{S,\sigma}(T)$ such that $u > \sigma_T$. Hence, $\sigma \notin C(N, V)$, as there exists $Q \in P(N \setminus S) \cup \{\emptyset\}$ and $v \in V(T \cup Q)$ such that $v \geq (u, \sigma_Q) > \sigma_{T \cup Q}$.

(1.b) CO-CONS: Let $(N, V) \in \mathcal{G}$ be a game with at least two players and let $\sigma \in V(N)$ be such that $(S, V_{S,\sigma}) \in \mathcal{G}$ and $\sigma_S \in C(S, V_{S,\sigma})$ for each $S \in P(N) \setminus \{N\}$. I have to show that $\sigma \in C(N, V)$. It is sufficient to show that σ is efficient in $V(N)$, as $V(S) \subseteq V_{S,\sigma}(S)$ and σ_S is efficient in $V_{S,\sigma}(S)$ for each $S \in P(N) \setminus \{N\}$. Suppose on the contrary that there exists $u \in V(N)$ such that $u > \sigma$. Let $i \in N$ be such that $u_i > \sigma_i$. Hence, $\sigma_i \notin C(\{i\}, V_{\{i\},\sigma})$ because $u_i \in V_{\{i\},\sigma}(\{i\})$. This is impossible.

2. $\Sigma \subseteq C$: Let Σ be a solution that satisfies both OPR and CONS, let $(N, V) \in \mathcal{G}$ and let $\sigma \in V(N)$. If $\sigma \notin C(N, V)$, then there exists $S \in P(N)$ and $u \in V(S)$ such that $u > \sigma_S$. Let $i \in S$ be such that $u_i > \sigma_i$. OPR implies that $\sigma_i \notin \Sigma(\{i\}, V_{\{i\},\sigma})$ because $u_i \in V_{\{i\},\sigma}(\{i\})$. Hence $\sigma \notin \Sigma(N, V)$ by CONS.
3. $C \subseteq \Sigma$: Let Σ be a solution that satisfies both OPR and CO-CONS. I prove by induction on the cardinality of N that $C(N, V) \subseteq \Sigma(N, V)$ for each game $(N, V) \in \mathcal{G}$. The result is obvious if there is just one player in the game, given OPR. Let n be a positive integer. Suppose that I already proved that $C(N, V) \subseteq \Sigma(N, V)$ for every game $(N, V) \in \mathcal{G}$ such that $\#N \leq n$. Let $(N, V) \in \mathcal{G}$ be such that $\#N = n + 1$ and let $\sigma \in C(N, V)$. By (1.a), $(S, V_{S,\sigma}) \in \mathcal{G}$ and $\sigma_S \in C(S, V_{S,\sigma})$ for each $S \in P(N) \setminus \{N\}$. By the induction hypothesis, $\sigma_S \in \Sigma(S, V_{S,\sigma})$ for each $S \in P(N) \setminus \{N\}$. By CO-CONS, $\sigma \in \Sigma(N, V)$. □

The reasoning is simple. If the players in the reduced games can buy up the cooperation of passive players, then coalitional stability amounts to OPR. Peleg's (1985) argument is similar but less straightforward because he assumes that the members of the grand coalition buy up the cooperation of all the passive players in the reduced games. For instance, he establishes the maximality of the core by combining the individual rationality constraints in all the reduced games with two players.

Peleg (1985, 1986) applies his axioms to the set of games with a non-empty core. Hence, his axiomatic characterizations are not completely independent of the core itself. Serrano and Volij (1998) can dispense with the non-emptiness axiom by altering the reduction process and by imposing OPR instead of 'individual rationality.'

The Davis–Maschler–Peleg reduction process can also be used to axiomatize the prekernel and the prenucleolus (Sobolev, 1975; Peleg, 1986; Serrano and Shimomura, 1998). It would be interesting to know whether similar results can be obtained with the Serrano–Volij reduction process. In view of Proposition 1, every solution that satisfies CONS (and OPR) is a core selection. Hence, CONS could actually be used only to axiomatize the intersection of the core with the prekernel and/or the intersection of the core with the prenucleolus.

3. The result

A game (N, V) is *convex-valued* if $V(S)$ is convex for each coalition $S \in P(N)$. Convex-valued games are the relevant models to consider when the players may agree on lotteries, and evaluate them according to the expected utility criterion.

I impose two additional regularity conditions. A game (N, V) is *non-level* if

$$[v_i \geq \max_{u \in V(\{i\})} u] \to [(\exists v' \in V(S)) : v'_{S\setminus\{i\}} \gg v_{S\setminus\{i\}}]$$

for each $v \in V(S)$, each $i \in S$ and each coalition $S \in P(N)$ with at least two members. Transferable utility is used in cooperative games in general, and in this paper in particular, to describe a situation where the utilities are transferable between the players in *any* amount, and at a *one to one* exchange rate. For exchange economies, for instance, it implies that the utility functions are quasi-linear in some good, usually called money. The non-levelness condition means that, starting from any utility profile $v \in V(S)$ that is individually rational for some member i of S, there is an alternative utility profile $v' \in V(S)$ that makes all the other members of S better off. Of course, this must be at the expense of player i if v is Pareto efficient in $V(S)$. Non-levelness thus means that there is a possibility of transferring *some* amount of utility from agent i to the other members of S at *some* exchange rate. It is thus much weaker than the property of transferable utility. This is why non-leveleness is sometimes referred to as a property of "minimal transferability" (see e.g. Moulin, 1988). For exchange economies, for instance, non-levelness is satisfied if the utility functions are strictly monotonic and the endowments are strictly positive. This rules out the possibility of satiation. As a consequence, if a vector λ is orthogonal to $V(S)$ at an efficient utility profile that is individually rational for all the players, then λ is strictly positive.

Next, a convex set is smooth at a point of its boundary if it admits a unique supporting hyperplane at that point. A convex-valued game (N, V) is *smooth* if $V(S)$ is smooth at each efficient utility profile, for each coalition S. The set of convex-valued games that are both non-level and smooth is denoted by $\mathcal{G}'$.

CONS is and CO-CONS are not well defined on $\mathcal{G}'$, as the reduction process does not preserve convexity. I suggest to adapt these two properties as follows.

Axiom 4. (Consistency$'$, CONS$'$) Let $(N, V) \in \mathcal{G}'$, let $S \in P(N)$ and let $\sigma \in \Sigma(N, V)$. Then $(S, V'_{S,\sigma}) \in \mathcal{G}'$ and $\sigma_S \in \Sigma(S, V'_{S,\sigma})$, where $(S, V'_{S,\sigma})$ is the reduced game with respect to S and σ defined as follows:[1]

$$V'_{S,\sigma}(T) := \text{co}[V_{S,\sigma}(T)]$$

for each $T \in P(S)$.

Axiom 5. (Converse consistency$'$, CO-CONS$'$) Let $(N, V) \in \mathcal{G}'$ be a game with at least two players and let $\sigma \in V(N)$. If $(S, V'_{S,\sigma}) \in \mathcal{G}'$ and $\sigma_S \in \Sigma(S, V'_{S,\sigma})$ for each $S \in P(N) \setminus \{N\}$, then $\sigma \in \Sigma(N, V)$.

CONS$'$ and CO-CONS$'$ are the analogue of CONS and CO-CONS obtained by convexifying the reduction process. The definition of $V'_{S,\sigma}(T)$ is more than a convenient mathematical construction to obtain a reduced game that belongs to $\mathcal{G}'$. Observe that the set $\{u \in \mathbb{R}^T | (\exists w \in V(T \cup Q)) : w_T = u \text{ and } w_Q \geq \sigma_Q\}$ is convex for each $Q \in P(N \setminus S) \cup \{\emptyset\}$. Hence, $v \in V'_{S,\sigma}(T)$ if and only if there exist a probability distribution α defined over $P(N \setminus S) \cup \{\emptyset\}$, a function $x : P(N \setminus S) \cup \{\emptyset\} \to \mathbb{R}^T$ and a function w that associates to every coalition $Q \in P(N \setminus S) \cup \{\emptyset\}$ an element $w(Q)$ in $V(T \cup Q)$ such that $v = \sum_{Q \in P(N\setminus S)\cup\{\emptyset\}} \alpha(Q)x(Q)$, $w_T(Q) = x(Q)$ and $w_Q(Q) \geq \sigma_Q$, for each $Q \in P(N \setminus S)$. I conclude that $V'_{S,\sigma}(T)$ is the set of feasible utility profiles for T if its members may use lotteries to determine the set of passive (i.e. in $N \setminus S$) players with whom to cooperate, given their utility 'price' σ.

The main purpose of the paper is to characterize the set of solutions that satisfy OPR, CONS$'$, and CO-CONS$'$. It turns out that there exists a unique such solution, and that it has been previously introduced in the literature under the name of "inner core" (see e.g. Shapley and Shubik, 1975; Myerson, 1991; Qin, 1993).

[1] 'co' denotes the convex hull operator. If A is a subset of some euclidian space, then co(A) is the set of vectors that can be written as a convex combination of finitely many elements of A.

The *inner core* is obtained by applying Shapley's (1969) fictitious-transfer procedure to the core defined on the class of games with transferable utility:

$$\mathrm{IC}(N, V) = \bigcup_{\lambda \in \mathbb{R}^N_{++}} \{\sigma \in V(N) | (\forall S \in P(N))(\forall u \in V(S)) : \sum_{i \in S} \lambda_i u_i \leq \sum_{i \in S} \lambda_i \sigma_i\}$$

for each $(N, V) \in \mathcal{G}'$. It is the set of feasible utility profiles σ for which there exists a vector of utility weights λ such that no coalition can improve upon σ even by making λ-weighted transfers of utilities between its members. By construction, the inner core is a subset of the core. The inclusion may be strict, as the following example shows.

Example 1. Let $(N, V) \in \mathcal{G}'$ be the game defined as follows: $N := \{1, 2, 3\}$, $V(\{i\}) = [0, 1]$ for each $i \in \{1, 2, 3\}$, $V(\{1, 2\}) = \{u \in \mathbb{R}^2_+ | u_1^2 + 10u_2^2 \leq 910\}$, $V(\{1, 3\}) = V(\{2, 3\}) = \{u \in \mathbb{R}^2_+ | u_1^2 + u_2^2 \leq 2\}$ and $V(\{1, 2, 3\}) = \{u \in \mathbb{R}^3_+ | u_1^2 + u_2^2 + u_3^2 \leq 300\}$. The payoff profile $\sigma := (10, 10, 10)$ belongs to the core but not to the inner core. To see that σ does not belong to the inner core, observe that any vector of weights supporting V(N) at σ is proportional to (1,1,1) and that coalition {1, 2} could improve upon σ if its members were able to transfer utility at this rate (for instance by achieving (30,1) and transferring 14 units of utility from player 1 to 2).

A key part of the proof of the main result relies on the following lemma that proposes a variant to Proposition 4 in de Clippel and Minelli (2005), dealing now with games instead of exchange economies. The proof is included for the sake of completeness.

Lemma 1. *Let $(N, V) \in \mathcal{G}'$ and let $\sigma \in V(N)$. If $\sigma_S \in C(S, V'_{S,\sigma})$ for each $S \in P(N)$, then $\sigma \in \mathrm{IC}(N, V)$.*

Proof. Notice that σ is efficient in $V(N)$ since $\sigma \in C(N, V)$. Let λ be the unique normalized vector that is orthogonal to $V(N)$ at σ. Notice also that $\sigma_i \geq \max_{v \in V(\{i\})} v$ for each $i \in N$. Hence $\lambda \gg 0$, because of the first regularity condition imposed on the games in $\mathcal{G}'$. Let S be a coalition. The utility profile σ_S is efficient in $V'_{S,\sigma}(S)$, as $\sigma_S \in C(S, V'_{S,\sigma})$. Let λ' be a vector that is orthogonal to $V'_{S,\sigma}(S)$ at σ_S. Let $U := \{u \in \mathbb{R}^S | (u, \sigma_{N \setminus S}) \in V(N)\}$. Observe that $U \subseteq V'_{S,\sigma}(S)$ and that $\sigma_S \in U$. Hence λ' is orthogonal to U at σ_S. The smoothness of $V(N)$ implies that λ' is proportional to λ_S (see Lemma A.3 in Appendix 4). In addition, $V(S) \subseteq V'_{S,\sigma}(S)$. Hence $\sum_{i \in S} \lambda_i u_i \leq \sum_{i \in S} \lambda_i \sigma_i$ for each $u \in V(S)$. □

Example 1 illustrates Lemma 1. The utility profile $\sigma = (10, 10, 10)$ belongs to core but not to the inner core. Hence there must exists a coalition $S \subsetneq N$ such that $\sigma_S \notin C(S, V'_{S,\sigma})$. Indeed, $\sigma_{\{1,2\}}$ is not efficient in $V'_{\{1,2\},\sigma}(\{1, 2\})$, as $(30, 1) \in V(\{1, 2\})$, $(\sqrt{79}, 11) \in V_{\{1,2\},\sigma}(\{1, 2\})$ and $(10, 10) < ((30, 1) + 10(\sqrt{79}, 11))/11$.

Lemma 1 does not hold for convex-valued games that do not satisfy the two regularity conditions, as the following example shows. It is similar to Example A.1 in de Clippel and Minelli (2005), but it is included for the sake of completeness.

Example 2. Let $\sigma = (3, 3, 3)$ and let $(\{1, 2, 3\}, V)$ be the game defined as follows: $V(\{i\}) = \{0\}$ for each $i \in \{1, 2, 3\}$, $V(\{1, 2\}) = V(\{2, 3\}) = \{u \in \mathbb{R}^2_+ | u_1 + 9u_2 \leq 9\}$, $V(\{1, 3\}) = \{u \in \mathbb{R}^2_+ | 9u_1 + u_2 \leq 9\}$ and $V(\{1, 2, 3\}) = \{u \in \mathbb{R}^3_+ | u \leq (3, 3, 3)\}$. It is easy to check that $\sigma_S \in C(S, V'_{S,\sigma})$ for each $S \in P(\{1, 2, 3\})$. Suppose that (3,3,3) belongs to the inner core of $(\{1, 2, 3\}, V)$. Let λ be the associated vector of utility weights. Observe that $(9, 0) \in V(\{1, 2\}) \cap V(\{2, 3\})$ and that $(0, 9) \in V(\{1, 3\})$. Hence $3\lambda_1 + 3\lambda_2 \geq 9\lambda_1$, $3\lambda_2 + 3\lambda_3 \geq 9\lambda_2$ and $3\lambda_1 + 3\lambda_3 \geq 9\lambda_3$. This implies that $6(\lambda_1 + \lambda_2 + \lambda_3) \geq 9(\lambda_1 + \lambda_2 + \lambda_3)$, which is impossible.

We are now ready to state and prove the main result.

Proposition 2. *The inner core is the maximal solution to satisfy OPR and CONS′ on $\mathcal{G}'$. It is also the minimal solution to satisfy OPR and CO-CONS′ on $\mathcal{G}'$. Hence it is the only solution to satisfy OPR, CONS′ and CO-CONS′ on $\mathcal{G}'$.*

Proof.

1. The inner core obviously satisfies OPR. It also satisfies CONS$'$ and CO-CONS$'$ as the following argument shows.

(1.a) CONS$'$: Let $(N, V) \in \mathcal{G}'$, let $S \in P(N)$ and let $\sigma \in \text{IC}(N, V)$. I first prove that $(S, V'_{S,\sigma}) \in \mathcal{G}'$. The convex hull of a compact set is a set that is both compact and convex. Let $T \in P(S)$ be a coalition with at least two members, let $v \in V'_{S,\sigma}(T)$, let (α, x, w) be some triple associated to v, as in the paragraph that follows the definition of CONS$'$ and CO-CONS$'$, and let $i \in T$. If $v_i \geq \max_{u \in V_{S,\sigma}(\{i\})'} u$, then there exists $\hat{Q} \in P(N \setminus S) \cup \{\emptyset\}$ such that $x_i(\hat{Q}) \geq \max_{u \in V(\{i\})} u$. Since V satisfies the first regularity condition, there exists $z \in V(T \cup \hat{Q})$ such that $z_{(T\cup\hat{Q})\setminus\{i\}} \gg x_{(T\cup\hat{Q})\setminus\{i\}}$. I obtain x' by modifying x as follows: $x'(\hat{Q}) := z_T$ and $x'(Q) := x(Q)$ for every $Q \in P(N \setminus S) \cup \{\emptyset\}$ different from $\hat{Q}$. Then $v' := \sum_{Q \in P(N\setminus S)\cup\{\emptyset\}} \alpha(Q)x'(Q) \in V'_{S,\sigma}(T)$ and $v'_{T\setminus\{i\}} \gg v_{T\setminus\{i\}}$. Suppose now that $v \in \mathbb{R}^{\text{T}}$ is efficient in $V'_{S,\sigma}(T)$ in order to check the second regularity condition. Let $Q \in P(N \setminus S) \cup \{\emptyset\}$. If $\alpha(Q) > 0$, then *x(Q)* is efficient in $\{u \in \mathbb{R}^{\text{T}} | (\exists w \in V(T \cup Q)) : w_T = u \text{and} w_Q \geq \sigma_Q\}$. Lemma A.3 in Appendix 4 implies that $\{u \in \mathbb{R}^{\text{T}} | (\exists w \in V(T \cup Q)) : w_T = u \text{and} w_Q \geq \sigma_Q\}$ is smooth at *x(Q)*. Therefore $V'_{S,\sigma}(T)$ is smooth at v.

I then prove that $\sigma_S \in \text{IC}(S, V'_{S,\sigma})$. Let $\lambda \in \mathbb{R}^N_{++}$ be a vector that supports σ as an inner core allocation for (N, V), let $T \in P(S)$, let $v \in V'_{S,\sigma}(T)$ and let (α, x, w) be some triple associated to v as before. I have: $\sum_{i \in T}\lambda_i v_i = \sum_{i \in T}\lambda_i \sum_{Q \in P(N\setminus S)\cup\{\emptyset\}} \alpha(Q)x_i(Q) = \sum_{Q \in P(N\setminus S)\cup\{\emptyset\}} \alpha(Q) \sum_{i \in T}\lambda_i w_i(Q)$. Hence there exists $Q \in P(N \setminus S) \cup \{\emptyset\}$ such that $\sum_{i \in T}\lambda_i v_i \leq \sum_{i \in T}\lambda_i w_i(Q)$. As $\sigma_Q \leq w_Q(Q)$ and $w(Q) \in V(T \cup Q)$, I have: $\sum_{i \in T}\lambda_i v_i + \sum_{i \in Q}\lambda_i \sigma_i \leq \sum_{i \in T\cup Q}\lambda_i w_i(Q) \leq \sum_{i \in T\cup Q}\lambda_i \sigma_i$. Hence $\sum_{i \in T}\lambda_i v_i \leq \sum_{i \in T}\lambda_i \sigma_i$. So $\sigma_S \in \text{IC}(S, V'_{S,\sigma})$.

(1.b) CO-CONS$'$: Let $(N, V) \in \mathcal{G}$ be a game with at least two players and let $\sigma \in V(N)$. Suppose that $\sigma_S \in \text{IC}(S, V'_{S,\sigma})$ for each $S \in P(N) \setminus \{N\}$. Hence $\sigma_S \in C(S, V'_{S,\sigma})$ for each $S \in P(N) \setminus \{N\}$. In addition, $\sigma \in C(N, V)$, as the core satisfies CO-CONS. Lemma 1 implies that $\sigma \in \text{IC}(N, V)$.

2. $\Sigma \subseteq \text{IC}$: Let Σ be a solution that satisfies both OPR and CONS$'$. It is easy to adapt item 2 in the proof of Proposition 1 to show that $\Sigma \subseteq C$. The following argument proves the stronger result $\Sigma \subseteq \text{IC}$. Let $(N, V) \in \mathcal{G}'$ and let $\sigma \in V(N)$. Suppose that $\sigma \notin \text{IC}(N, V)$. Lemma 1 implies that there exists $S \in P(N)$ such that $\sigma_S \notin C(S, V'_{S,\sigma})$. Hence $\sigma_S \notin \Sigma(S, V'_{S,\sigma})$ and $\sigma \notin \Sigma(N, V)$ by CONS$'$.
3. $\text{IC} \subseteq \Sigma$: Let Σ be a solution that satisfies both OPR and CO-CONS$'$. I prove by induction on the cardinality of N that $\text{IC}(N, V) \subseteq \Sigma(N, V)$ for each game $(N, V) \in \mathcal{G}'$. The result is obvious for one-player games, given OPR. Let n be a positive integer. Suppose that I already proved that $\text{IC}(N, V) \subseteq \Sigma(N, V)$ for every game $(N, V) \in \mathcal{G}'$ with at most n players. Let $(N, V) \in \mathcal{G}'$ be a game with $n + 1$ players and let $\sigma \in \text{IC}(N, V)$. By (1.a), $(S, V'_{S,\sigma}) \in \mathcal{G}'$ and $\sigma_S \in \text{IC}(S, V'_{S,\sigma})$ for each $S \in P(N) \setminus \{N\}$. By the induction hypothesis, $\sigma_S \in \Sigma(S, V'_{S,\sigma})$ for each $S \in P(N) \setminus \{N\}$. By CO-CONS$'$, $\sigma \in \Sigma(N, V)$. □

The axioms are independent. For each non-negative integer k, let Σ_k be the solution defined as follows: $\Sigma_k(N, V) := \text{IC}(N, V)$ for each game $(N, V) \in \mathcal{G}'$ with $\#N < k$ and $\Sigma_k(N, V) = \emptyset$ for each game $(N, V) \in \mathcal{G}'$ with $\#N \geq k$. Σ_0 (the empty solution) satisfies both CONS$'$ and CO-CONS$'$, but not OPR. For each $k \geq 1$, Σ_k satisfies both OPR and CONS$'$, but not CO-CONS$'$. Finally, the core satisfies both OPR and CO-CONS$'$, but not CONS$'$.

The regularity assumptions are important. The inner core does not satisfy CO-CONS$'$ on the class of all convex-valued games. Consider the game V and the payoff vector σ defined in Example 2 for instance. I have: $\sigma \notin \text{IC}(N, V)$ although $\sigma_S \in \text{IC}(S, V'_{S,\sigma})$ for each coalition S different from the grand coalition. Also, the inner core is not the maximal solution to satisfy OPR and CONS$'$ on the class of convex-valued games. I refer again to the game V and the payoff vector σ defined in Example 2. Consider for instance the solution Σ that coincides with the inner core for all convex-valued games except $(\{1, 2, 3\}, V)$ where the solution is $\{\sigma\}$. It is easy to check that Σ satisfies both OPR and CONS$'$. Yet it is larger than the inner core.

I proposed a first axiomatization of the inner core in de Clippel (2002). This previous result is not related at all to Proposition 2. Indeed, it showed that the inner core is a natural extension of the core defined for TU-games to some class of convex-valued NTU-games, by adapting Aumann's (1985) axiomatization of the Shapley (1969) NTU value. The key properties were the conditional sure-thing and the conditional decreasingness axioms. They linked the solution of NTU-games with the solution of some supporting TU-games. The key properties in Proposition 2 are the consistency and the converse consistency axioms. They link coalitional stability with one-person rationality. This approach leads to a full axiomatization of the inner core that does not impose on the solution to coincide with the (inner) core on the class of TU-games.

Finally, it is relevant to note that Peleg (1985) shows that his converse reduced game property is redundant for his axiomatic result, if the set of potential players is infinite (see his Section 7.1). I do not know whether a similar conclusion can be reached for Proposition 2.

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Appendix A

Lemma 2. *(Intermediate Value Theorem for Correspondences) Let β be a real number and let $F : [a, c] \to \mathbb{R}$ be a correspondence with non-empty convex values and a compact graph. If there exists a couple $(\alpha, \gamma) \in F(a) \times F(c)$ such that $\alpha < \beta < \gamma$, then there exists $b \in [a, c]$ such that $\beta \in F(b)$.*

Proof. The set $S := \{x \in [a, c] | (\exists \chi \in F(x)) : \chi \leq \beta\}$ is non-empty (as it contains a) and bounded above by c. Let b be its supremum. I show that $\beta \in F(b)$. Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in S such that $x_k \to b$ and let $(\chi_k)_{k \in \mathbb{N}}$ be a sequence such that $\chi_k \leq \beta$ and $\chi_k \in F(x_k)$ for each $k \in \mathbb{N}$. I may assume without loss of generality (because F has a compact graph) that $\chi_k \to \chi$ for some $\chi \in F(b)$. Notice that $\chi \leq \beta$. Let $(\hat{x}_k)_{k \in \mathbb{N}}$ be a sequence in $[b, c]$ such that $\hat{x}_k \to b$ and let $(\hat{\chi}_k)_{k \in \mathbb{N}}$ be a sequence such that $\hat{\chi}_k \geq \beta$ and $\hat{\chi}_k \in F(x_k)$ for each $k \in \mathbb{N}$. I may assume without loss of generality (because F has a compact graph) that $\hat{\chi}_k \to \hat{\chi}$ for some $\hat{\chi} \in F(b)$. Notice that $\hat{\chi} \geq \beta$. In addition, $F(b)$ is a convex set. Hence $\beta \in F(b)$. □

Let V be a compact and convex subset of $\mathbb{R}^N$, let $i \in N$ and let $r \in \mathbb{R}$. Let $f : \mathbb{R}^{N\setminus\{i\}} \to \mathbb{R}^N$ be the function defined as follows: $f_i(\phi) := r$ and $f_j(\phi) := \phi_j$ if $j \in N \setminus \{i\}$, for each $\phi \in \mathbb{R}^{N\setminus\{i\}}$. Let U be the slice of V obtained by intersecting V with the set of vectors in $\mathbb{R}^N$ whose ith component equals r:

$$U := \{\phi \in \mathbb{R}^{N\setminus\{i\}} | f(\phi) \in V\}.$$

It is a convex set. The next proposition states that, under mild conditions, any vector λ that is orthogonal to U at some vector ϕ^* can be extended into a vector that is orthogonal to V at $f(\phi^*)$.

Lemma 3. *Let ϕ^* be a vector that belongs to the boundary of U and let $\lambda \in \mathbb{R}^{N\setminus\{i\}} \setminus \{0\}$ be a vector that is orthogonal to U at ϕ^*. If there exists a couple $(v, v') \in V \times V$ such that $v_i < r < v'_i$, then there exists $\mu \in \mathbb{R}^N$ that is orthogonal to V at $f(\phi^*)$ such that $\mu_{N\setminus\{i\}} = \lambda$.*

Proof. Let $\mu : \mathbb{R} \to \mathbb{R}^N$ be the function defined as follows: $\mu_i(x) := x$ and $\mu_j(x) := \lambda_j$ for each $j \in N \setminus \{i\}$ and each $x \in \mathbb{R}$. Let $F : \mathbb{R} \to \mathbb{R}$ be the correspondence defined as follows:

$$F(x) := \text{proj}_{\mathbb{R}^{\{i\}}}\{\chi \in V | \mu(x) \text{ is orthogonal to} V \text{at} \chi\}$$

for each $x \in \mathbb{R}$. Observe that F has non-empty convex values. Also, I have that

$$\sum_{j \in N} \mu_j(x) v_j \leq \sum_{j \in N} \mu_j(x) \chi_j \leq x\chi_i + \max_{u \in V} \sum_{j \in N\setminus\{i\}} \lambda_j u_j$$

which implies that

$$x(v_i - \chi_i) \leq \max_{u \in V} \sum_{j \in N\setminus\{i\}} \lambda_j (u_j - v_j)$$

for each $x \in \mathbb{R}$ and each $\chi \in V$ such that $\mu(x)$ is orthogonal to V at χ. Hence, there exists $\underline{x} \in \mathbb{R}_-$ and $\chi_i \in F(\underline{x})$ such that $\chi_i < r$. Similarly,

$$x(v'_i - \chi_i) \le \max_{u \in V} \sum_{j \in N\setminus\{i\}} \lambda_j(u^*_j - v'_j)$$

for each $x \in \mathbb{R}$ and each $\chi \in V$ such that $\mu(x)$ is orthogonal to V at χ. Hence, there exists $\bar{x} \in \mathbb{R}_+$ and $\chi_i \in F(\bar{x})$ such that $r < \chi_i$. In addition, the graph of the correspondence F restricted to $[\underline{x}, \bar{x}]$ is compact. By Lemma A.1, there exists $\hat{x} \in [\underline{x}, \bar{x}]$ such that $r \in F(\hat{x})$. Hence, there exists $\hat{\phi} \in U$ such that $\mu(\hat{x})$ is orthogonal to V at $f(\hat{\phi})$. This implies that $\mu(\hat{x})$ is orthogonal to V at $f(\phi^*)$, as $\sum_{j \in N}\mu_j(\hat{x})f_j(\hat{\phi}) = \mu_i(\hat{x})r + \sum_{j \in N\setminus\{i\}}\lambda_j\hat{\phi}_j \le \mu_i(\hat{x})r + \sum_{j \in N\setminus\{i\}}\lambda_j\phi^*_j = \sum_{j \in N}\mu_j(\hat{x})f_j(\phi^*)$. □

Lemma A.2 is not valid without the existence of a couple $(v, v') \in V \times V$ such that $v_i < r < v'_i$, as the next example shows.

Example 3. The set $V := \{v \in \mathbb{R}^N | \sum_{j \in N} v_j^2 \le 1\}$ is compact and convex. Let $i \in N$, let $r := 1$ and let $\phi^* := 0 \in \mathbb{R}^{S\setminus\{i\}}$. Notice that $U = \{0\}$ and so $\phi^* \in \partial U$. If a vector μ is orthogonal to V at $f(\phi^*)$, then $\mu_{N\setminus\{i\}} = 0$. Hence it is impossible to extend any vector λ orthogonal to U at ϕ^* into some vector μ orthogonal to V at $f(\phi^*)$.

Lemma 4. *If u is efficient in V, V is smooth at u, and there exists a couple $(v, v') \in V \times V$ such that $v_i < u_i < v'_i$, then U is smooth at $u_{N\setminus\{i\}}$.*

Proof. Let μ be the unique normalized vector that is orthogonal to V at u. By Lemma A.2, any vector λ that is orthogonal to U at $u_{N\setminus\{i\}}$ must be proportional to $\mu_{N\setminus\{i\}}$. Hence U is smooth at $u_{N\setminus\{i\}}$. □

Example A.1 shows that Lemma A.3 is not valid without the existence of a couple $(v, v') \in V \times V$ such that $v_i < u_i < v'_i$.

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