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An axiomatization of the Nash bargaining solution

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Abstract I prove that 'Disagreement Point Convexity' and 'Midpoint Domination' characterize the Nash bargaining solution on the class of two-player bargaining problems and on the class of smooth bargaining problems. I propose an example to show that these two axioms do not characterize the Nash bargaining solution on the class of bargaining problems with more than two players. I prove that the other solutions that satisfy these two properties are not lower hemi-continuous. These different results refine the analysis of Chun (Econ Lett 34:311–316, 1990). I also highlight a rather unexpected link with the result of Dagan et al. (Soc Choice Welfare 19:811–823, 2002).

1 Introduction

A bargaining problem specifies the set of utility vectors that are achievable if the players cooperate, as well as the utility vector that prevails in case of disagreement. Nash (1950) characterizes a unique single-valued solution that satisfies the following list of elementary properties or axioms: 'Efficiency' (EFF), 'Symmetry' (SYM), 'Scale Covariance' (SC) and 'Independence of Irrelevant Alternatives' (IIA).

'Midpoint Domination' (MD) requires that any reasonable agreement Pareto dominates the outcome of the random dictatorship procedure. This property appears in the literature at the beginning of the eighties. Moulin (1983) (see Thomson (1994), Sect. 4.1) proves that the Nash bargaining solution is the only solution to satisfy MD and IIA.

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IIA involves comparisons between bargaining problems with different feasible sets: the solution has to be invariant to some contractions of the set of achievable contracts. Chun and Thomson (1990) and Peters and van Damme (1991) propose a dual approach involving the disagreement point. Disagreement point convexity (DPC) for instance requires the solution to be invariant to movements of the disagreement point towards the solution. Peters and van Damme prove that the Nash bargaining solution is the only single-valued solution to satisfy SYM, SC, DPC, 'Strong Individual Rationality', 'Disagreement Point Continuity' and 'Invariance with respect to Non-Individually Rational Alternatives' (INIR).

I prove that the Nash bargaining solution is the only solution to satisfy MD and DPC on the class of two-player bargaining problems and on the class of smooth bargaining problems. I propose an example to show that these two axioms do not characterize the Nash bargaining solution on the class of bargaining problems with more than two players. I prove that the other solutions that satisfy these two properties are not lower hemi-continuous.

These results imply that the theorems of Chun (1990) can be extended to multi-valued solutions and that his efficiency axiom is redundant. More interestingly, his continuity axiom also appears to be redundant on the two important sub-classes of bargaining problems discussed previously. I also highlight a rather unexpected link with the result of Dagan et al. (2002). They prove, as a variant of Peters and van Damme (1991), that the Nash bargaining solution is the only solution to satisfy SYM, EFF, SC, INIR, DPC, 'Single-Valuedness in Symmetric Problems' (SV) and 'Twisting' (TW) on the class of two-player bargaining problems.[1] I prove that SYM, EFF, SC, INIR, SV and TW together imply MD. I also show by means of examples that the converse is not true.

2 Definitions

Let n be a positive integer and let $N := \{1, \ldots, n\}$ be the set of players. Vectors in $\mathbb{R}^N$ are compared according to Pareto's ordering: $x \geq y$ if $x_i \geq y_i$ for each $i \in N$, and $x >> y$ if $x_i > y_i$ for each $i \in N$. The vector x is strictly positive if $x >> 0$. The set of strictly positive vectors is denoted by $\mathbb{R}^N_{++}$. The inner product of two vectors x and y is denoted by $x \cdot y$ $(= \sum_{i \in N} x_i y_i)$. Let X be a subset of $\mathbb{R}^N$. Then $x \in X$ is *efficient* (in X) if there does not exist $x' \in X$ different from x such that $x' \geq x$. A vector $\lambda \in \mathbb{R}^N$ is *orthogonal* to X at a vector x^* of its boundary if $\lambda \cdot x \leq \lambda \cdot x^*$, for each $x \in X$.

A *bargaining problem* is a couple (V, d) where $V \subseteq \mathbb{R}^N$ is the set of utility vectors that the players can achieve through cooperation and $d \in V$ is the utility vector that prevails in case of disagreement. The players have a strict interest to cooperate ($d << v$ for some $v \in V$) and the set V of feasible agreements is

[1] To be precise, Dagan et al. consider a class of bargaining problems that is slightly larger than the class of bargaining problems considered in the present paper. Indeed, they do not require that the utilities are transferable in the sense of condition (1), introduced in the next section.

compact, convex and non-level. *Non-level* means that the utilities are transferable in the following sense:[2]

$$[v_i > d_i] \rightarrow [(\exists v' \in V)(\forall j \in N \backslash \{i\}) : v'_j > v_j] \tag{1}$$

for each $v \in V$ and each $i \in N$. If the agreement v is strictly better than the disagreement outcome for player i, then there exists another feasible agreement v' that makes all the other players strictly better off, obviously at the expense of player i if v is efficient. A utility vector v is *individually rational* if $v \geq d$. It is *strictly individually rational* if $v >> d$. Condition (1) implies that, if a vector λ is orthogonal to V at an efficient utility vector that is strictly individually rational, then λ is strictly positive. A bargaining problem (V, d) is *smooth* if V admits a unique supporting hyperplane at each utility vector on its boundary.

A *solution* is a function that associates a nonempty set of feasible utility vectors to each bargaining problem. For instance, the *Nash bargaining solution* selects the (unique) utility vector that maximizes the product of the utility gains of the players over the set of feasible agreements that are individually rational, i.e.

$$\Sigma_{\text{Nash}}(V, d) := \arg \max_{\{v \in V | v \geq d\}} \prod_{i \in N} (v_i - d_i). \tag{2}$$

The Nash bargaining solution is single-valued, as the sets of achievable utility vectors are convex.

3 Main results

Here are restrictions that could be imposed on a solution Σ. They are supposed to hold for each bargaining problem (V, d), for each two-player bargaining problem (V, d), or for each smooth bargaining problem (V, d), depending on the context.

Disagreement point convexity (DPC) If $\sigma \in \Sigma(V, d)$, then $\sigma \in \Sigma(V, \pi d + (1 - \pi)\sigma)$ for each $\pi \in]0, 1[$.

If σ is a reasonable agreement for a bargaining problem (V, d), then so should it remain for the modified problem $(V, \pi d + (1 - \pi)\sigma)$. The solution has to be invariant to movements of the disagreement point towards the final agreement. The axiom was first introduced by Peters and van Damme (1991). Chun and Thomson (1990) apply a similar axiom imposing some linearity on the solution with respect to the disagreement point. I refer to these papers for justifications in terms of invariance with respect to resolution of the uncertainty about the disagreement point. Dagan et al. (2002) relate the axiom to properties of some non-cooperative bargaining models.

[2] This differs from the definition of transferable utility in cooperative games where it is actually assumed that the utilities are transferable between the players at a one to one exchange rate.

Midpoint domination (MD) If $\sigma \in \Sigma(V,d)$, then $\sigma \geq \sum_{i \in N} a^i(V,d)/n$, where $a^i(V,d) := \arg\max_{\{v \in V | v_{-i} \geq d_{-i}\}} v_i$, for each $i \in N$.

If player i had all the bargaining power, the outcome would be $a^i(V,d)$. A natural reference point when the players have equal bargaining abilities is obtained by mixing these extreme points, using a uniform probability distribution. It is only because of its lack of efficiency that the random dictatorship principle has to be amended. The objective of most bargaining solutions is to specify a way to split the remaining surplus when the midpoint is not efficient. For instance, the discrete Raiffa bargaining solution (see Luce and Raiffa 1957, pages 136–137) considers the midpoint as a partial agreement and applies recursively the random dictatorship argument up to the Pareto frontier. The solution of Kalai and Smorodinsky (1975) follows the direction defined by the disagreement point and the midpoint up to the Pareto frontier. I prove hereafter that the Nash solution itself satisfies the axiom. Though natural and powerful, MD has not been used frequently in the literature. The only references that I am aware of are Moulin (1983) and Chun (1990).

Lower hemi-continuity (LHC) Let $\sigma \in \Sigma(V,d)$ and let $[(V^k, d^k)]_{k \in \mathbb{N}}$ be a sequence of bargaining problems that converges[3] towards the bargaining problem (V,d) when k tends to infinity. Then, there exists a sequence $(\sigma^k)_{k \in \mathbb{N}}$ of vectors in $\mathbb{R}^N$ that converges towards σ and such that $\sigma^k \in \Sigma(V^k, d^k)$ for each $k \in \mathbb{N}$.

If σ is a reasonable agreement for the bargaining problem (V,d) and if the bargaining problem (V^k, d^k) is close enough to the bargaining problem (V,d), then there exists a reasonable agreement σ^k for (V^k, d^k) that is close to σ.

Theorem 1 *The Nash bargaining solution is the only solution to satisfy 'Disagreement Point Convexity' (DPC) and 'Midpoint Domination' (MD) on the class of two-player bargaining problems.*

Theorem 2 *The Nash bargaining solution is the only solution to satisfy 'Disagreement Point Convexity' (DPC) and 'Midpoint Domination' (MD) on the class of smooth bargaining problems.*

Theorem 3 *The Nash bargaining solution is the only solution to satisfy 'Disagreement Point Convexity' (DPC), 'Midpoint Domination' (MD) and 'Lower Hemi-Continuity' (LHC) on the class of bargaining problems.*

The proof of these three theorems is based on the following characterization of the Nash bargaining solution, suggested by Harsanyi (1959). It is obtained by separating V from the set of utility vectors whose Nash product is greater than its evaluation at the solution. Both sets are indeed convex.

[3] I.e. the sequence $(d^k)_{k \in \mathbb{N}}$ converges towards d in the usual sense and the sequence $(V^k)_{k \in \mathbb{N}}$ converges towards V according to Hausdorff's topology.

Lemma (Harsanyi 1959) *Let (V,d) be a bargaining problem and let $\sigma \in V$. Then, $\Sigma_{Nash}(V,d) = \{\sigma\}$ if and only if $\sigma >> d$ and there exists $\lambda \in \mathbb{R}^N_{++}$ such that*

$$\text{(a)} \quad \lambda \cdot \sigma = \max_{v \in V} \lambda \cdot v$$

$$\text{(b)} \quad (\forall (i,j) \in N \times N) : \lambda_i(\sigma_i - d_i) = \lambda_j(\sigma_j - d_j).$$

The lemma can be interpreted as follows: the Nash bargaining solution specifies the only feasible agreement that satisfies simultaneously the utilitarian and the egalitarian objectives for some re-scaling of the individual utilities (see Shapley 1969; Myerson 1991, Sect. 8.3).

Proof of Theorem 1 I start by proving that the Nash bargaining solution satisfies both DPC and MD. Let (V,d) be a two-player bargaining problem, let $\Sigma_{Nash}(V,d) = \{\sigma\}$, let $\lambda \in \mathbb{R}^N_{++}$ be the vector appearing in the lemma, and let $V^\lambda := \{x \in \mathbb{R}^N | \lambda \cdot x \leq \lambda \cdot \sigma\}$. Observe that

$$\frac{a^1(V^\lambda,d) + a^2(V^\lambda,d)}{2} = \left(\frac{d_1}{2} + \frac{\lambda \cdot \sigma - \lambda_2 d_2}{2\lambda_1}, \frac{d_2}{2} + \frac{\lambda \cdot \sigma - \lambda_1 d_1}{2\lambda_2}\right) = \sigma. \quad (3)$$

The second equality follows from condition b of the lemma. On the other hand, $V \subseteq V^\lambda$, thanks to condition a of the lemma. Hence $a^1(V,d) \leq a^1(V^\lambda,d)$, $a^2(V,d) \leq a^2(V^\lambda,d)$, and the Nash bargaining solution satisfies MD. Let now $\pi \in]0,1[$. Clearly, σ satisfies condition a of the lemma for λ and for the bargaining problem $(V, \pi d + (1-\pi)\sigma)$, as the condition does not depend on the disagreement point. On the other hand, $\lambda_i(\sigma_i - (\pi d_i + (1-\pi)\sigma_i)) = \pi\lambda_i(\sigma_i - d_i)$ for each $i \in N$. Hence σ also satisfies condition a of the lemma for λ and for the bargaining problem $(V, \pi d + (1-\pi)\sigma)$. The lemma implies that $\Sigma_{\text{Nash}}(V, \pi d + (1-\pi)\sigma) = \{\sigma\}$. Hence the Nash bargaining solution satisfies DPC.

Let Σ be a solution that satisfies both DPC and MD, let (V,d) be a two-player bargaining problem and let $\sigma \in \Sigma(V,d)$. I will use the lemma to prove that $\sigma \in \Sigma_{\text{Nash}}(V,d)$. This will imply that $\Sigma(V,d) = \Sigma_{\text{Nash}}(V,d)$, as $\Sigma(V,d)$ is nonempty and the Nash bargaining solution is single-valued. MD implies that $\sigma >> d$, as there exists $v \in V$ such that $v >> d$. Let $\lambda := (\sigma_2 - d_2, \sigma_1 - d_1) \in \mathbb{R}^2_{++}$. Condition b of the lemma is trivially satisfied. Let $v \in V$. I prove that $\lambda \cdot v \leq \lambda \cdot \sigma$. The result is obvious if $v \leq \sigma$. Observe also that σ is efficient. Otherwise, the condition $\sigma \in \Sigma(V, \pi d + (1-\pi)\sigma)$ imposed by DPC is incompatible with MD when π is small. Hence there remain two cases to consider: (1) $v_1 < \sigma_1$ and $v_2 > \sigma_2$; (2) $v_1 > \sigma_1$ and $v_2 < \sigma_2$. The analysis being similar in both cases, I assume that $v_1 < \sigma_1$ and $v_2 > \sigma_2$. Suppose in addition for the moment that $v_1 > d_1$. Let $\pi := (\sigma_1 - v_1)/(\sigma_1 - d_1) \in]0,1[$, so that $\pi d_1 + (1-\pi)\sigma_1 = v_1$. By DPC, $\sigma \in \Sigma(V, \pi d + (1-\pi)\sigma)$. Notice that $a^2(V, \pi d + (1-\pi)\sigma) \geq v$. Hence $\sigma_2 \geq [\pi d_2 + (1-\pi)\sigma_2 + v_2]/2$, by MD. Developing the terms, I get $v_2 - \sigma_2 \leq \pi(\sigma_2 - d_2)$. Given the definitions of λ and π, this amounts to $\lambda_2(v_2 - \sigma_2) \leq \lambda_1(\sigma_1 - v_1)$, or $\lambda \cdot v \leq \lambda \cdot \sigma$, as required.

If $v_1 \leq d_1$, then there exists $\alpha \in]0,1[$ such that $\alpha v_1 + (1-\alpha)\sigma_1 > d_1$. Hence, by the previous reasoning, $\lambda \cdot (\alpha v + (1-\alpha)\sigma) \leq \lambda \cdot \sigma$. Rearranging the terms, I obtain again $\lambda \cdot v \leq \lambda \cdot \sigma$. □

Proof of Theorem 2 It is easy to extend the arguments developed in the Proof of Theorem 1 to show that the Nash bargaining solution satisfies both DPC and MD on the class of smooth bargaining problems.

Let Σ be a solution that satisfies both DPC and MD, let (V,d) be a smooth bargaining problem and let $\sigma \in \Sigma(V,d)$. I will use the lemma to prove that $\sigma \in \Sigma_{\mathrm{Nash}}(V,d)$. This will imply that $\Sigma(V,d) = \Sigma_{\mathrm{Nash}}(V,d)$, as $\Sigma(V,d)$ is nonempty and the Nash bargaining solution is single-valued. MD implies that $\sigma >> d$, as there exists $v \in V$ such that $v >> d$. Observe also that σ is efficient. Otherwise, the condition $\sigma \in \Sigma(V, \pi d + (1-\pi)\sigma)$ imposed by DPC is incompatible with MD when π is small. Let $\lambda \in \mathbb{R}^N_{++}$ be a vector that is orthogonal to V at σ. Condition a of the lemma is trivially satisfied. I check condition b. Combining DPC and MD, I have

$$\sigma \geq \sum_{j \in N} \frac{a^j(V, \pi d + (1-\pi)\sigma)}{n} \tag{4}$$

for each $\pi \in [0,1]$. Notice that $a^j_i(V, \pi d + (1-\pi)\sigma) = \pi d_i + (1-\pi)\sigma_i$ for all $(i,j) \in N \times N$ such that $i \neq j$. Equation (4) therefore implies

$$(n-1)(\sigma_i - d_i) \geq \frac{a^i_i(V, \pi d + (1-\pi)\sigma) - \sigma_i}{\pi} \tag{5}$$

for each $i \in N$ and each $\pi \in [0,1]$. Remember that V is smooth and that λ is orthogonal to V at σ. The implicit function theorem implies that $a^i(V,x)$ is differentiable with respect to x_j at σ and that $\partial a^i_i(V,\sigma)/\partial x_j = -\lambda_j/\lambda_i$ for each $j \in N\backslash\{i\}$. Taking the limit as π tends to zero and applying the chain rule of differential calculus, I obtain

$$(n-1)(\sigma_i - d_i) \geq \sum_{j \in N\backslash\{i\}} \frac{\lambda_j(\sigma_j - d_j)}{\lambda_i} \tag{6}$$

for each $i \in N$. Hence,

$$\lambda_i(\sigma_i - d_i) \geq \frac{\lambda \cdot (\sigma - d)}{n} \tag{7}$$

for each $i \in N$. Taking the sum over i, I conclude that the inequality cannot be strict and hence the left-hand term does not depend on i. Condition b of the lemma is therefore satisfied. □

Proof of Theorem 3 It is easy to extend the arguments developed in the proof of theorem 1 to show that the Nash bargaining solution satisfies both DPC and

MD on the class of bargaining problems with any number of players. It is also easy to check that the Nash bargaining solution is lower hemi-continuous.[4]

Let Σ be a solution that satisfies DPC, MD, and LHC. Let (V, d) be a bargaining problem and let $\sigma \in \Sigma(V, d)$. I will use Theorem 2 and LHC to prove that $\sigma \in \Sigma_{\text{Nash}}(V, d)$. This will imply that $\Sigma(V, d) = \Sigma_{\text{Nash}}(V, d)$, as $\Sigma(V, d)$ is nonempty and the Nash bargaining solution is single-valued. As V is compact, there exists a real number r large enough such that V is included in the ball $W := \{x \in \mathbb{R}^N \,|\, \|x - d\|_2 \leq r\}$ of center d and radius r. Let V^k be the set defined as follows:

$$V^k := \frac{1}{k} W + \left(1 - \frac{1}{k}\right) V, \tag{8}$$

for each positive integer k. Notice that (V^k, d) is a smooth bargaining problem, for each $k \in \mathbb{N}$, and that the sequence $[(V^k, d)]_{k \in \mathbb{N}}$ converges towards (V, d) when k tends to infinity. By LHC, there exists a sequence $(\sigma^k)_{k \in \mathbb{N}}$ that converges towards σ and such that $\sigma^k \in \Sigma(V^k, d)$, for each $k \in \mathbb{N}$. Theorem 2 implies that $\sigma^k \in \Sigma_{\text{Nash}}(V^k, d)$, for each $k \in \mathbb{N}$. Hence $\sigma \in \Sigma_{\text{Nash}}(V, d)$, as the Nash bargaining solution is (upper hemi-) continuous. □

The Nash bargaining solution does not necessarily satisfy MD if the utilities are not transferable in the sense of condition (1). For instance, let $n = 2$, let $d = (0, 0)$ and let V be the convex hull of the vectors $(0, 0)$, $(1, 1)$ and $(2, 0)$. The Nash bargaining solution is $\{(1, 1)\}$, but $[a^1(V, d) + a^2(V, d)]/2 = (3/2, 1/2)$. Notice that Chun (1990) considers a variant of MD where $a^i(V, d)$ is replaced by:

$$\hat{a}^i(V, d) := \arg \max_{\{v \in V | v_{-i} = d_{-i}\}} v_i \tag{9}$$

for each $i \in N$. Obviously, $a^i(V, d) = \hat{a}^i(V, d)$ when V is non-level. On the other hand, the Nash bargaining solution satisfies the modified version of MD even if the utilities are not transferable in the sense of condition (1). It is easy to adapt the three theorems to this larger class of bargaining problems, replacing MD by its modified version. Nevertheless, $\hat{a}^i(V, d)$ is hard to interpret, as it does not necessarily coincide with the agreement that player i would impose should he have all the bargaining power.

The axioms appearing in the three theorems are independent. The egalitarian solution satisfies DPC and LHC, but not MD. The solution of Kalai and Smorodinsky (1975) satisfies MD and LHC, but not DPC. Finally, the Nash bargaining solution is not the only solution to satisfy DPC and MD on the class of bargaining problems when there are more than two players, as the following example shows. Let $n = 3$, let $\hat{V}$ be the convex hull of the vectors $(0, 0, 0)$, $(15, 0, 0)$, $(0, 15, 0)$, $(10, 10, 0)$ and $(0, 0, 10)$, and let Σ be the solution defined as follows: $\Sigma(V, d) := \Sigma_{\text{Nash}}(V, d) \cup \{(5, 5, 5)\}$ if $(V, d) = (\hat{V}, \pi(0, 0, 0) + (1 - \pi)(5, 5, 5))$ for some $\pi \in [0, 1]$ and $\Sigma(V, d) := \Sigma_{\text{Nash}}(V, d)$ for each other

4 Or simply a continuous function, as the Nash bargaining solution is single-valued.

bargaining problem (V, d). Notice that $(5,5,5) \notin \Sigma_{\text{Nash}}(V, \pi(0,0,0) + (1-\pi)(5,5,5))$ for each $\pi \in]0,1[$. The solution Σ obviously satisfies DPC. It also satisfies MD because $(5,5,5) \geq \sum_{i=1}^{3} a^i(\hat{V}, \pi(0,0,0)+(1-\pi)(5,5,5))/3$ for each $\pi \in [0,1]$. Indeed, $a^1 = (5+10\pi, 5(1-\pi), 5(1-\pi))$, $a^2 = (5(1-\pi), 5+10\pi, 5(1-\pi))$ and $a^3 = (5(1-\pi), 5(1-\pi), 10-5(1-\pi))$.

4 Related literature

I already discussed part of the literature in the introduction. I further discuss in this section how my results relate to Chun (1990, Theorems 1 and 3) and to Dagan et al. (2002).

Theorem 3 shows that Chun's results can be extended to multi-valued[5] solutions and that his efficiency axiom is redundant. More interestingly, Theorems 1 and 2 show that his continuity axiom is also redundant on two important sub-classes of bargaining problems. To the best of my knowledge, Theorems 1 and 2 are the only axiomatizations of the Nash bargaining solution that do not involve any axiom relating the solution of bargaining problems with different feasible sets.

I devote the rest of the section to explain how theorem 1 relates to a previous result obtained by Dagan et al. (2002). They axiomatize the Nash bargaining solution on a large[6] class of two-player bargaining problems. Here are the axioms they consider, in addition to DPC. They are supposed to hold for each two-player bargaining problem (V, d).

Symmetry (SYM) A bargaining problem (V, d) is symmetric if $d_1 = d_2$ and $V = \{(v_2, v_1) | v \in V\}$. If (V, d) is symmetric, then $\Sigma(V, d) = \{(\sigma_2, \sigma_1) | \sigma \in \Sigma(V, d)\}$.

Efficiency (EFF) Any element of $\Sigma(V, d)$ is efficient.

Scale covariance (SC) $\Sigma(\alpha V + \beta) = \alpha \Sigma(V, d) + \beta$, for each $\alpha \in \mathbb{R}^2_{++}$ and each $\beta \in \mathbb{R}^2$.

Single-valuedness in symmetric problems (SV) If (V, d) is symmetric, then $\Sigma(V, d)$ is a singleton.

Independence of non-individually rational alternatives (INIR) If a bargaining problem (V', d) is such that $\{v \in V' | v \geq d\} = \{v \in V | v \geq d\}$, then $\Sigma(V', d) = \Sigma(V, d)$.

Twisting (TW) Let $\sigma \in \Sigma(V, d)$, let $i \in \{1, 2\}$ and let (V', d) be a bargaining problem such that $V \setminus V' \subseteq \{x \in \mathbb{R}^2 | x_i > \sigma_i\}$ and $V' \setminus V \subseteq \{x \in \mathbb{R}^2 | x_i < \sigma_i\}$. Then, $\sigma'_i \leq \sigma_i$ for some $\sigma' \in \Sigma(V', d)$.

[5] On the contrary, the restriction to single-valued solutions is crucial in Nash (1950) and Moulin (1983). For instance, the solution that associates to every bargaining problem the set of efficient allocations that Pareto dominate the midpoint satisfies the natural adaptation of Nash and Moulin's axioms when multi-valued solutions are considered.

[6] Their class of games is larger than the one considered in the present paper, as they do not restrict the feasible sets to be non-level.

The next proposition shows that these six axioms together imply MD, the converse being not true.

Proposition *SYM, EFF, SC, SV, INIR and TW together imply MD on the class of two-player bargaining problems. On the other hand, MD does not imply any of the six previous axioms.*

Proof Let (V, d) be a two-player bargaining problem and let $\sigma \in \Sigma(V, d)$. Suppose that $\sigma \neq a^1(V, d)$. Let V' be the set of individually rational utility pairs that is bounded above by the line going through the points σ and $a^1(V, d)$. By the first four axioms, $\Sigma(V', d) = \{(\frac{d_1 + a_1^1(V,d)}{2}, x)\}$ for some $x \in \mathbb{R}$. By INIR, $\sigma \in \Sigma(\hat{V}, d)$ where $\hat{V}$ is the set of individually rational utility pairs that belong to V. By TW, $\sigma_1 \geq (d_1 + a_1^1(V, d))/2$. A similar argument implies that $\sigma_2 \geq (d_2 + a_2^2(V, d))/2$ if $\sigma \neq a^2(V, d)$. It is then easy to conclude.

Here are examples of solutions that satisfy MD but violate some of the above axioms. The solution that is obtained by projecting vertically (upward) the midpoint on the Pareto frontier of the feasible set does not satisfy SYM, the solution that selects the midpoint does not satisfy EFF, the solution that equally split the surplus above the midpoint does not satisfy SC, the solution that selects the set of efficient utility vectors that dominates the midpoint does not satisfy SV, the solution that maximizes the product of the utility gains with respect to the downward vertical projection of the disagreement point on the boundary of the feasible set and over the set of feasible utility vectors that Pareto dominate the midpoint does not satisfy INIR, and finally the solution that selects the utility vectors on the Pareto frontier that are the closest to the midpoint does not satisfy TW. □

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