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Impartial division of a dollar

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Abstract

For impartial division, each participant reports only her opinion about the fair relative shares of the other participants, and this report has no effect on her own share. If a specific division is compatible with all reports, it is implemented.

We propose a family of natural methods meeting these requirements, for a division among four or more participants. No such method exists for a division among three participants.

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1. Introduction

How should a dollar, or any amount of any divisible commodity, be divided if subjective claims are to be respected? If the claims are objective, the most common rule is proportionality.[1] No such simple rule is available, however, when the claims are at least in part subjective and there is no impartial observer who can resolve disagreements.

We propose division rules for aggregating possibly conflicting evaluations of claims by eliciting individual reports that we call *impartial* in view of the following two properties:

- everyone reports an evaluation of the (relative) shares that *other* agents deserve; no one makes any statement about *her own* share of the dollar;

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[1] It goes back to Aristotle's *Nichomachean Ethics*. A recent axiomatic literature discusses alternative norms for fair division under objective claims: see [5] for a survey.

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- the share of any participant is determined *exclusively* by the reports of other agents, her own report has *no influence* on her own share.

The first property takes literally the old adage that a man is never a good judge of his own cause. It eliminates *overt* partiality, whereas the second property rules out the *covert* form of partiality whereby a strategic participant uses her report to indirectly increase her share. For an agent who cares only about her own share, it is a weakly dominant strategy to report truthfully her evaluation of others' relative claims.

We call our third requirement the *consensus* property:

- if the profile of opinions points to a *consensual* division (if there is a way to divide the dollar that agrees with all individual reports), then this is the outcome.

Of the three properties above, only the third links the substantive content of the reports to the actual shares of the dollar. It is a very weak link, because it puts no restriction on the outcome when there is even a modicum of disagreement among the participants. Yet, in combination with the first two properties, the consensus property has much bite.

Our model is very general, in that it requires no assumption about the nature of individual claims or about the origin of the disagreement. Individual claims may be derived from effort applied toward the creation of the surplus (think of partners dividing a profit at the end of the year); they may measure the relative needs for the resource (relief distribution after a catastrophic loss); or represent exogenous rights (contestable claims in a bankruptcy or inheritance situation); or a combination of these factors. Objective claims may exist yet be imperfectly known (some records are lost), requiring the participants to make educated guesses. Finally we may be dividing a *cost*, in which case the claims turn into individual *liabilities*, with a similar array of possible interpretations.

Our problem bears a tenuous relation to the Condorcet Jury Problem and more generally to the literature on the pooling of expert opinions.[2] There like here, one aggregates conflicting impartial opinions; but in our case the "experts" themselves are the beneficiaries of the outcome. Our approach is orthogonal to bargaining theory, where individual messages are explicitly partial and meant to improve one's share of the dollar.

The follow-up paper by Tideman and Plassmann [6] elaborates on the numerical and statistical properties of the division methods that are discussed here from an axiomatic perspective.

1.1. Overview of the results

Our model requires at least three agents. With exactly three, there is a unique impartial and consensual division rule (Proposition 1), but that rule distributes *exactly* the dollar only when the three reports are consistent; otherwise it distributes strictly less. In contrast, with four or more agents, we propose many anonymous, impartial, and consensual rules that always distribute the dollar exactly (Theorem 2 in Section 5).

The first step toward this result is to construct a family of inexact, wasteful rules (distributing one dollar or less), in which agent i's share is derived from aggregate reports on the ratio of j's share to i's share for all $j \neq i$. Different methods can be used to aggregate the reports (e.g. median, maximum, geometric mean and arithmetic mean). Theorem 1 in Section 4 offers a characterization of these rules based on a separability property reminiscent of Gorman's work [2].

[2] Recent contributions include [3,1,4].

We construct exact rules from inexact ones by dividing the dollar into n equal parts (n is the number of agents), designating for each part a different *residual* agent i, dividing inexactly that part among the $n-1$ other agents while ignoring agent i's opinion, and giving the residual share to agent i.

2. The model

Let $N=\{1,\ldots,n\}$ be a set of at least three agents ($n \geqslant 3$). For each subset M of N, define $\mathcal{R}[M]$, the set of *consistent share ratios* for subset M, to be the following subset of $\mathbb{R}^{M\times M}$:

$$\mathcal{R}[M]=\{r \in \mathbb{R}^{M\times M} | \forall i,j,k \in M, r_{ij}>0, \text{ and } r_{ij}r_{jk}r_{ki}=1\}.$$

In particular $r_{ii}=1$ and $r_{ij}r_{ji}=1$.

Definition 1. Agent i's *evaluation* of her partners is $r^i \in \mathcal{R}[N\setminus\{i\}]$.

For instance, $r^i_{jk}=2$ means that agent j deserves twice as much as agent k, according to agent i. The definition of $\mathcal{R}[M]$ imposes natural consistency requirements on the agents' evaluations. For instance, "j deserves twice as much as k" is equivalent to "k deserves half of what j gets."

The assumption $r^i_{jk}>0$, for all i, j, k, means that in everyone's evaluation, everyone else deserves a positive share of the pie. With some qualifications, our results extend to the case where an agent can recommend a null share for some of the other agents: see Section 6.

Agent i's evaluation does not suggest anything about the payoff she deserves. On the other hand, it determines her view of what the other agents' payoffs should be. For any $i \in N$ and any payoff $\alpha \in [0,1]$ for i, the system:

$$\sum_{j\in N} x_j = 1-\alpha \quad \text{and} \quad \frac{x_j}{x_k}=r^i_{jk} \quad \text{for all } j,k \in N\setminus\{i\} \tag{1}$$

has the unique solution:

$$x_j=\frac{1-\alpha}{1+\sum_{k\in N\setminus\{i,j\}} r^i_{kj}} \quad \text{for all } j \in N\setminus\{i\}.$$

Definition 2. An *evaluation profile* is a list $r=(r^i)_{i\in N}$ of evaluations, one for each agent. An evaluation profile r is *consensual* if it is possible to share the dollar and agree with all the ratios in r, i.e. if there exists $x \in \mathbb{R}^N$ such that

$$\sum_{i\in N} x_i = 1 \quad \text{and} \quad r^k_{ij}=\frac{x_i}{x_j} \quad \text{for all } k \in N \text{ and all } i,j \in N\setminus\{k\}. \tag{2}$$

Lemma 1. *The evaluation profile r is consensual if and only if there exists a set of consistent share ratios $t \in \mathcal{R}[N]$ such that $r^i_{jk}=t_{jk}$, for each i in N and each j,k in $N\setminus\{i\}$. This is equivalent to $r^1_{23}r^2_{31}r^3_{12}=1$ if $n=3$, and to*

$$r^k_{ij}=r^l_{ij} \quad \textit{for all } i,j \in N \;\; \textit{all } k,l \in N\setminus\{i,j\}$$

if $n \geqslant 4$. *In addition, the unique solution of system* (2) *is*

$$x_i = \frac{1}{1 + \sum_{j \in N \setminus \{i\}} t_{ji}} \quad \textit{for all } i \in N. \tag{3}$$

We omit the straightforward proof.

Definition 3. A *division rule f* assigns to each evaluation profile r a vector $f(r)$ in $\mathbb{R}_+^N$. The rule f is *exact* if $\sum_{i \in N} f_i(r) = 1$, for all r. It is gain-feasible, or simply *feasible*, if $\sum_{i \in N} f_i(r) \leqslant 1$, for all r. It is *cost-feasible* if $\sum_{i \in N} f_i(r) \geqslant 1$, for all r.

The dollar to be divided can be a *gain*, or a *cost*. If the division rule is exact, both applications are feasible. For inexact rules, the domain of feasibility depends on whether the dollar is a gain or a cost. Although our main interest is in exact rules, gain-feasible rules are also relevant (Propositions 1 and 2): they are a key ingredient in the construction of the exact rules of Theorem 2. On the other hand, cost-feasible rules play no role in our model (as explained in Propositions 1 and 2), so we focus on the gain interpretation: when we speak below of a feasible rule, we always mean gain-feasible.

Definition 4. The division rule f is *consensual* if it chooses the consensus division (3) when the evaluation profile r is consensual. It is *impartial* if the share of an agent is independent of her own report. That is, if for all pairs of profiles $\{r, s\}$ and all i, we have:

$$\{r^j = s^j \text{ for all } j \in N \setminus \{i\}\} \Rightarrow f_i(r) = f_i(s).$$

Consensuality restricts the choice of a rule only in the knife-edge case where all reports agree with respect to the relative shares of all pairs of agents. Suppose now that the reports agree only with respect to a particular pair (i, j) of agents: $r_{ij}^k = r_{ij}^l$ for all $k, l \in N \setminus \{i, j\}$. In the spirit of the independence properties familiar in axiomatic work, it would be natural to require the actual shares to reflect this common value: $\frac{f_i(r)}{f_j(r)} = r_{ij}$, even in the absence of agreement with respect to other pairs. But this strengthening of consensuality is too demanding. Fix $a > 0$, and consider the following reports:

$$r_{i,i+1}^j = a \quad \text{for all } i, j \ \text{ such that } j \neq i, i+1, \tag{4}$$

where we use the convention $n+1 = 1$. The above independence property would require $\frac{x_i}{x_{i+1}} = a$, for all i, which is impossible if $a \neq 1$.

Proposition 1. *Suppose* $n = 3$. *There is a unique impartial and consensual division rule* f^*:

$$f^*(r) = \left(\frac{1}{1 + r_{31}^2 + r_{21}^3}, \frac{1}{1 + r_{12}^3 + r_{32}^1}, \frac{1}{1 + r_{23}^1 + r_{13}^2} \right) \quad \textit{for all } r.$$

This rule is anonymous and feasible. Moreover for all evaluation profiles r

$$\sum_{i=1}^{3} f_i^*(r) = 1 \iff r \textit{ is consensual.}$$

Proof. It is straightforward to check that f^* satisfies the properties of impartiality and consensuality. Consider now an impartial and consensual rule *f*. Let *r* be an evaluation profile and define $t^1 \in \mathcal{R}[N \setminus \{1\}]$ as follows: $t^1_{23} = r^3_{21} r^2_{13}$ and $t^1_{32} = 1/t^1_{23}$. By impartiality:

$$f_1(r) = f_1(t^1, r^2, r^3).$$

Because the profile (t^1, r^2, r^3) is consensual, consensuality and Lemma 1 imply

$$f_1(t^1, r^2, r^3) = \frac{1}{1 + r^2_{31} + r^3_{21}} = f^*_1(r).$$

As the choice of coordinate 1 was arbitrary, we conclude $f = f^*$. Feasibility of f^*, and the fact that $\sum_{i=1}^3 f^*_i(r) < 1$ if *r* is not consensual, follow from Lemma 3 in Appendix A.1. □

A consequence of Proposition 1 is that, among three agents, no impartial and consensual division rule is exact or even cost-feasible. This impossibility no longer holds with four or more agents: in Section 5 we construct impartial, consensual and exact division rules for such problems. Our construction starts in the next section with a family of feasible yet inexact rules generalizing f^* to any number of agents.

3. Aggregated ratios

The key to Proposition 1 is that with only three agents, any evaluation profile can be made consensual by changing the report of just one agent. With four or more agents this is no longer true. The axioms of consensuality and impartiality allow for many (an infinity of) division rules.

We extend the canonical rule f^* for the three-agent case to any *n*. Every pair of agents $\{i, j\}$ is now evaluated by $n - 2$ other agents. We propose to aggregate these ratios r^k_{ij} for different *k*'s into a single representative ratio r_{ij}.

Definition 5. Given an integer *m*, an *m-aggregator* is a symmetric, continuous and non-decreasing mapping $\rho : \mathbb{R}^m_{++} \to \mathbb{R}_{++}$, such that

$$\rho(a, \ldots, a) = a \quad \text{for all } a > 0.$$

Familiar examples of aggregators are the arithmetic, geometric, and harmonic means, and the rank-order aggregators. For all $z \in \mathbb{R}^m_{++}$, let z^* be the vector obtained from *z* by rearranging its coordinates increasingly: $z^*_1 \leqslant z^*_2 \leqslant \cdots \leqslant z^*_m$. The *k*th rank order aggregator is $\rho^{*k}(z) = z^*_k$. The max and min aggregators ρ^{*m} and ρ^{*1} are clearly the smallest and largest aggregators.

Definition 6. Let ρ be an $(n - 2)$-aggregator. For each evaluation profile *r*, write $\rho(r_{ji})$ instead of $\rho((r^k_{ji})_{k \in N \setminus \{i,j\}})$. Then define the division rule f^ρ as follows:

$$f^\rho_i(r) = \frac{1}{1 + \sum_{j \in N \setminus \{i\}} \rho(r_{ji})} \quad \text{for all } i \in N, \text{ and all profiles } r. \tag{5}$$

We use three additional properties of division rules in the axiomatic discussion below.

Definition 7. A division rule *f* is *anonymous* if it is a symmetric mapping with respect to all permutations of *N*. It is *continuous* if the share of every agent depends continuously upon the profile of individual reports. It is *non-decreasing* if for all $i, k \in N, i \neq k$, and all evaluation profiles r, s :

$$\{r^j = s^j, r^k_{jj'} = s^k_{jj'} \text{ and } r^k_{ij} \geqslant s^k_{ij} \text{ for all } j, j' \in N \backslash \{i, k\}\} \Rightarrow \{f_i(r) \geqslant f_i(s)\}.$$

In words, non-decreasingness requires that agent i's share does not decrease when some agent k changes her report about pairs involving i in favor of i, while k's report about the pairs of agents other than i, and the reports of all agents other than k are unchanged.

Proposition 2. *The division rule f^ρ*

(i) *is impartial*, *consensual*, *anonymous*, *continuous*, *and non-decreasing*;
(ii) *is not cost-feasible*:

$$\inf_r \sum_{i=1}^n f_i^\rho(r) = 0;$$

(iii) *is feasible if and only if ρ satisfies*

$$\rho(z)\rho\left(\frac{1}{z}\right) \geqslant 1 \quad \textit{for all } z \in \mathbb{R}^{n-2}_{++} \quad \textit{where } \frac{1}{z} = \left(\frac{1}{z_1}, \ldots, \frac{1}{z_{n-2}}\right). \tag{6}$$

Proof. The rules f^ρ are consensual by Lemma 1. The other properties in the first statement are obvious.

For the second statement, fix $a > 0$, and consider the evaluation profile (4). The unanimity property of ρ implies $\rho(r_{i+1,i}) = \frac{1}{a}$ for all i, and therefore $f_i^\rho(r) \leqslant \frac{1}{1+\rho(r_{i+1,i})} \leqslant a$. Choosing a arbitrarily small proves the claim.

For the third statement, assume first that ρ satisfies inequalities (6). Fix an evaluation profile r, then apply the first statement of Lemma 3 (in Appendix A.1) to $y_{ij} = \rho(r_{ij})$, taking into account the fact that $r^k_{ij} r^k_{ji} = 1$. This yields $\sum_{i \in N} f_i^\rho(r) \leqslant 1$. Conversely, we must show that ρ satisfies (6) if f^ρ is feasible. Fix $z_1, \ldots, z_{n-2} \in \mathbb{R}^{n-2}_{++}$, and consider the following profile of reports:

$$r^1_{2i} = r^2_{1i} = \lambda \quad \text{for all } i \geqslant 3, \quad r^k_{2i} = \lambda, r^k_{1i} = \lambda z_k, \quad \text{for all } k \geqslant 3 \text{ and all } i \geqslant 3$$

the other coordinates being deduced from the consistency condition that defines the evaluations. For all $i \geqslant 3$, $\rho(r_{2i}) = \lambda$, and therefore $f_i^\rho(r) \to 0$ as $\lambda \to +\infty$, whereas

$$f_1^\rho(r) \to \frac{1}{1 + \rho\left(\dfrac{1}{z_1}, \ldots, \dfrac{1}{z_{n-2}}\right)} \quad \text{and} \quad f_2^\rho(r) \to \frac{1}{1 + \rho(z_1, \ldots, z_{n-2})}.$$

Now feasibility implies inequality (6) at once. □

If $n = 3$, there is only one trivial $(n-2)$-aggregator ρ, and it meets property (6). Thus Proposition 2 repeats part of Proposition 1. For $n \geqslant 4$, condition (6) places some restrictions on ρ. The harmonic mean fails it, but the geometric and arithmetic means meet it. The kth rank order aggregator ρ^{*k} meets (6) if and only if $\lfloor \frac{n}{2} \rfloor \leqslant k \leqslant n-2$, where $\lfloor x \rfloor$ denotes the largest integer bounded above by x. Indeed inequality (6) reads $z^*_k \geqslant z^*_{n-1-k} \Leftrightarrow k \geqslant \frac{n-1}{2}$.

To further illustrate the impact of property (6), it is useful to introduce convex combinations of rank order aggregators as well as "geometric" combinations. For any set of convex weights λ, $\lambda \in \mathbb{R}_+^{n-2}$ and $\sum_{i=1}^{n-2} \lambda_i = 1$, define the two $(n-2)$-aggregators σ_λ and π_λ

$$\sigma_\lambda(z) = \sum_{i=1}^{n-2} \lambda_i z_i^*; \quad \pi_\lambda(z) = \prod_{i=1}^{n-2} (z_i^*)^{\lambda_i}. \tag{7}$$

Lemma 2. *The $(n-2)$-aggregator σ_λ satisfies condition* (6) (*and the corresponding rule f^{σ_λ} is feasible*) *if and only if*

$$\sum_{i=1}^{k} \lambda_i \leqslant \sum_{i=1}^{k} \lambda_{n-i-1} \quad \textit{for all } k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1. \tag{8}$$

The same statement holds for the aggregator π_λ.

The proof is in Appendix A. We discuss now some examples.

If n is odd, the median aggregator is the rank order aggregator $\rho^{*\frac{n-1}{2}}$ and we already know that the corresponding rule is feasible. If n is even, there is an *upper median* $\rho^{*\frac{n}{2}}$ and a *lower median* $\rho^{*(\frac{n}{2}-1)}$. Lemma 2 says that $\rho^{*\frac{n}{2}}$, as well as $\frac{1}{2}(\rho^{*(\frac{n}{2}-1)} + \rho^{*\frac{n}{2}})$ and $\sqrt{\rho^{*(\frac{n}{2}-1)}\rho^{*\frac{n}{2}}}$ satisfy (6), but $\rho^{*(\frac{n}{2}-1)}$ does not.

Recall that the geometric mean is not larger than the arithmetic mean. Similarly $\pi_\lambda(z) \leqslant \sigma_\lambda(z)$ for all z (because log is a concave function). This implies $f^{\sigma_\lambda} \leqslant f^{\pi_\lambda}$, which means that the rule f^{π_λ} Pareto dominates the rule f^{σ_λ}, in particular it is less "wasteful" than f^{σ_λ}.

Remark 1. Participants will prefer a rule that wastes as little money as possible, and to this end an aggregator ensuring $\rho(z)\rho(\frac{1}{z}) \equiv 1$ is maximal: no other aggregator ρ' meeting (6) Pareto dominates ρ. The following fact is obvious: among the geometric aggregators π_λ the property $\{\pi_\lambda(z)\pi_\lambda(\frac{1}{z}) = 1$ for all $z\}$ holds if and only if λ is symmetric, i.e., $\lambda_i = \lambda_{n-i-1}$, for all $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1$. Examples include the usual geometric mean, the median $\rho^{*\frac{n-1}{2}}$ if n is odd, the geometric average of the two medians $\sqrt{\rho^{*(\frac{n}{2}-1)}\rho^{*\frac{n}{2}}}$ if n is even, as well as the geometric mean of max and min. On the other hand, it is easily seen that for odd n the only "convex" aggregator σ_λ satisfying $\sigma_\lambda(z)\sigma_\lambda(\frac{1}{z}) \equiv 1$ is the median, and that for even n this property is out of reach. In this limited sense, geometric aggregators are therefore more efficient than convex ones.

Remark 2. The canonical three-person rule in Proposition 1 is non-wasteful only if the reports are consensual. Many of the rules introduced in Definition 6 share this property, that we dub *consensus-rewarding*:

$$\text{for all } r : \sum_{i=1}^{n} f_i(r) = 1 \quad \Longleftrightarrow \quad r \text{ is consensual.} \tag{9}$$

If the participants engage in extensive discussions before sending their reports, property (9) gives them a strong collective incentive to reach a consensus. We prove in Appendix A.3 the following fact: if λ is a set of convex weights meeting (8) and such that $\lambda_{n-2} > 0$, then the division rule f^{σ_λ} is consensus-rewarding.

For instance the arithmetic mean, and the half-sum of max and min yield consensus-rewarding rules. On the other hand, if λ meets (8) the geometric aggregator π_λ does not yield a consensus-rewarding rule (see Appendix A.3 for details).

Note that the two properties discussed in Remarks 1 and 2 are incompatible for the division rules of Definition 6. Pick a division rule f^ρ where ρ satisfies $\rho(z)\rho(\frac{1}{z}) \equiv 1$ and check that property (9) fails at the following non-consensual profile r:

$$r^1_{2i} = r^2_{1i} = 2, \quad r^1_{ij} = r^2_{ij} = 1, \quad r^i_{kl} = 1 \quad \text{for all } i, j \in \{3, \dots, n\} \text{ and all } k, l$$

because for all $i, j \in \{3, \dots, n\}$ we have $\rho(r_{12}) = \rho(r_{ij}) = 1, \rho(r_{1i}) = \rho(r_{2i}) = \rho(2, 1, \dots, 1) = a$, and $\rho(r_{i1}) = \rho(r_{i2}) = \rho(\frac{1}{2}, 1, \dots, 1) = \frac{1}{a}$ so that the share ratios $\rho(r_{kl})$ are consistent.

In the next section, we characterize the rules of Definition 6 by a natural property of separability.

4. Separability and the aggregated ratios

For any impartial division rule, agent i's payoff depends only on the reported ratios that concern him directly, namely the numbers r^k_{ji} for all $j, k \in N \setminus \{i\}$. Indeed the other ratios r^k_{jl} are deduced from the consistency properties of $\mathcal{R}[N \setminus \{k\}]$, via $r^k_{jl} = \frac{r^k_{ji}}{r^k_{li}}$. The next definition qualifies the functional dependence of agent i's payoff on this vector.

Any of the rules f^ρ in Definition 6 provides the intuition for Separability. Consider two profiles of reports r^{-1} and $\hat{r}^{-1}$ by the agents other than 1, that differ only in components r^i_{21}, for $i = 3, \dots, n$. As $\rho(r_{j1}) = \rho(\hat{r}_{j1})$ for all $j = 3, \dots, n$, we can tell if agent 1's share goes up or down simply by comparing $\rho(r_{21})$ and $\rho(\hat{r}_{21})$, and for this we only need to know the numbers r^i_{21} and $\hat{r}^i_{21}$, for $i = 3, \dots, n$. Thus the impact on agent 1's share of a change in the reports of the relative shares of 1 and 2 can be evaluated independently of the rest of the reports. The separability property only requires that if a change in the reports r^i_{21} and $\hat{r}^i_{21}$ leaves agent 1's share unchanged, it does so independently of the rest of the reports.

Definition 8. An impartial division rule f is separable if for all $i \in N$, and all profiles $r, \hat{r}, s, \hat{s}$, for which there exists $j \in N \setminus \{i\}$ such that

$$r^l_{ki} = \hat{r}^l_{ki} \quad \text{and} \quad s^l_{ki} = \hat{s}^l_{ki} \quad \text{for all } k \neq i, j, \text{ and all } l \neq i, k, \tag{10}$$

$$r^l_{ji} = s^l_{ji} \quad \text{and} \quad \hat{r}^l_{ji} = \hat{s}^l_{ji} \quad \text{for all } l \neq i, j \tag{11}$$

we have: $f_i(r) = f_i(\hat{r}) \Leftrightarrow f_i(s) = f_i(\hat{s})$.

Theorem 1. *Assume that $n \geqslant 4$. An impartial division rule f is anonymous, consensual, continuous, feasible, non-decreasing, and separable if and only if $f = f^\rho$ (Definition 6) for some $(n-2)$-aggregator ρ that satisfies property (6).*

Proof. If. We only need to show that f^ρ is separable. The other properties have either already been discussed in Section 3, or are straightforward. Consider some evaluation profiles $r, \hat{r}, s, \hat{s}$ and a pair of agents i, j, as in Definition 8. Condition (10) implies $\{f^\rho_i(r) = f^\rho_i(\hat{r}) \Leftrightarrow \rho(r_{ji}) = \rho(\hat{r}_{ji})\}$ and $\{f^\rho_i(s) = f^\rho_i(\hat{s}) \Leftrightarrow \rho(s_{ji}) = \rho(\hat{s}_{ji})\}$. Condition (11) implies $\rho(r_{ji}) = \rho(s_{ji})$ and $\rho(\hat{r}_{ji}) = \rho(\hat{s}_{ji})$.

Only if. For each pair i, j in N and each $x \in \mathbb{R}_{++}^{N \setminus \{i,j\}}$, let $s(x, ij)$ be the following evaluation profile:

$$s^i_{kk'}(x, ij) = 1 \quad \text{for all } k, k' \neq i,$$

$$s^k_{ij}(x, ij) = x_k \quad \text{for all } k \neq i, j,$$

$$s^k_{ij'}(x, ij) = 1 \quad \text{for all } j' \neq j \text{ and } k \neq i, j'.$$

The rest of the profile is determined by the consistency properties of evaluations (Definition 1). Define a real-valued function g_{ij} as follows:

$$g_{ij}(x) = f_i(s(x, ij)) \quad \text{for all } x \in \mathbb{R}_{++}^{N \setminus \{i,j\}}.$$

The functions g_{ij} are all identical because f is anonymous. Let g be this common function. Anonymity implies that g is symmetric. In addition, g inherits from f the properties of non-decreasingness and continuity. For any $a, a > 0$, let $c(a) \in \mathbb{R}_{++}^{N \setminus \{i,j\}}$ be the vector $c_k(a) = a$ for all $k \in N \setminus \{i, j\}$. For all i, j, it is easy to construct a consensual evaluation profile $\hat{s}(c(a), ij)$ such that $\hat{s}^k(c(a), ij) = s^k(c(a), ij)$, for all $k \neq i$. Consensuality and impartiality imply

$$g(c(a)) = f_i(\hat{s}(c(a), ij)) = \frac{1}{(n-1) + 1/a} \tag{12}$$

(Lemma 1). Because g is non-decreasing, $g(x) \leqslant g(c(\max_{N \setminus \{i,j\}} x_k))$, hence Eq. (12) implies $g(x) \leqslant \frac{1}{n-1}$.

Now define ρ on $\mathbb{R}_{++}^{n-2}$ as follows:

$$\rho(x) = \frac{1}{g\left(\dfrac{1}{x}\right)} - (n-1).$$

The range of ρ is $\mathbb{R}_{++}$ because that of g is $]0, \frac{1}{n-1}[$. Next ρ inherits from g the properties of continuity, non-decreasingness and symmetry. Finally $\rho(c(a)) = a$ for $a > 0$ (12). Thus ρ is an $(n-2)$-aggregator (Definition 5).

Let r and $\hat{r}$ be two evaluation profiles and let i be an agent. Following an argument of Gorman [2], we now prove that

$$\{\rho(r_{ji}) = \rho(\hat{r}_{ji}) \text{ for all } j \in N \setminus \{i\}\} \;\Rightarrow\; f_i(r) = f_i(\hat{r}). \tag{13}$$

Let $j \in N \setminus \{i\}$, and let t be the evaluation profile defined as follows:

$$t^l_{ij} = \hat{r}^l_{ij} \quad \text{for all } l \neq i, j,$$

$$t^l_{ik} = r^l_{ik} \quad \text{for all } k \neq i, j \text{ and all } l \neq i, k.$$

If $\rho(r_{ji}) = \rho(\hat{r}_{ji})$, then $\rho(r_{ji}) = \rho(t_{ji})$, and hence $g(r_{ij}) = g(t_{ij})$, or $f_i(s(r_{ij}, ij)) = f_i(s(t_{ij}, ij))$. Impartiality and separability imply $f_i(r) = f_i(t)$. An argument by induction, changing j at each step, leads to (13).

Now we pick $i \in N$, and r an evaluation profile, both arbitrary. Define $\hat{r}$ as follows, for all $j, k \in N \setminus \{i\}$:

$$\hat{r}^k_{ji} = \rho(r_{ji}); \quad \hat{r}^i_{jk} = \frac{\hat{r}^k_{ji}}{\hat{r}^j_{ki}}.$$

Clearly $\rho(r_{ji}) = \rho(\hat{r}_{ji})$ for $j \in N \setminus \{i\}$, so formula (13) yields $f_i(r) = f_i(\hat{r})$. On the other hand, $\hat{r}$ is consensual (Lemma 1), and hence

$$f_i(\hat{r}) = \frac{1}{1 + \sum_{j \in N \setminus \{i\}} \rho(r_{ji})} = f_i^{\rho}(r).$$

Finally, the feasibility of f implies that ρ satisfies (6). $\square$

The properties appearing in Theorem 1 are logically independent. To prove this, we construct seven different division rules, each one of them satisfying all but one of the axioms.

1. Consider the division rule that splits the dollar equally among all the participants, independently of their reports. It is not consensual, but it does satisfy all the other properties listed in Theorem 1.

2. Write σ^* and π^* respectively for the arithmetic and geometric mean (corresponding in (7) to the uniform vector $\lambda_i = \frac{1}{n-2}$ for all i). The rule $f = \frac{1}{2}(f^{\pi^*} + f^{\sigma^*})$ satisfies all the properties, except Separability.

3. Define the $(n-2)$-aggregator ρ:

$$\rho(z) = \begin{cases} \pi^*(z) & \text{if } \pi^*(z) \leqslant 1 \\ \sigma^*(z) & \text{if } \pi^*(z) > 1 \end{cases} \qquad \text{for all } z.$$

The rule f^{ρ} satisfies all the properties (remember that $\sigma^* \geqslant \pi^*$), except Continuity.

4. We know from Proposition 2 that the rule f^{ρ} is not feasible if ρ does not satisfy (6). Taking ρ as the harmonic mean, for instance, we find a rule that satisfies all the properties, except Feasibility.

5. Consider the following mapping ρ:

$$\rho(z) = \sigma^*(z) + \left(\max_{i \in N} z_i - \min_{i \in N} z_i\right) \qquad \text{for all } z.$$

It satisfies all the properties of an $(n-2)$-aggregator except for non-decreasingness. Defining f^{ρ} as in Definition 6, we have a division rule failing non-decreasingness as well, and satisfying all other properties (e.g., feasibility because $\rho \geqslant \sigma^*$).

6. Pick any rule f^{ρ} in Theorem 1 and project it on the simplex as $f(r) = f^{\rho}(r) / \sum_{i \in N} f_i^{\rho}(r)$ for all r. The projection preserves all the properties except Impartiality. Note that Separability does not apply here, because we chose (for simplicity) to define it only for impartial rules.

7. We now construct a non-anonymous rule meeting all the other axioms. This construction will be useful in the next section as well. Fix an agent $j \in N$, an $(n-2)$-aggregator ρ and an $(n-3)$-aggregator τ both satisfying (6). We use the notation introduced in Definition 6: $\rho(r_{ji}) = \rho((r_{ji}^k)_{k \in N \setminus \{i,j\}})$. Similarly, $\tau(r_{ki}^{-j}) = \tau((r_{ki}^l)_{l \in N \setminus \{i,j,k\}})$. Let $f^{\rho,\tau}[-j]$ be a division rule similar to f^{ρ} (see (5)), but in which agent j's report is omitted. There are $(n-2)$ reports comparing j to another agent, and $(n-3)$ reports comparing agents different from j, thus the

following expressions are well defined for all r:

$$f_j^{\rho,\tau}[-j](r) = \frac{1}{1+\sum_{k\in N\setminus\{j\}} \rho(r_{kj})}, \tag{14}$$

$$f_i^{\rho,\tau}[-j](r) = \frac{1}{1+\rho(r_{ji})+\sum_{k\in N\setminus\{i,j\}} \tau(r_{ki}^{-j})} \quad \text{for all } i \neq j. \tag{15}$$

Feasibility of the division rule $f^{\rho,\tau}[-j]$ follows from Lemma 3 in Appendix A.1. The other properties are easy to check.

5. Four agents or more: exact rules

The properties of consensuality and impartiality are far from characterizing a unique division rule when there are four agents or more. The "aggregated ratios" rules characterized in Theorem 1 are a rich family of such rules, none of which is exact.

Our construction of exact rules relies on the non-anonymous rules defined above. Assume $n \geqslant 4$. Choose an $(n-2)$-aggregator ρ, and an $(n-3)$-aggregator τ. For any choice of j in N, Eqs. (14) and (15) define the feasible but inexact rule $f^{\rho,\tau}[-j]$. We turn this rule into an exact rule $f^{\rho,\tau}\{-j\}$ by preserving the shares of agents other than j while making j the *residual claimant,* i.e., for all profiles r:

$$\begin{aligned} f_j^{\rho,\tau}\{-j\}(r) &= 1 - \sum_{i\in N\setminus\{j\}} f_i^{\rho,\tau}[-j](r); \\ f_i^{\rho,\tau}\{-j\}(r) &= f_i^{\rho,\tau}[-j](r) \quad \text{for all } i \neq j. \end{aligned} \tag{16}$$

The exact and anonymous rule f is simply the average of the n rules $f^{\rho,\tau}\{-j\}$. For all i and r:

$$\begin{aligned} f_i(r) &= \frac{1}{n}\sum_{j\in N} f_i^{\rho,\tau}\{-j\}(r) \\ &= \frac{1}{n}\left(1 - \sum_{j\in N\setminus\{i\}} f_j^{\rho,\tau}[-i](r)\right) + \frac{1}{n}\sum_{j\in N\setminus\{i\}} f_i^{\rho,\tau}[-j](r). \end{aligned} \tag{17}$$

Theorem 2. *Assume that $n \geqslant 4$. If the $(n-2)$-aggregator ρ and the $(n-3)$-aggregator τ both satisfy inequality* (6), *then Eq.* (17) *defines an exact, anonymous, impartial, consensual, and continuous division rule.*

Proof. For any $i, j, i \neq j$, the shares $f_i^{\rho,\tau}[-j](r)$ are positive by definition of $f^{\rho,\tau}[-j]$. Feasibility of $f^{\rho,\tau}[-i]$ (discussed after (15)) implies $f_i^{\rho,\tau}\{-i\}(r) \geqslant 0$ for all r. Therefore $f_i(r) > 0$ for all i and r.

The rule $f^{\rho,\tau}\{-j\}$ is exact by construction, and obviously continuous, impartial, and non-decreasing. To check that it is consensual, suppose r is associated with the consensual division x. Then, for each j different from i, $\rho(r_{ij}) = \frac{x_i}{x_j}$ and $\tau(r_{kj}^{-i}) = \frac{x_k}{x_j}$, so that $f_j^{\rho,\tau}[-i](r) = x_j$, and i's residual share is $1 - \sum_{j\in N\setminus\{i\}} x_j = x_i$.

The rule f is the arithmetic average of the n distinct i-rules, hence it is anonymous. The other four properties are preserved under convex combinations. □

We emphasize that there are many other rules satisfying the five properties in Theorem 2. First, these properties are stable under convex combinations, and we have many choices of the aggregators ρ and τ. Next we can construct rules that satisfy the five properties, in which aggregators are replaced by averaging at a different level. Suppose for simplicity that $n = 4$. When agent 1 is the residual claimant, let agent 2's share be the average of the two terms $\frac{1}{1+r_{12}^i+r_{32}^4+r_{42}^3}$, for $i = 3, 4$. The other agents' shares are deduced by symmetry. Now $f_1(r)$ takes the form $f_1(r) = \frac{1}{4} + \frac{1}{8}(T - T')$, where T is the sum of six terms like $\frac{1}{1+r_{21}^3+r_{31}^4+r_{41}^3}$, and T' that of six terms like $\frac{1}{1+r_{12}^3+r_{32}^4+r_{42}^3}$.

We note that we can also apply our division rules to situations in which some agents are unable to evaluate the relative contributions of some of the other agents, provided that for each pair of agents i, j, there are at least two agents other than i and j who are able to evaluate both i and j. Whatever aggregator we use is simply applied to the information from all agents who are able to evaluate both i and j.

Remark 3. The rules constructed in Theorem 2 share an appealing approximation property: if the reports are almost consensual, then the outcome is similarly close to the almost-consensus. Say that the evaluation profile r is ε-consensual if there exists a vector $x \in R^N$ such that $\sum_{i\in N} x_i = 1$, and

$$r_{ij}^k \leqslant (1+\varepsilon)\frac{x_i}{x_j} \quad \text{for all } k \in N \text{ and all } i, j \in N \setminus \{k\}.$$

If the evaluation profile r is ε-consensual with respect to x, and f is the rule (17), then $|f(r) - x| = O(\varepsilon)$.[3] In words, if all opinions are "close" to an underlying compromise x, our methods implement a division which is comparably close. The easy proof is omitted.

Remark 4. The (inexact) rule $f^{\max}$, which is the smallest of all separable rules in Proposition 2, is also a lower bound for all the rules f constructed in Theorem 2:

$$f_i(r) \geqslant f_i^{\max}(r) = \frac{1}{1+\sum_{j\in N\setminus\{i\}} \max_{k\in N\setminus\{i,j\}} r_{ji}^k} \quad \text{for all } i \text{ and } r. \tag{18}$$

Fix i and r. Since $\rho(r_{ji}) \leqslant \max_{k\in N\setminus\{i,j\}} r_{ji}^k$ for all $j \in N \setminus \{i\}$, and $f^{\rho,\tau}[-i]$ is feasible we have $1-\sum_{j\in N\setminus\{i\}} f_j^{\rho,\tau}[-i](r) \geqslant f_i^{\rho,\tau}[-i](r) \geqslant f_i^{max}(r)$. On the other hand, $\tau(r_{ki}^{-j}) \leqslant \max_{l\in N\setminus\{i,k\}} r_{ki}^l$ for all $k \in N \setminus \{i, j\}$ therefore $f_i^{\rho,\tau}[-j](r) \geqslant f_i^{max}(r)$, for all $j \in N \setminus \{i\}$. Inequality (18) then follows from (17).

6. Zero shares

Definition 3 allows the division rule to assign a null share to some agents, however Definition 1 requires $r_{ji}^k > 0$, i.e., agent i is not allowed to report that j's claim to the dollar is null. A consequence of this restriction is that for any aggregated ratios rule f^ρ (Definition 6) and any rule f in Theorem 2, individual shares are always strictly positive.

Our model can be adapted to accommodate null shares in the reports as well as in the actual shares. We briefly sketch this extension, of which the details are available upon request from the

[3] That is to say, $\sup_{\varepsilon>0} \sup_r \frac{|f(r)-x|}{\varepsilon} < \infty$, where the second supremum bears on all r that are ε-consensual.

authors. Say agent i thinks that within $N\backslash\{i\}$, only the agents in the subset S^i deserve a positive share. His evaluations r^i_{jk} are then constrained as follows:

$$r^i_{jk} = 0 \text{ if } j \in N\backslash(S^i \cup \{i\}),\ \ k \in S^i, \quad r^i_{jk} = \infty \text{ if } j \in S^i,\ \ k \in N\backslash(S^i \cup \{i\}),$$
$$0 < r^i_{jk} < \infty \quad \text{if } j, k \in S^i.$$

Note that r^i_{jk} is not defined if j, k are both in $N\backslash(S^i \cup \{i\})$.

After extending the definitions of impartiality and consensuality in the obvious way, we define as before the method f^ρ by

$$f^\rho_i(r) = \frac{1}{1 + \sum_{j \in N\backslash\{i\}} \rho(r_{ji})} \quad \text{for all } i \in N \text{ and all profiles } r, \tag{19}$$

where the aggregator $\rho(z_1, \ldots, z_{n-2})$ is now defined over $[\mathbb{R}_+ \cup \{\infty\}]^{n-2}$. Observe that $\rho(r_{ji}) = \rho((r^k_{ji})_{k \in N\backslash\{i,j\}})$ is not defined if for some $k \in N\backslash\{i, j\}$, we have $j, i \in N\backslash(S^k \cup \{k\})$. However in that case $r^k_{li} = \infty$ for any agent $l \in S^k$, therefore if the aggregator ρ satisfies

$$\text{for all } z \in [0, \infty]^{n-2} \ \max_i z_i = \infty \Rightarrow \rho(z) = \infty \tag{20}$$

then the above expression can be simply defined as $f^\rho_i(r) = \frac{1}{\infty} = 0$. Naturally ρ must also satisfy property (6). Note that for a convex aggregator σ_λ given by (7), property (20) amounts to $\lambda_{n-2} > 0$.

We can then define the exact methods of Theorem 2 just as before. We can also choose the aggregators ρ and τ in such a way that the only case in which an agent i gets a zero share is if the reports by all other agents are fully consensual, and they agree that agent i should get nothing.

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Appendix A.

A.1. A mathematical inequality

We establish in this section a mathematical inequality that plays a crucial role in proving most of our results (see Propositions 1 and 2, Theorem 2, and inequality (18)).

Lemma 3. *Fix $N = \{1, \ldots, n\}$ where $n \geqslant 1$ and a matrix $y \in \mathbb{R}^{N \times N}_{++}$ such that $y_{ij} y_{ji} \geqslant 1$, for all i, j (in particular $y_{ii} \geqslant 1$, for all i). Then*

$$\sum_{i \in N} \frac{1}{\sum_{j \in N} y_{ji}} \leqslant 1, \tag{21}$$

$$\sum_{i \in N} \frac{1}{\sum_{j \in N} y_{ji}} = 1 \quad \Leftrightarrow \quad y_{ij} y_{jk} y_{ki} = 1 \quad \text{for all } i, j, k. \tag{22}$$

Proof. *Step* 1: *inequality* (21)[4]

Notice first that it is enough to prove (21) when $y_{ij}y_{ji} = 1$ for all i, j. The desired inequality is obvious if $n = 1$, and we proceed by induction on n. Assume (21) holds for n.

Fix $N = \{1, \dots, n\}$ and $y \in \mathbb{R}_{++}^{N\cup\{0\}\times N\cup\{0\}}$ such that $y_{ij}y_{ji} \geqslant 1$, for all $i, j \in N \cup \{0\}$. Set $z_i = \sum_{j\in N} y_{ji}$ and $x_i = y_{0i}$. We want to prove

$$\frac{1}{1+\sum_{i\in N} \dfrac{1}{x_i}} + \sum_{i\in N} \frac{1}{z_i + x_i} \leqslant 1 \quad \Leftrightarrow \quad \sum_{i\in N} \frac{1}{z_i + x_i} \leqslant \frac{\sum_{i\in N} \dfrac{1}{x_i}}{1+\sum_{i\in N} \dfrac{1}{x_i}}. \tag{23}$$

The induction hypothesis implies $\sum_{i\in N} \frac{1}{z_i} \leqslant 1$. We compute now, for x fixed in $\mathbb{R}_{++}^N$, the maximum of $\sum_{i\in N} \frac{1}{z_i+x_i}$ when $z \in \mathbb{R}_{++}^N$ is such that $\sum_{i\in N} \frac{1}{z_i} \leqslant 1$. Changing variables to $w_i = \frac{1}{z_i}$, this is equivalent to the maximization problem

$$\max_{w\geqslant 0, \sum_{i\in N} w_i \leqslant 1} \sum_{i\in N} \frac{w_i}{1 + x_i w_i} = \max_{w\geqslant 0, \sum_{i\in N} w_i = 1} \sum_{i\in N} \frac{w_i}{1 + x_i w_i}. \tag{24}$$

The function $\psi(w) = \sum_{i\in N} \frac{w_i}{1+x_i w_i}$ is concave and differentiable on $\mathbb{R}_{+}^N$. At the point w^* defined by

$$w_i^* = \frac{\dfrac{1}{x_i}}{\sum_{j\in N} \dfrac{1}{x_j}} \quad \text{for all } i$$

the gradient of ψ is diagonal:

$$\nabla\psi(w^*) = \frac{1}{\left(1 + \dfrac{1}{\sum_{j\in N} \dfrac{1}{x_j}}\right)^2}(1, 1, \dots, 1).$$

Pick any w in $\mathbb{R}_{+}^N$ such that $\sum_{i\in N} w_i = \sum_{i\in N} w_i^* = 1$ and apply the concavity of ψ at w^*:

$$\psi(w) - \psi(w^*) \leqslant \nabla\psi(w^*) \cdot (w - w^*) = 0.$$

This shows that w^* is an (in fact, the unique) optimal solution of program (24) and $\psi(w^*) = \frac{\sum_{i\in N} \frac{1}{x_i}}{1+\sum_{i\in N} \frac{1}{x_i}}$ is its maximum value. The proof of inequality (23) and of step 1 is complete.

Step 2: *property* (22)

We can assume $n \geqslant 2$ because the statement is obvious for $n = 1$. For a set of share ratios y as in the premises of Lemma 3, the property $\{y_{ij}y_{jk}y_{ki} = 1$, for all $i, j, k\}$ is what we called the consistency of these ratios in Section 2, and it is equivalent to the existence of $x \in \mathbb{R}_{++}^N$ such that $y_{ij} = \frac{x_i}{x_j}$ for all i, j. As in Lemma 1 this implies the left-hand side equality in (22).

[4] We are grateful to the referee for suggesting the induction argument of this proof.

Conversely assume y meets the left-hand-side equality in (22). By step 1, y is a maximizer of $\phi(y) = \sum_{i\in N} \frac{1}{\sum_{j\in N} y_{ji}}$ over the set of those $y \in \mathbb{R}^{N\times N}_{++}$ such that $y_{ij}y_{ji} = 1$, for all i, j.

When we replace y_{ij} by $1/y_{ji}$ for all $j > i$, the maximization of ϕ on this subset of $\mathbb{R}^{N\times N}_{++}$ becomes an unrestricted maximization of the function

$$\sum_{i\in N} \frac{1}{1 + \sum_{j<i} \frac{1}{y_{ij}} + \sum_{j>i} y_{ji}}$$

over the $\frac{n(n-1)}{2}$-dimensional vectors $(y_{ij})_{\{j>i\}}$. An optimal solution y must satisfy the first order conditions, namely:

$$\frac{1}{(1 + \sum_{k\in N\setminus\{i\}} y_{ki})^2} - \frac{1}{y_{ji}^2 (1 + \sum_{k\in N\setminus\{j\}} y_{kj})^2} = 0 \quad \text{for all } i, j \text{ s.t. } j > i$$

$$\Leftrightarrow \quad y_{ji} = \frac{1 + \sum_{k\in N\setminus\{i\}} y_{ki}}{1 + \sum_{k\in N\setminus\{j\}} y_{kj}} \quad \text{for all } i, j. \tag{25}$$

Setting $x_i = \frac{1}{1+\sum_{k\in N\setminus\{i\}} y_{ki}}$ for all i we see that $y_{ji} = \frac{x_j}{x_i}$ has the desired consistency property. $\square$

A.2. *Proof of Lemma 2*

We use the notations of Definition 7 and Lemma 2.

Step 1: *geometric aggregator* π_λ

Inequalities (6) write

$$\prod_{i=1}^{n-2} (z_i^*)^{\lambda_i - \lambda_{n-i-1}} \geqslant 1 \quad \text{for all } z \in \mathbb{R}^{n-2}_{++}. \tag{26}$$

Applying this inequality to $z = (1, \ldots, 1, 2)$ gives $\lambda_{n-2} \geqslant \lambda_1$; applying it to $z = (1, \ldots, 1, 2, 2)$ gives $\lambda_{n-2} + \lambda_{n-3} \geqslant \lambda_1 + \lambda_2$, and so on. This is system (8).

Conversely, suppose (8) holds and assume to fix ideas n is even, $n - 2 = 2m$ (the entirely similar proof for n odd is omitted). Set

$$\mu_k = \sum_{i=1}^{k} (\lambda_{2m-i+1} - \lambda_i) \quad \text{for } k = 1, \ldots, m.$$

To prove (26) we assume $z_i^* = z_i$ without loss of generality, and we rearrange (26) as

$$\left(\frac{z_{m+1}}{z_m}\right)^{\lambda_{m+1}-\lambda_m} \left(\frac{z_{m+2}}{z_{m-1}}\right)^{\lambda_{m+2}-\lambda_{m-1}} \cdots \left(\frac{z_{2m}}{z_1}\right)^{\lambda_{2m}-\lambda_1} \geqslant 1$$

$$\Leftrightarrow \left(\frac{z_{m+1}}{z_m}\right)^{\mu_m} \left(\frac{z_{m+2}}{z_{m-1}} \frac{z_m}{z_{m+1}}\right)^{\mu_{m-1}} \left(\frac{z_{m+3}}{z_{m-2}} \frac{z_{m-1}}{z_{m+2}}\right)^{\mu_{m-2}} \cdots \left(\frac{z_{2m}}{z_1} \frac{z_2}{z_{2m-1}}\right)^{\mu_1} \geqslant 1$$

and the desired conclusion follows from our assumptions $\mu_k \geqslant 0$ for all k and $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_{2m}$.

Step 2: *convex aggregators* σ_λ

If system (8) holds for λ, the method f^{π_λ} is feasible. Moreover $\pi_\lambda \leqslant \sigma_\lambda$ implies $f^{\sigma_\lambda} \leqslant f^{\pi_\lambda}$ and f^{σ_λ} is feasible as well. Conversely, pick λ such that σ_λ meets property (6).

Fix $\varepsilon > 0$, and let k be an integer such that $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor - 1$. Set $a = \sum_{i=1}^{k} \lambda_i$, $b = \sum_{i=n-k-1}^{n-2} \lambda_i$, and $c = 1 - a - b$. Finally, define z as follows:

$$z_i = \begin{cases} 1 - \varepsilon & \text{if } i \leqslant k, \\ 1 & \text{if } k + 1 \leqslant i \leqslant n - k - 2, \\ 1 + \varepsilon & \text{if } n - k - 1 \leqslant i \leqslant n - 2. \end{cases}$$

Compute

$$\sigma_\lambda(z)\sigma_\lambda\left(\frac{1}{z}\right) = ((1-\varepsilon)a + c + (1+\varepsilon)b)\left(\frac{a}{1+\varepsilon} + c + \frac{b}{1-\varepsilon}\right).$$

The right-hand side equals 1 for $\varepsilon = 0$, hence by (6) its derivative at $\varepsilon = 0$ is non-negative. This implies $a \leqslant b$, i.e., the kth inequality in system (8). $\square$

A.3. Proof of Remark 2

Take λ satisfying (8) and $\lambda_{n-2} > 0$. Check first that for all z

$$\sigma_\lambda(z)\sigma_\lambda\left(\frac{1}{z}\right) = 1 \iff z_1 = z_2 = \cdots = z_{n-2}. \tag{27}$$

Consider the convex vector of weights

$$\lambda'_i = \lambda'_{n-i-1} = \frac{\lambda_i + \lambda_{n-i-1}}{2}.$$

Clearly λ' meets (8) (as equalities) hence $\sigma_{\lambda'}(z)\sigma_{\lambda'}(\frac{1}{z}) \geqslant 1$ for all z. Observe next that $\sigma_\lambda(z) \geqslant \sigma_{\lambda'}(z)$ for all z: indeed it is enough to check these inequalities at those vectors z of the form $z = (0, \ldots, 0, 1, \ldots, 1)$. Thus $\sigma_\lambda(z)\sigma_\lambda(\frac{1}{z}) = 1$ implies $\sigma_{\lambda'}(z)\sigma_{\lambda'}(\frac{1}{z}) = 1$, so by Schwarz's inequality and the fact that λ' is symmetric, the vectors $(\sqrt{\lambda'_i z^*_i})$ and $(\sqrt{\lambda'_i / z^*_i})$ must be parallel. This implies at once that $z^*_1 = z^*_{n-2}$, hence $z_1 = z_2 = \cdots = z_{n-2}$.

Now we fix a profile r and apply Lemma 3 again. The equality $\sum_{i=1}^{n} f_i^{\sigma_\lambda}(r) = 1$ holds only if $\sigma_\lambda(r_{ij})\sigma_\lambda(r_{ji}) = 1$ for all i, j. In view of (27) this implies that r^k_{ij} is independent of $k \in N \setminus \{i, j\}$. Thus r is consensual by Lemma 1.

For the geometric aggregator π_λ, it is clear that the equality $\pi_\lambda(z)\pi_\lambda(\frac{1}{z}) = 1$ does not imply $z_1 = z_2 = \cdots = z_{n-2}$, which is why the corresponding division rule is not a consensus-rewarding rule.

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