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JOURNAL OF Mathematical ECONOMICS
Journal of Mathematical Economics 40 (2004) 799–813
www.elsevier.com/locate/jmateco

Two-person bargaining with verifiable information

Geoffroy de Clippel [a], Enrico Minelli [b,*]

[a] Aspirant FNRS, CORE, Université catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium
[b] Chercheur qualifié FNRS, CORE, Université catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium

Received 29 January 2003; accepted 8 July 2003
Available online 18 January 2004

Abstract

We study Myerson's incomplete information bargaining solution under the assumption of verifiable types. For the case of an informed principal, in which one individual has all the bargaining power, we provide exact characterizations both from the non-cooperative and from the cooperative perspective. We then show that the axiomatic characterization can be extended to the case in which both individuals have some bargaining power. The 'contract curve' is obtained by varying the relative bargaining power of the players. This new solution concept refines Wilson's coarse core by taking into account important aspects of negotiation at the interim stage.

Keywords: Bargaining; Verifiable information

1. Introduction

A two-person bargaining problem with incomplete information is a collection

$$G = (D, d^*, T_1, T_2, u_1, u_2, q).$$

The set D is the set of joint decisions. The set T_i is the set of possible types of individual i. If we let $T = T_1 \times T_2$ be the set of possible states, then $q \in \Delta(T)$ denotes the common prior that determines the interim beliefs of the individuals. Without loss of generality we assume that for $i = 1, 2$ and all $t_i \in T_i$, $q(t_i) > 0$. Utility functions are: $u_i : D \times T \to R$. We assume that the disagreement outcome d^* gives zero utility to each individual in each state: $u_i(d^*, t) = 0$. A mechanism is $\mu : T \to D$.

* Corresponding author. Tel.: +32-10-47-43-21; fax: +32-10-47-43-01.
E-mail addresses: declippel@core.ucl.ac.be (G. de Clippel), minelli@core.ucl.ac.be (E. Minelli).

0304-4068/$ – see front matter
doi:10.1016/j.jmateco.2003.07.001

A theory of bargaining should specify, for each possible problem G a set of mechanisms which represent reasonable agreements, given the individuals' bargaining power. For the case of complete information, when the type set of each individual is a singleton, the problem is well understood (Nash, 1950). The presence of asymmetric information raises two distinct conceptual issues. On one hand, when individuals have different information at the stage of implementation of the mechanism, they might strategically manipulate the terms of the agreement. Attention should therefore be restricted to mechanisms which are incentive compatible. If bargaining takes place at the interim stage, when individuals already know their type, an additional problem arises, due to the fact that any action may reveal some private information. This second problem has received a lot of attention in the literature on competitive equilibria with rational expectations (Radner, 1979), but much less in the context of cooperative games with incomplete information. In two seminal papers, Myerson (1983, 1984) proposed solutions for two-person bargaining problems with incomplete information which take into account both aspects. The first paper deals with the case in which one individual has all the bargaining power, the second with a situation in which the two players have equal bargaining abilities.

In this note, we focus on the revelation of private information at the bargaining stage. To do this, we assume that when agreements are implemented each individual can costlessly verify the true state. Types are thus verifiable in a particularly strong sense: all information is public at the implementation stage. For example, one individual may be a manufacturer with information on aggregate demand, and the other a retailer with information on local demand. Or, one may be a firm with private information over its future profits and the other an uninformed investor. More generally, two individuals are bargaining over an asset, each one may have some private information on the value of the asset at the time of bargaining, and contracts can be contingent on the value, which will be observable at the time of implementation.

Under complete information, the relevance of Nash's solution to the bargaining problem is reinforced by the fact that it can be justified by three different points of view. First, one can interpret it constructively, as in Shapley's (1969) procedure: the solution is obtained by considering a family of supporting linear games that are easier to solve. Second, one can provide an axiomatic justification, as in Nash's (1950) paper. Finally, the solution can be characterized as the 'good' equilibrium outcome of some explicit bargaining procedure (Nash, 1953). Myerson's objective is to extend this program to the case of incomplete information. By introducing the idea of 'virtual utility', he generalizes Shapley's (1969) procedure. He then obtains a partial axiomatization, showing that his solution is the minimal one to satisfy a set of axioms. For the case in which one individual has all the bargaining power, he also studies a specific bargaining procedure, and shows that his solution can be sustained as an equilibrium outcome. He also discusses equilibrium refinements, but does not obtain a non-cooperative characterization of his solution.

In the more specialized setting with verifiable types, we are able to further pursue this program. The ex-post 'virtual utility' of a player reduces in this case to a rescaling of his true utility and one obtains a very simple expression for Myerson's solution. For any possible distribution of bargaining power between the two players we obtain a full axiomatization of the solution in terms of a system of axioms that, beyond independence of irrelevant alternatives, includes its dual version: independence of irrelevant expansions. For the principal–agent case, we adapt the specific bargaining procedure proposed by Myerson and we exactly characterize the solution as the set of equilibrium allocations which are robust to objections by the principal, even when he has a substantial power to influence the out of equilibrium beliefs of the agent. We conclude our analysis of two-person bargaining by defining the ‘contract curve’ as the union of Myerson’s solutions over all the possible relative bargaining powers. This new solution concept refines Wilson’s (1978) coarse core. We explain on an example why allocations in the coarse core that do not belong to the contract curve cannot be considered as reasonable bargaining outcomes.

2. Model

With verifiable types, incentive constraints are irrelevant and all that we need to describe a bargaining problem is a set of achievable utility profiles for each possible state. The model can thus be simplified. For a given information structure (T_1, T_2, q), a *bargaining problem with verifiable types* is a correspondence $V : T \to R^2$ such that, for each $t \in T$, $V(t)$ is a closed, comprehensive, convex and non-level (each point on the frontier of $V(t)$ admits only strictly positive normal vectors) strict subset of R^2 which includes the disagreement payoff, $0 \in V(t)$. An *allocation rule* (for V) is a function $x : T \to R^2$ such that $x(t) \in V(t)$ for each $t \in T$. The players may agree on any allocation rule. If they fail to agree, then the disagreement outcome is enforced. A *solution* associates, to each bargaining problem V, a set $\sigma(V)$ of allocation rules for V.

3. Informed principal

Myerson (1983) proposed a solution for the case in which one individual, the principal, has all the bargaining power. The study of mechanism design by an informed principal has then been pursued by Maskin and Tirole (1990, 1992). For the case of verifiable types, we may consider the following extensive form game. First nature chooses the type of each player. Then player 1 proposes an allocation rule. Finally, player 2 chooses whether to accept player 1’s proposition. After player 2’s move, the state is revealed. If player 1’s proposal has been accepted the corresponding ex post allocation is realized, otherwise both players obtain their disagreement utility. For simplicity, we assume that player 2 accepts the proposal when he is indifferent between accepting and rejecting. A strategy for individual 1 is a proposal of an allocation rule. This choice is contingent on his type. A strategy for individual 2 is a choice of acceptance or rejection. This choice is contingent on his type and the proposed allocation rule. A belief for an individual is a probability distribution on the possible types of the other, conditional on his own type. A *weak sequential equilibrium* (WSE) (Myerson, 1991) is a profile of strategies and an updated belief for individual 2 such that:

(1) individual 2’s updated belief is *weakly consistent*, i.e. it is derived from his interim belief and individual 1’s proposal by applying Bayes rule when possible,
(2) individual 2’s participation decision is optimal given his updated belief,
(3) individual 1’s proposal is optimal given his interim belief and the strategy of individual 2.

Each profile of strategies generates an allocation rule. An *equilibrium allocation* is an allocation rule generated by a profile of strategies which is part of a weak sequential equilibrium.

An allocation rule x is *safe* if it is ex-post individually rational, i.e. $x(t) \geq 0$ for each $t \in T$. For instance, the constant allocation rule which gives the disagreement payoff in each state is safe. The *best safe* allocation rule is defined as $x^{\text{BS}}(t) = \arg\max\{v_1 | v \in V(t) \text{ and } v_2 \geq 0\}$, for each $t \in T$. In the next proposition we provide a characterization of the set of equilibrium allocations. A similar characterization was obtained by Maskin and Tirole (1992, Theorems 1 and 1*) without the assumption of verifiable types. With verifiable types there is no continuation game after the acceptance of the mechanism; this allows for a much simpler proof.

Proposition 1. *Let x be an allocation rule. Then, x is an equilibrium allocation if and only if $E(x_i|t_i) \geq E(x_i^{\text{BS}}|t_i)$, for each $t_i \in T_i$, $i = 1, 2$.*

Proof. If x is an equilibrium allocation, for each t_2, $E(x_2|t_2) \geq E(x_2^{\text{BS}}|t_2) = 0$, because player 2 can always refuse player 1's proposal. Also, if there exists t_1 such that $E(x_1|t_1) < E(x_1^{\text{BS}}|t_1)$, then x cannot be an equilibrium allocation. Indeed, individual 1 would have a profitable deviation in which he proposes x^{BS} when his type is t_1. In the other direction, let individual 1 propose, whatever his type, the allocation rule x. After observing x, the updated belief of individual 2 coincides with his interim belief and, using the fact that for each possible type t_2, $E(x_2|t_2) \geq E(x_2^{\text{BS}}|t_2) = 0$, acceptance of x is a best response. If the proposal of individual 1 is $x' \neq x$, we let individual 2's belief and action when he is of type t_2 be as follows. If there exists t_1 such that $x_2'(t_1, t_2) < 0$, his updated belief is $q(t_1|t_2, x') = 1$ and he chooses rejection. Otherwise he accepts whatever his beliefs. Given the specified strategy of individual 2, any $x' \neq x$ would generate an allocation x'' such that $E(x_1''|t_1) \leq E(x_1^{\text{BS}}|t_1) \leq E(x_1|t_1)$. □

An allocation rule x is *individually rational* if it gives a non-negative interim payoff to both players, i.e. if $E(x_i|t_i) \geq 0$ for each $t_i \in T_i$ and each $i \in \{1, 2\}$. An allocation rule x *dominates* an allocation rule x' *for the principal* if $E(x_1|t_1) > E(x_1'|t_1)$, for each $t_1 \in T_1$. An allocation rule x is *efficient for the principal* if it is individually rational and there does not exist any other individually rational allocation rule that dominates it for the principal. Following Myerson, we say that an allocation rule is a *strong solution* if it is both safe and efficient for the principal. When a strong solution exists, it is the natural solution in view of Proposition 1. For instance, if the subordinate has no private information, the best safe allocation rule is efficient for the principal. When there is private information on both sides, there may be no strong solution, as the following example shows.

Example 2. Let us consider $T_1 = \{1a, 1b\}$, $T_2 = \{2a, 2b\}$ with q the uniform probability distribution. The bargaining problem is the following:

$$V(1a, 2b) = V(1b, 2a) = \{v \in R^2 | v_1 + v_2 \leq 2\}$$
$$V(1a, 2a) = V(1b, 2b) = \{v \in R^2 | v_2 \leq \min[2 - v_1, 3 - 2v_1]\}.$$

The best safe allocation rule gives all the surplus to the principal in each state: $x^{\text{BS}}(1a, 2b) = x^{\text{BS}}(1b, 2a) = (2, 0)$, $x^{\text{BS}}(1a, 2a) = x^{\text{BS}}(1b, 2b) = (3/2, 0)$. It is dominated for the principal by the individually rational allocation rule x^* defined as: $x^*(1a, 2b) = x^*(1b, 2a) = (3, -1)$, $x^*(1a, 2a) = x^*(1b, 2b) = (1, 1)$.

When there is no strong solution, the set of equilibrium allocations, as characterized in Proposition 1, is quite large and in particular contains allocation rules which are not efficient for the principal. To select among them, we may consider how the principal could eliminate some equilibria by arguing with the subordinate on the unreasonableness of his out of equilibrium beliefs. Grossman and Perry (1986) defined a refinement of WSE, *perfect sequential equilibrium* (PSE). We present their idea in terms of robustness of an equilibrium allocation to certain objections that the principal may have. Let us say that the principal has an *objection* against an allocation rule x if there exists an allocation rule x' such that:

(1) $T_1' := \{t_1 \in T_1 | E(x_1'|t_1) > E(x_1|t_1)\} \neq \emptyset$;
(2) $(\forall t_2 \in T_2) : E(x_2'|t_2, T_1') \geq 0$.

Individually rational allocation rules against which the principal has no objection are efficient for the principal. In particular, the principal has an objection against x^{BS} in the previous example. An objection allows the principal to credibly reveal to the subordinate the information that his true type is in the set T_1'. This may allow a substantial refinement of the set of reasonable allocation rules. In the example though, any equilibrium allocation which is efficient for the principal admits no objection.[1]

We may further refine the set of equilibrium allocations by considering more sophisticated concepts of objection. For instance, the updated beliefs of the subordinate after receiving an unexpected offer could be proportional to the relative gains obtained by the various types of the principal by deviating, thus giving some cardinal content to the utilities. Those types gaining relatively more are seen as being more likely to deviate. In order to deal with the plethora of possible refinements, it is worth looking for a very strong one. The principal has a *weak objection* against an allocation rule x if there exists an allocation rule x' and a probability distribution $\theta \in \Delta(T_1)$ such that:

(1) $(\forall t_1 \in T_1) : \theta(t_1) > 0 \Rightarrow E(x_1'|t_1) > E(x_1|t_1)$;
(2) $(\forall t_2 \in T_2) : \sum_{t_1 \in T_1} \theta(t_1) q(t_1|t_2) x_2'(t) \geq 0$.

The principal has the power to influence the beliefs of the subordinate as he wants after a deviation, except that he cannot pretend that types who do not gain by deviating are possible. This idea appears in Myerson (1989, 2002). Most reasonable concepts of objection that we could come up with give less power to the principal than the concept of weak objection.

Following Myerson (1991), we may say that the set of equilibrium allocation determines an *upper solution* to the principal–agent problem. Allocation rules that do not belong to this set should never emerge from the principal–agent relationship. Symmetrically, the set of equilibrium allocations against which the principal has no weak objection may be thought of as determining a *lower solution*. Any allocation rule that belong to this set may emerge from the principal–agent relationship. Non-emptiness of lower solutions are important results.

[1] This observation holds true whenever the principal has only two possible types. It fails in Example 2 if we add a third type for the principal and consider $V(1c, 2a) = V(1c, 2b) = \{v \in R^2 | v_1 + v_2 \leq 0\}$.

Proposition 3. *There always exists some equilibrium allocation against which the principal has no weak objection.*

This proposition is a corollary of the next two results.

After proving Proposition 4, it will not be difficult to check that, in Example 2, x^* is the only individually rational allocation rule that is immune to weak objection from the principal. For the moment, let us consider the allocation x defined by: $x(1a, 2a) = (1, 1)$, $x(1b, 2a) = (3, -1)$, $x(1a, 2b) = (13/4, -5/4)$, and $x(1b, 2b) = (3/4, 5/4)$. The principal has no objection against x, as it satisfies the condition in Proposition 1 and is efficient for the principal. Nevertheless, (x', θ) is a weak objection against x, where $\theta = (5/11, 6/11)$ and x' is defined by: $x'(1a, 2a) = (79/100, 121/100)$, $x'(1b, 2a) = (361/120, -121/120)$, $x'(1a, 2b) = (349/100, -149/100)$, and $x'(1b, 2b) = (91/120, 149/120)$.

In fact, an allocation rule x is an equilibrium allocation against which the principal has no weak objection if and only if x belongs to the 'weighted-utility solution for the principal–agent problem' which is nothing more than the virtual utility solution defined in Myerson (1983) expressed in our specific framework with verifiable information. An allocation rule x is in the *weighted-utility solution* for the principal–agent problem, $\sigma^{\mathrm{M}}(V)$, if there exists $\gamma \in R^{T_1}_{++} \times R^{T_2}_{++}$ such that:

(1) $(\forall t_1 \in T_1) : E(x_1|t_1) = [\sum_{t_2 \in T_2} q(t_2|t_1) \max_{v \in V(t)} (\gamma_1(t_1)v_1 + \gamma_2(t_2)v_2)]/\gamma_1(t_1)$;
(2) $(\forall t_2 \in T_2) : E(x_2|t_2) = 0$.

This solution can be interpreted in terms of the Shapley (1969) procedure. For any vector of utility weights $\gamma \in R^{T_1}_{++} \times R^{T_2}_{++}$, let us consider the extended bargaining problem V^γ in which players can transfer utility at rates $\gamma_i(t_i)$ in state t, i.e. $V^\gamma(t) := \{x \in R^2 | \gamma_1(t_1)v_1 + \gamma_2(t_2)v_2 \leq \sup_{x \in V(t)} [\gamma_1(t_1)v_1 + \gamma_2(t_2)v_2]\}$, for each $t \in T$. V^γ is easy to solve as it has a strong solution, giving all the surplus to the principal in each state of the world:

$$x^{\mathrm{BS}}_\gamma(t) := \left(\frac{\max_{v \in V(t)} (\gamma_1(t_1)v_1 + \gamma_2(t_2)v_2)}{\gamma_1(t_1)}, 0 \right),$$

for each $t \in T$. Indeed, premultiplying $E(x^{\mathrm{BS}}_{\gamma,1}|t_1)$ by $q(t_1)$ and summing over t_1 it is easy to see that x^{BS}_γ is efficient for the principal. Conditions 1 and 2 above state that x is *interim equivalent* to x^{BS}_γ in the sense that $E(x_i|t_i) = E(x^{\mathrm{BS}}_{\gamma,i}|t_i)$ for each $t_i \in T_i$ and each $i \in \{1, 2\}$. So, x may be considered as a reasonable solution for V^γ. As x is an allocation rule for $V \subseteq V^\gamma$, it is a reasonable solution for V.

Proposition 4. *Let x be an allocation rule. Then, x is an equilibrium allocation against which the principal has no weak objection if and only if x belongs to $\sigma^{\mathrm{M}}(V)$.*

Proof. Let x be an allocation rule in the weighted-utility solution for the principal–agent problem. Let γ be the associated supporting vector. We have $E(x_2|t_2) = 0 = E(x^{\mathrm{BS}}_2|t_2)$ for all $t_2 \in T_2$, and $E(x_1|t_1) = E(x^{\mathrm{BS}}_{\gamma,1}|t_1) \geq E(x^{\mathrm{BS}}_1|t_1)$ for all $t_1 \in T_1$. Therefore, by Proposition 1, x is an equilibrium allocation. Let us assume that the principal has a weak objection (x', θ) against x. Then, we have:

$$\begin{aligned}
&\sum_{t\in T}\theta(t_1)q(t)\max_{v\in V(t)}(\gamma_1(t_1)v_1+\gamma_2(t_2)v_2)\\
&\quad=\sum_{t_1\in T_1}\theta(t_1)q(t_1)\gamma_1(t_1)E(x_1|t_1)<\sum_{t_1\in T_1}\theta(t_1)q(t_1)\gamma_1(t_1)E(x'_1|t_1)\\
&\qquad+\sum_{t_2\in T_2}q(t_2)\gamma_2(t_2)\sum_{t_1\in T_1}\theta(t_1)q(t_1|t_2)x'_2(t)\\
&\quad=\sum_{t\in T}q(t)\theta(t_1)(\gamma_1(t_1)x'_1(t)+\gamma_2(t_2)x'_2(t))
\end{aligned}$$

So, $\max_{v\in V(t)}(\gamma_1(t_1)v_1+\gamma_2(t_2)v_2) < \gamma_1(t_1)x'(t)+\gamma_2(t_2)x'_2(t)$, for some $t\in T$. This is absurd.

Let x be an equilibrium allocation against which the principal has no weak objection. Then, by non-levelness, $E(x_2|t_2)=0$ for each $t_2\in T_2$. Let $S(x)$ be the set of interim utility allocations w such that

(1) $(\forall t_1\in T_1): w_1(t_1)\le\theta(t_1)(E(x'_1|t_1)-E(x_1|t_1))$;
(2) $(\forall t_2\in T_2): w_2(t_2)\le\sum_{t_1\in T_1}\theta(t_1)q(t_1|t_2)x'_2(t)$

for some $\theta\in\Delta_{++}(T_1)$ and some allocation rule x'. We notice that $0\in S(x)$, by taking $x'=x$ and θ the uniform distribution. If there exists $y\in S(x)$ such that $y\gg 0$, then the principal has a weak objection against x. On the other hand, $S(x)$ is convex and comprehensive. As far as convexity is concerned, let w' and w'' be two elements of $S(x)$. Let (θ',x') and (θ'',x'') be the associated supporting vectors. Let $\alpha\in]0,1[$. For checking that $\alpha w'+(1-\alpha)w''\in S(x)$, it suffices to consider the vectors $\theta^\alpha:=\alpha\theta'+(1-\alpha)\theta''$ and

$$x^\alpha(t):=\frac{\alpha\theta'(t_1)x'(t)+(1-\alpha)\theta''(t_1)x''(t)}{\theta^\alpha(t_1)}$$

for each $t\in T$. By a separation argument, there exists $\lambda\in(R_+^{T_1}\times R_+^{T_2})\setminus\{0\}$ such that $\lambda.w\le 0$ for each $w\in S(x)$. So, we have:

$$\begin{aligned}
&\sum_{t_1\in T_1}\lambda_1(t_1)\theta(t_1)E(x'_1|t_1)\\
&\quad+\sum_{t_2\in T_2}\lambda_2(t_2)\sum_{t_1\in T_1}\theta_1(t_1)q(t_1|t_2)x'_2(t)\le\sum_{t_1\in T_1}\lambda_1(t_1)\theta(t_1)E(x_1|t_1)
\end{aligned}$$

for each $\theta\in\Delta_{++}(T_1)$ and each allocation rule x'. Rearranging the terms, the previous expression becomes:

$$\sum_{t_1\in T_1}\theta(t_1)\sum_{t_2\in T_2}[\lambda_1(t_1)q(t_2|t_1)x'_1(t)+\lambda_2(t_2)q(t_1|t_2)x'_2(t)]\le\sum_{t_1\in T_1}\theta(t_1)\lambda_1(t_1)E(x_1|t_1)$$

for each $\theta\in\Delta_{++}(T_1)$ and each allocation rule x'. So,

$$\sum_{t_2\in T_2}[\lambda_1(t_1)q(t_2|t_1)x'_1(t)+\lambda_2(t_2)q(t_1|t_2)x'_2(t)]\le\lambda_1(t_1)E(x_1|t_1)$$

for each allocation rule x' and each $t_1 \in T_1$. Dividing both sides of the inequality by $q(t_1)$, we have that there exists $\gamma \in (R_+^{T_1} \times R_+^{T_2}) \setminus \{0\}$ such that

$$\sum_{t_2 \in T_2} q(t_2|t_1)[\gamma_1(t_1)x_1'(t) + \gamma_2(t_2)x_2'(t)] \leq \gamma_1(t_1)E(x_1|t_1)$$

for each allocation rule x' and each $t_1 \in T_1$. We conclude that there exists $\gamma \in R_{++}^{T_1} \times R_{++}^{T_2}$ such that

$$E(x_1|t_1) = \frac{\sum_{t_2 \in T_2} q(t_2|t_1) \max_{v \in V(t)} [\gamma_1(t_1)v_1 + \gamma_2(t_2)v_2]}{\gamma_1(t_1)}.$$

Indeed, multiplying the penultimate expression by $q(t_1)$, summing over t_1, and adding on the left-hand side the term $\sum_{t_2 \in T_2} q(t_2)\gamma_2(t_2)E(x_2|t_2)$ (= 0), we obtain that $\sum_{t \in T} q(t)$ $(\gamma_1(t_1)x_1'(t) + \gamma_2(t_2)x_2'(t)) \leq \sum_{t \in T} q(t)(\gamma_1(t_1)x_1(t) + \gamma_2(t_2)x_2(t))$, for each allocation rule x'. Hence, $\gamma_1(t_1)x_1(t) + \gamma_2(t_2)x_2(t) = \max_{v \in V(t)} [\gamma_1(t_1)v_1 + \gamma_2(t_2)v_2]$, for each $t \in T$. As $V(t)$ is assumed to be non-level for each $t \in T$, each component of γ is strictly positive. □

As in Shapley (1969), the definition of the solution in terms of supporting virtual games suggests a technique to prove non-emptiness based on a fixed point argument on the interim utility weights. An *utility allocation* is $w \in R^{T_1} \times R^{T_2}$. It is *feasible for V* if there exists an allocation rule $x \in \times_{t \in T} V(t)$ such that $w_i(t_i) = \sum_{t_{-i}} q(t_{-i}|t_i)x_i(t)$, for each $t_i \in T_i$, $i = 1, 2$. Let W be the set of feasible utility allocations. We assume that W is contained in a half space. Bargaining problems which do not satisfy this assumption are uninteresting because there are unbounded benefits from cooperation. The fact that each $V(t)$ is contained in a half space is not sufficient.

Proposition 5. *For any bargaining problem V, $\sigma^{\mathrm{M}}(V)$ is non-empty.*

Proof. We give the proof for the smooth case: for all t, $V(t)$ admits a unique supporting hyperplane at each point on its frontier. For the extension to non-smooth problems, see de Clippel (2002). Let $\hat{W} \subset W$ be the set of utility allocations that are individually rational and efficient for the principal. Using smoothness we can define a continuous function $\lambda : \hat{W} \to \Delta_{++}(T_1)$ which associates to each point of the frontier its normal vector. Let the function $\phi : \mathrm{Im}(\lambda) \to R_+^{T_1}$ be:

$$\phi(\lambda) = \left(\frac{\sum_{t_2} q(t_2|t_1) \max_{v \in V(t)} [\gamma_1(t_1)v_1 + \gamma_2(t_2)v_2]}{\gamma_1(t_1)} \right)_{t_1 \in T_1},$$

with $\gamma_i(t_i) = \lambda_i(t_i)/q(t_i)$. Finally, let $\pi : R_+^{T_1} \setminus \{0\} \to \hat{W}$ be the function that associates to each utility allocation w_1 for player 1 the intersection of $\hat{W}$ with the ray going through 0 and w_1. Notice that $\hat{W}$ is isomorphic to $\Delta(T_1)$, and we can apply Brouwer's fixed point theorem to the continuous function $\pi \circ \phi \circ \lambda$ from $\hat{W}$ into itself. A fixed point of this function is a feasible utility allocation. Any allocation rule generating it is an element of $\sigma^{\mathrm{M}}(V)$. □

In Example 2, x^* is the only allocation rule in $\sigma^{\mathrm{M}}(V)$. Indeed, it is easy to check that, if $\gamma \in R_{++}^{T_1} \times R_{++}^{T_2}$ supports some allocation rule in the weighted-utility solution associated to the principal–agent model, then $\gamma_1(t_1) = \gamma_2(t_2)$ for each $t \in T$.

The solution $\sigma^{\mathrm{M}}(V)$ has been defined by an application of Shapley's (1969) procedure. Proposition 4 provides a non-cooperative foundation by showing that it corresponds to the 'lower solution' of a 'natural' bargaining procedure. We now complete the implementation of the program set forth in the introduction by providing an exact axiomatic characterization for the class of smooth bargaining problems. With respect to Myerson (1983), our axiomatic system is simpler mainly because the set of feasible utility allocations is non-level. For each pair (V, V') of bargaining problems, we say that V' is an *extension* of V ($V \subseteq V'$) if $V(t) \subseteq V'(t)$ for each $t \in T$. Let σ be a solution. We may impose the following properties.

Axiom 1 (Strong solution—SS). Let V be a bargaining problem. If x^{BS} is efficient for the principal, then $x^{\mathrm{BS}} \in \sigma(V)$.

Axiom 2 (Independence of irrelevant alternatives—IIA). Let V and V' be two bargaining problems and let x be an allocation rule for V' such that $x \in \sigma(V')$. If $V \subseteq V'$ and x is an allocation rule for V, then $x \in \sigma(V)$.

Axiom 3 (Interim equivalence—IE). Let V be a bargaining problem. Let x and x' be two allocation rules and $x \in \sigma(V)$. If x and x' are interim equivalent, then $x' \in \sigma(V)$.

The axioms are motivated and justified as follows. SS states that strong solutions are reasonable solutions. A first justification comes from Proposition 6 where it is shown that strong solutions are immune to weak objections from the principal. Another justification is given in Remark 8. IIA is a natural analogue of the independence of irrelevant alternatives property introduced by Nash (1950). IE imposes that it is not the allocation rule itself but the interim allocation it supports that makes it a reasonable solution or not. All the solution concepts introduced before in this chapter satisfy this property.

Proposition 6. *σ^{M} is the minimal solution satisfying SS, IIA and IE.*

Proof. We first check that σ^{M} satisfies the axioms. IIA and IE follow directly from the definition of σ^{M}. As far as SS is concerned, let us assume that x^{BS} is efficient for the principal in a bargaining game V. Then x^{BS} is interim efficient in the sense that there does not exist any allocation rule x such that $E(x_i|t_i) \geq E(x_i^{\mathrm{BS}}|t_i)$ for each $t_i \in T_i$ and each $i \in \{1, 2\}$ with at least one strict inequality. So, there exists $\lambda \in R_{++}^{T_1} \times R_{++}^{T_2}$ (using the fact that $V(t)$ is non-level for each $t \in T$) such that

$$\sum_{i\in\{1,2\}} \sum_{t_i\in T_i} \lambda_i(t_i) E(x_i|t_i) \leq \sum_{t_1\in T_1} \lambda_1(t_1) E(x_1^{\mathrm{BS}}|t_1)$$

for each allocation rule x. Then,

$$\gamma_1(t_1) x_1^{\mathrm{BS}}(t) = \max_{v\in V(t)} [\gamma_1(t_1) v_1 + \gamma_2(t_2) v_2]$$

for each $t \in T$, where $\gamma_i(t_i) := \lambda_i(t_i)/q(t_i)$ for each $t_i \in T_i$ and each $i \in \{1, 2\}$. So, $x^{\mathrm{BS}} \in \sigma^{\mathrm{M}}(V)$.

We now check that σ^{M} is the minimal solution satisfying the axioms. Let σ be any solution satisfying them, let V be any bargaining problem, let $x \in \sigma^{\mathrm{M}}(V)$, and let $\gamma \in R^{T_1}_{++} \times R^{T_2}_{++}$ be the associated supporting vector. Let V^γ be the bargaining problem defined by $V^\gamma(t) := \{v \in R^2 | \gamma_1(t_1)v_1 + \gamma_2(t_2)v_2 \leq \max_{v\in V(t)} [\gamma_1(t_1)v_1 + \gamma_2(t_2)v_2]\}$ for all $t \in T$. The best safe allocation rule in V^γ is

$$x^{\mathrm{BS}}(t) = \left(\frac{\max_{v\in V(t)} [\gamma_1(t_1)v_1 + \gamma_2(t_2)v_2]}{\gamma_1(t_1)}, 0\right)$$

for each $t \in T$. It is efficient for the principal in V^γ. By SS, $x^{\mathrm{BS}} \in \sigma(V^\gamma)$. By IE, $x \in \sigma(V^\gamma)$. By IIA, $x \in \sigma(V)$. □

The proof is similar to a part of the arguments developed by Nash (1950) for the bargaining problem under complete information. He obtains though an exact characterization result as he looks for solutions of cardinality one. Here, we cannot do the same, as $\sigma^{\mathrm{M}}(V)$ can be multivalued. This is obvious in terms of allocation rules. It is a bit less obvious in terms of interim utility allocations.

Example 7. Let us consider $T_1 = \{1a, 1b\}$, $T_2 = \{2a, 2b\}$ with q the uniform probability distribution. The bargaining problem is the following:

$$\begin{aligned}
V(1a, 2a) &= \{v \in R^2 | v_2 \leq \min[\tfrac{1}{2}(-v_1 + 3), -2v_1 + 3]\} \\
V(1a, 2b) &= \{v \in R^2 | v_2 \leq \min[\tfrac{1}{3} - v_1, -v_1 + 2]\} \\
V(1b, 2a) &= \{v \in R^2 | v_2 \leq \min[-v_1 + 2, -2v_1 + 5]\} \\
V(1b, 2b) &= \{v \in R^2 | v_2 \leq \min[-v_1 + 2, \tfrac{1}{3}(-4v_1 + 7)]\}.
\end{aligned}$$

As for the game in Example 2, $x^* \in \sigma^{\mathrm{M}}(V)$. The associated interim utility level for both types of the principal is 2. It is easy to check on the other hand that the allocation rule x' defined by $x'(1a, 2a) = (1, 1)$, $x'(1a, 2b) = (2, -(2/3))$, $x'(1b, 2a) = (3, -1)$, and $x'(1b, 2b) = (5/4, 2/3)$, also belongs to $\sigma^{\mathrm{M}}(V)$ with $\gamma_1(1.a) = 1$, $\gamma_1(1.b) = 4$, $\gamma_2(2.a) = 2$, $\gamma_2(2.b) = 3$. The associated interim utility levels for the principal are $3/2$ for type $1.a$ and $17/8$ for type $1.b$.

We may adapt the dual arguments developed for instance by Thomson (1981) on the class of smooth bargaining problems. A bargaining problem V is *smooth* if, for all t, $V(t)$ admits a unique supporting hyperplane at each point on its frontier.

Axiom 4 (Efficiency—EFF). Let V be a bargaining problem. Then, $\sigma(V)$ specifies allocation rules that are efficient for the principal.

Axiom 5 (Best safe lower bound—BSLB). Let V be a bargaining problem, let x be an allocation rule for V and let x^{BS} be the associated best safe allocation rule. If $x \in \sigma(V)$, then x weakly interim Pareto dominates x^{BS}.

Axiom 6 (Independence over irrelevant expansions—IIE). Let V and V' be two bargaining problems and let x be an allocation rule for V such that $x \in \sigma(V)$. If $V \subseteq V'$ and x is efficient for the principal in V', then $x \in \sigma(V')$.

BSLB is justified for at least two reasons. First, it is a necessary requirement in view of Proposition 1. Second, in a context with verifiable types, it seems plausible that the players have the possibility to delay the agreement. They know that they will agree on $x^{\mathrm{BS}}(t)$ if state t occurs, for each $t \in T$. It is only in cases where there is a mutual interest to agree on some allocation rule at the interim stage that the players will do so.

Remark 8. IE and BSLB imply SS for solutions that are non-empty[2] valued.

IIE is a dual version of the IIA property. It is important to note that σ^{M} doesn't satisfy IIE on the class of all bargaining problems as the following example shows.

Example 9. Let us consider $T_1 = \{1a, 1b\}$, $T_2 = \{2a, 2b\}$ with q the uniform probability distribution. Let V be a slight modification of the bargaining problem introduced in Example 2:

$$V(1a, 2a) = \{v \in R^2 | v_2 \leq \min[2 - v_1, 3 - 2v_1]\}.$$
$$V(1a, 2b) = V(1b, 2a) = \{v \in R^2 | v_1 + v_2 \leq 2\}$$
$$V(1b, 2b) = \{v \in R^2 | v_2 \leq \min[\tfrac{1}{2}(3 - v_1), 3 - v_1]\}.$$

The allocation x^* defined, as in Example 2, by $x^*(1a, 2b) = x^*(1b, 2a) = (3, -1)$, $x^*(1a, 2a) = x^*(1b, 2b) = (1, 1)$ remains efficient for the principal in the linear extension V' defined by

$$V(1a, 2a) = \{v \in R^2 | 2v_1 + v_2 \leq 3\}.$$
$$V(1a, 2b) = V(1b, 2a) = \{v \in R^2 | v_1 + v_2 \leq 2\}$$
$$V(1b, 2b) = \{v \in R^2 | v_1 + 2v_2 \leq 3\}.$$

We obtain a contradiction as σ^{M} satisfies BSLB (cf. the next proposition) and x^* does not dominate the best safe allocation rule for the principal in V'.

Proposition 10. *σ^{M} is the maximal solution satisfying EFF, BSLB and IIE on the class of smooth bargaining problems.*

Proof. We first check that σ^{M} satisfies the axioms. The solution that specifies, for each bargaining problem, the set of individually rational allocation rules that are imune to weak objections from the principal clearly satisfies EFF and BSLB. So does σ^{M} thanks

[2] Non-emptiness is a necessary requirement for any definitive theory. Nevertheless, it is maybe more like an utopia for the moment. Many insightful concepts don't satisfy this requirement (e.g. the core and the Nash equilibrium). Nevertheless, it is usually imposed as an axiom in bargaining theory, as in Aumann (1985) for instance. In fact, even Nash (1950, 1953) imposes it. Indeed, he restricts his domain of solutions to those that specify a set of cardinality one for each bargaining problem.

to Proposition 4. IIE follows directly from the definition of σ^{M} and from the smoothness assumption.

We now check that σ^{M} is the maximal solution satisfying the axioms. Let σ be any solution satisfying them, let V be a bargaining problem and let $x \in \sigma(V)$. By EFF, there exists $\gamma \in R_{++}^{T_1} \times R_{++}^{T_2}$ such that x is efficient for the principal in the bargaining problem V^{γ}, where $V^{\gamma}(t) := \{x \in R^2 | \gamma_1(t_1)v_1 + \gamma_2(t_2)v_2 \leq \sup_{x \in V(t)} [\gamma_1(t_1)v_1 + \gamma_2(t_2)v_2)]\}$, for each $t \in T$. V^{γ} has a strong solution, giving all the surplus to the principal in each state of the world:

$$x_{\gamma}^{\mathrm{BS}}(t) := \left(\frac{\max_{v \in V(t)} (\gamma_1(t_1)v_1 + \gamma_2(t_2)v_2)}{\gamma_1(t_1)}, 0 \right),$$

for each $t \in T$. We conclude by applying BSLB. □

We obtain an exact axiomatization of $\sigma^{\mathrm{M}}(V)$ by combining Propositions 6 and 10.

We conclude this section with a comparison between our results and those of Myerson (1983). In his more general context, Myerson defines virtual utility by taking into account the dual variables associated with the incentive compatibility constraints. Once this is done, the Shapley (1969) procedure may be adapted to define a virtual utility solution for the principal–agent problem, and one may look for non-cooperative and cooperative characterizations. The presence of incentive constraints creates some difficulties. Even if the ex-post problems are extremely well behaved, for example TU, as in the bilateral trade example of Myerson (1991, 10.3), the set of feasible interim utilities need not be comprehensive nor smooth. Myerson enlarges the definition of virtual utility solution by a closure argument and is so able to prove an existence theorem and an analog of Proposition 6, characterizing his solution as the minimal one satisfying some axioms, including one of Domination, which is disputable. As in Example 3, the lack of smoothness prevents the use of the IIE property to obtain a full axiomatic characterization. Moreover, the concept of weak objection is not useful when incentives matter. In the bilateral trade example quoted above there is a strong solution when the probability of a good type seller is low, corresponding to the least cost separating equilibrium. Nevertheless, the principal has a weak objection in which he proposes a pooling mechanism and pretends that he is of a good type with a high probability. This may explain why Myerson (1983) is not able to provide a non-cooperative characterization.

4. Power on both sides

Myerson (1984) proposed a solution for two person bargaining problems in which both players have some power. If $\beta = (\beta_1, \beta_2) \in \Delta(\{1, 2\})$ is the vector of individuals' relative strengths, the weighted-utility solution $\sigma_{\beta}^{\mathrm{M}}(V)$ is the set of allocation rules x for which there exists $\gamma \in R_{++}^{T_1} \times R_{++}^{T_2}$ such that:

$$E(x_i|t_i) = \frac{\sum_{t_{-i} \in T_{-i}} q(t_{-i}|t_i)\beta_i \max_{v \in V(t)} (\gamma_1(t_1)v_1 + \gamma_2(t_2)v_2)}{\gamma_i(t_i)}$$

for each $t_i \in T_i$ and each $i \in \{1, 2\}$.

We may consider the following adaptation of SS, BSLB and IIE. Let $x^{\mathrm{BS},i}$ denote the best safe allocation rule when player i is the principal.

Axiom 1′ (Strong solution–SS’). Let V be a bargaining problem. If $\beta_1 x^{\mathrm{BS},1} + \beta_2 x^{\mathrm{BS},2}$ is interim efficient, then $\beta_1 x^{\mathrm{BS},1} + \beta_2 x^{\mathrm{BS},2} \in \sigma(V)$.

Axiom 4′ (efficiency—EFF′). Let V be a bargaining problem. Then, $\sigma(V)$ specifies allocation rules that are interim efficient.

Axiom 5′ (Best safe lower bound—BSLB′). Let V be a bargaining problem and let x be an allocation rule for V. If $x \in \sigma(V)$, then x weakly interim Pareto dominates $\beta_1 x^{\mathrm{BS},1} + \beta_2 x^{\mathrm{BS},2}$.

Axiom 6′ (Independence over irrelevant expansions—IIE′). Let V and V' be two bargaining problems and let x be an allocation rule for V such that $x \in \sigma(V)$. If $V \subseteq V'$ and x is interim efficient in V', then $x \in \sigma(V')$.

In the context of smooth bargaining problems with verifiable types, an exact characterization can be obtained, as in Proposition 6.

Proposition 6′. *$\sigma_\beta^{\mathrm{M}}$ is the minimal solution satisfying SS’, IIA and IE.*

Proposition 10′. *$\sigma_\beta^{\mathrm{M}}$ is the maximal solution satisfying EFF′, BSLB′ and IIE′ on the class of smooth bargaining problems.*

As opposed to the principal–agent case, we are not able, in general, to obtain a non-cooperative characterization of $\sigma_\beta^{\mathrm{M}}$. A special case is when utility is transferable in the ex-post problems $V(t)$; then the mixture of the best safe allocations is efficient and $\sigma_\beta^{\mathrm{M}}$ can be obtained as the unique equilibrium allocation of the game in which player i is chosen to be the principal with probability β_i.

Varying β leads to a new solution, called the *contract curve*: $C^{\mathrm{M}}(V) = \bigcup_{\beta \in \Delta(\{1,2\})} \sigma_\beta^{\mathrm{M}}$. For the case of verifiable types that we are considering, a classical notion of interim core is the coarse core of Wilson (1978) which, for two-player problems, is the set of interim efficient and interim individually rational allocations. The following example illustrates how $C^{\mathrm{M}}(V)$ refines the coarse core.

Example 11. Let us consider $T_1 = \{1a, 1b\}$, $T_2 = \{*\}$ with q the uniform probability distribution. The bargaining problem is the following: $V(1a) = V(1b) = \{v \in R^2 | v_1 + v_2 \le 100\}$. The coarse core (which in this example coincides also with Wilson’s (1978) fine core) is the simplex in W with vertices $(w_1(1a), w_1(1b); w_2) = (200, 0; 0), (0, 200; 0), (0, 0; 100)$. In the interim utility space, the strong solution for player 1 (resp. 2) is $w^{\mathrm{BS},1} = (100, 100; 0)$, $w^{\mathrm{BS},2} = (0, 0; 100)$, and the set $C^{\mathrm{M}}(V)$ is the line on the simplex connecting these two points.

In the coarse core, like in the standard definition of the core for complete information games, one tests the stability of given status quo allocations with respect to a given class of deviations. What Myerson's theory suggests is that this might not be the correct approach to bargaining with incomplete information. One should take into account that status quo allocations emerge themselves from the bargaining process and as such bear an informational content. In the example, an interim allocation such as $(200, 0; 0)$, which is in the coarse core, should not be expected as a reasonable outcome of a bargaining process. Such an outcome can be obtained only if individual 2 has no bargaining power and individual 1 is the principal and proposes the allocation rule $x(1a) = (200, -100)$, $x(1b) = (0, 100)$. But then individual $1b$ would have a profitable deviation: the best safe allocation rule $x(1a) = x(1b) = (100, 0)$. Knowing this, individual 2 should deduce that the proposal generating $(200, 0; 0)$ comes from $1a$ and therefore reject it.

Another instructive difference with the coarse core is that the solution $C^{\mathrm{M}}(V)$ is not based only on the set of feasible interim allocation W and the disagreement point. Indeed, if we modify the example by setting $V(1a) = \{v \in R^2 | v_1 + v_2 \leq 120\}$, $V(1b) = \{v \in R^2 | v_1 + v_2 \leq 80\}$, the set W remains the same, but the image in the interim utility space of $C^{\mathrm{M}}(V)$ is now the segment joining $(120, 80; 0)$ and $(0, 0; 100)$. This reflects the new intertype compromise, taking into account the increased strength of $1a$ over $1b$. The fact that an interim solution should take into account also characteristics of the ex-post games is actually a more general insight emerging from the approach developed by Myerson. Indeed this differentiates $\sigma_{\beta}^{\mathrm{M}}$, from other proposals, like those in Harsanyi and Selten (1972) or Myerson (1979).

Acknowledgements

We thank Professors Françoise Forges, Jean-François Mertens, Salvatore Modica and Roger Myerson for very useful comments and discussions. This paper presents research results of the programme 'Actions de recherche concertées' of the 'Communauté française de Belgique'. The scientific responsibility is assumed by the authors.

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