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Two remarks on the inner core

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Abstract

For the case of smooth concave exchange economies, we provide a characterization of the inner core as the set of feasible allocations such that no coalition can improve on it, even if coalitions are allowed to use some random plans. For the case of compactly generated games, we discuss Myerson's definition of the inner core, and we characterize it using lexicographic utility weight systems.

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Introduction

Shapley (1969) proposed a general procedure that allows to extend TU solution concepts to NTU games by considering fictitious transfers of utility. When applied to the TU core, this procedure leads to the inner core.

We first study the inner core in the case of well-behaved exchange economies with concave utility functions. In this context, it appears to be a relevant refinement of the core. First, it contains the set of competitive equilibria. As a consequence, convergence and equivalence results for the core can immediately be extended to the inner core. Second, we show that the inner core coincides with the set of feasible allocations such that no coalition

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can improve on it, even if coalitions are allowed to use some random plans. This characterization illustrates how the inner core exploits the cardinal content of the preferences, and gives a new argument to support it as an interesting solution concept for economies with concave utility.

In a second section, we study the inner core for general compactly generated games. We provide an example to show why, in this more general context, the characterization in terms of random improvements requires centralized plans, involving a mediator which randomly selects the improving coalition. We also discuss Myerson's idea to take some topological closure of Shapley's inner core in order to correctly take into account the case of zero utility weights in Shapley's procedure, and we show how this larger set can be characterized by using lexicographic utility weight systems.

1. Concave exchange economies

Individuals are a finite set $I = \{1, 2, \dots, I\}$. Commodities are a finite set $L = \{1, 2, \dots, L\}$. Each individual is characterized by a continuous, concave and strictly increasing utility function $u^i : \mathbb{R}^L_+ \to \mathbb{R}$ and by a strictly positive initial endowment $e^i \in \mathbb{R}^L_{++}$. A concave exchange economy is thus $E = \{I, L, (u^i, e^i)_{i \in I}\}$.

A feasible allocation for the economy E is a vector $x = (x^i)_{i \in I} \in \mathbb{R}^{LI}_+$ such that $\sum_i x^i \leqslant \sum_i e^i$. An allocation x is an *equilibrium allocation* if it is feasible and there exists a price vector $p \in \mathbb{R}^L_+$ such that, for all i, $x^i \in \arg\max\{u^i(y) \mid py \leqslant pe^i\}$. The set of equilibrium allocations of the economy E is $W(E)$.

A coalition is a (non empty) subset of individuals $S \subset I$. Coalition S can improve on a given allocation x if there exists $y = (y^i)_{i \in S}$ such that $\sum_{i \in S} y^i \leqslant \sum_{i \in S} e^i$ and for all $i \in S$ $u^i(y^i) > u^i(x^i)$. An allocation is Pareto-optimal (respectively individually rational) if it is feasible and it cannot be improved upon by I (respectively by any singleton coalition). An allocation x is a *core allocation* if it is feasible and no coalition can improve on it. The core of the economy E is the set $C(E)$ of its core allocations.

In the context of concave exchange economies, the inner core was introduced by Shapley and Shubik (1975). For a given vector of utility weights $\lambda \in \mathbb{R}^I_{++}$, a coalition S can λ-improve on an allocation x if there exists $y = (y^i)_{i \in S}$ such that $\sum_{i \in S} y^i \leqslant \sum_{i \in S} e^i$ and $\sum_{i \in S} \lambda^i u^i(y^i) > \sum_{i \in S} \lambda^i u^i(x^i)$. An allocation x is an *inner core allocation* if it is feasible and there exists $\lambda \in \mathbb{R}^I_{++}$ such that no coalitions can λ-improve on it. The inner core of the economy E is the set $IC(E)$ of its inner core allocations.

Proposition 1. $W(E) \subset IC(E) \subset C(E)$.

Proof. The second inclusion is easy to prove by contraposition. Indeed, if S can improve on an allocation x, it can also λ-improve on it for any $\lambda \in \mathbb{R}^I_{++}$. We now prove the first inclusion. Let x be an equilibrium allocation at a price p. Consider one individual i and define the following set:

$$C^i = \left\{(u, m) \in \mathbb{R}^2 \mid \exists y \in \mathbb{R}^L_+ \colon\ u \leqslant u^i(y) - u^i\left(x^i\right),\ m \leqslant p\left(e^i - y\right)\right\}.$$

By the concavity of u^i, this set is convex. On the other hand, $C^i \cap \mathbb{R}^2_{++} = \emptyset$, as x^i is optimal for individual i on his budget set. By the separating hyperplane theorem there exists a non-zero non-negative vector $(\alpha^i, \beta^i) \in \mathbb{R}^2$ such that:

$$\alpha^i u^i(x^i) \geqslant \alpha^i u^i(y^i) - \beta^i p(y^i - e^i)$$

for all $y^i \in \mathbb{R}^L_+$. Strict positivity of the endowment guarantees that $pe^i > 0$, so that, from the above inequality, $\alpha^i > 0$. We can then assume that $\alpha^i = 1$. Moreover, strict monotonicity of the utility function implies that $\beta^i > 0$. Let $\lambda^i = 1/\beta^i$. Summing over all $i \in I$ we obtain

$$\sum_i \lambda^i u^i(x^i) \geqslant \sum_i \lambda^i u^i(y^i) - p \sum_i (y^i - e^i)$$

for all $y \in \mathbb{R}^{LI}_+$. If a coalition S could λ-improve on x with $(z^i)_{i \in S}$, then the previous inequality would be violated by taking y with $y^i = z^i$ for all $i \in S$ and $y^i = x^i$ for all $i \in I \setminus S$. □

Competitive equilibria exist in concave exchange economies. The inner core is thus non empty. Convergence and equivalence results for the core can be immediately extended to the inner core (see Qin, 1994a).

Remark 1. Strict monotonicity of utility functions and strict positivity of the endowments are not needed for the results above. Indeed, the weaker assumptions of local non satiation and indecomposability (see, e.g., Mas-Collel et al., 1995, for definitions) are enough to guarantee that equilibria exists and that, at an equilibrium, for all i $\beta^i > 0$ and $pe^i > 0$, so that our argument goes through.

The following example, inspired by the Banker game of Owen (1972), shows that the inclusions in the proposition may be strict.

Example 1. There are three individuals and three commodities. Initial endowments are $e^1 = (1, 0, 0)$, $e^2 = (0, 10, 0)$ and $e^3 = (0, 0, 10)$. Utility functions are $u^1(x) = x_3$, $u^2(x) = 10x_1 + x_2 - 10$, $u^3(x) = x_2 + x_3 - 10$. In this economy, Pareto optimality requires that the first good goes to individual 2. Once this is done, any reallocation of good 2 (respectively 3) between individuals 2 (respectively 1) and 3 is consistent with Pareto optimality. Notice that the only subcoalition which can do better than individual rationality is coalition $\{1, 2\}$, by giving a positive payoff to individual 2. Without the presence of individual 3, though, 1 and 2 cannot share the surplus they generate. In the grand coalition, on the other hand, individual 3 acts as an intermediary: 2 can transfer to him some of his endowment of good 2 and 3 can then compensate 1 in terms of good 3. The core is the set of all Pareto-optimal and individually rational allocations: $x^1 = (0, 0, a)$, $x^2 = (1, b, 0)$, $x^3 = (0, 10-b, 10-a)$, with $0 \leqslant a, b \leqslant 10$ and $a + b \leqslant 10$. In the utility space, it corresponds to the points (u^1, u^2, u^3) in the convex hull of $(10, 0, 0)$, $(0, 10, 0)$, $(0, 0, 10)$. We now argue that no point in the interior of this triangle can be generated by an inner core allocation. Indeed, if $\lambda \neq (1/3, 1/3, 1/3)$ the grand coalition can λ-improve on any such point by choosing a feasible allocation that maximizes the utility of the individual(s) with the highest weight.

If, on the other hand, $\lambda = (1/3, 1/3, 1/3)$, coalition $\{1, 2\}$ can λ-improve on any feasible allocation which gives a positive payoff to individual 3 by choosing $x^1 = (0, 0, 0)$, $x^2 = (1, 10, 0)$. In fact, one can show that the inner core reduces to the subset of core allocations such that $a + b = 10$ or $[a > 0$ and $b = 0]$. We now compute the competitive equilibrium. Prices must be strictly positive, and we can normalize the price of good 1 to $p_1 = 10$. If $p_2 < p_3$ then individual 3's demand for good 2 exceeds the total endowment of that good. If $p_2 > p_3$, then individual 3 keeps his initial endowment, and individual 1's demand for good 3 cannot be met. Thus $p_2 = p_3 = p$. If $p < 1$, then there is no demand for good 1. If $p > 1$, individual 2's demand for good 1 exceeds the total endowment of that good. The only equilibrium allocation is thus the optimal allocation with $a = 10$, supported by prices $(10, 1, 1)$.

In the next proposition we characterize the inner core in terms of a notion of improvement in which coalitions are allowed to use lotteries. More precisely, we say that coalition S can *L-improve* on a given allocation x if there exist $\pi \in [0, 1]$, $y = (y^i)_{i\in S}$ and $z = (z^i)_{i\in S}$ such that $\sum_{i\in S} y^i \leqslant \sum_{i\in S} e^i$, $\sum_{i\in S} z^i \leqslant \sum_{i\in S} x^i$ and for all $i \in S$ $\pi u^i(y^i) + (1-\pi)u^i(z^i) > u^i(x^i)$. Given a proposed allocation x, individuals in S are able to write contracts between themselves which specify a probability π of refusal, an allocation y to be realized in this case, and a reallocation among themselves of the commodities they would receive in case of acceptance of x. A standard improvement is the special case in which $\pi = 1$. The notion of L-improvement involves the computation of expected utilities, and exploits the cardinal content of the concave utility functions. To prove our characterization result we need uniqueness of the (normalized) vector of utility weights associated to any Pareto-optimal and individually rational allocation. Therefore we impose differentiability and a weak form of interiority.

Regularity. For all $i \in I$, u^i is C^2 on $\mathbb{R}^L_{++}$ and $(u^i)^{-1}(c) \subset \mathbb{R}^L_{++}$ for all $c \geqslant u^i(e^i)$.

Proposition 2. *Under Regularity, $x \in IC(E)$ if and only if x is a feasible allocation such that no coalition can L-improve on it.*

Proof. ($\Rightarrow$) If x is in the inner core then it is feasible and there exists $\lambda \in \mathbb{R}^I_{++}$ such that no coalition can λ-improve on x. Suppose that a coalition S has an L-improvement, that is a (π, y, z) such that $\sum_{i\in S} y^i \leqslant \sum_{i\in S} e^i$, $\sum_{i\in S} z^i \leqslant \sum_{i\in S} x^i$ and for all $i \in S$ $\pi u^i(y^i) + (1-\pi)u^i(z^i) > u^i(x^i)$. If we premultiply each term by λ^i and sum over all $i \in S$, we obtain:

$$\pi \sum_{i\in S} \lambda^i u^i(y^i) + (1-\pi) \sum_{i\in S} \lambda^i u^i(z^i) > \sum_{i\in S} \lambda^i u^i(x^i).$$

Adding $(1-\pi)\sum_{i\in I\setminus S} \lambda^i u^i(x^i)$ to both sides, and rearranging terms, we have

$$\pi \sum_{i\in S} \lambda^i u^i(y^i) + (1-\pi)\left[\sum_{i\in S} \lambda^i u^i(z^i) + \sum_{i\in I\setminus S} \lambda^i u^i(x^i)\right]$$
$$> \pi \sum_{i\in S} \lambda^i u^i(x^i) + (1-\pi) \sum_{i\in I} \lambda^i u^i(x^i).$$

Observe that $\sum_{i\in S} y^i \leqslant \sum_{i\in S} e^i$ and that the allocation that gives z^i to members of S and x^i to individuals in $I \setminus S$ is feasible for the grand coalition I. We conclude that the preceding inequality contradicts the fact that no coalition can λ-improve on x.

($\Leftarrow$) Let x be a feasible allocation such that no coalition can L-improve on it. In particular, given the Regularity conditions, there exists a unique (normalized) strictly positive vector $\lambda \in \mathbb{R}^I$ of utility weights such that x maximizes $\sum_{i\in I} \lambda^i u^i(y^i)$ over the set of feasible allocations. This vector is such that, for all i and $j \in I$,

$$\lambda^i/\lambda^j = \frac{\partial u^j(x^j)}{\partial x_1^j} \bigg/ \frac{\partial u^i(x^i)}{\partial x_1^i}.$$

Fix a coalition S and define the following sets of utility payoffs:

$$V(S) = \left\{ u \in \mathbb{R}^S \;\middle|\; \exists y \in \mathbb{R}_+^{LS}\colon \sum_{i\in S} y^i \leqslant \sum_{i\in S} e^i \wedge u \leqslant \bigl(u^i\bigl(y^i\bigr)\bigr)_{i\in S} \right\},$$

$$V_x(S) = \left\{ u \in \mathbb{R}^S \;\middle|\; \exists y \in \mathbb{R}_+^{LS}\colon \sum_{i\in S} y^i \leqslant \sum_{i\in S} x^i \wedge u \leqslant \bigl(u^i\bigl(y^i\bigr)\bigr)_{i\in S} \right\}.$$

Concavity of the utility functions implies that these two sets are convex. So, the fact that S cannot L-improve on x implies that there does not exist[1] $u \in Con(V(S) \cup V_x(S))$ such that $u^i > u^i(x^i)$ for $i \in S$. By a standard separation argument there exists a non zero vector $\lambda_S \in \mathbb{R}_+^S$ of utility weights for the coalition such that $\sum_{i\in S} \lambda_S^i u^i(x^i) \geqslant \sum_{i\in S} \lambda_S^i u^i$ for all $u \in Con(V(S) \cup V_x(S))$. In particular, $\sum_{i\in S} \lambda_S^i u^i(x^i) \geq \sum_{i\in S} \lambda_S^i u^i(y^i)$ for all $(y^i)_{i\in S}$ such that $\sum_{i\in S} y^i \leqslant \sum_{i\in S} e^i$. To finish the proof, we argue that λ_S is proportional to the restriction of λ to S. Indeed, $(u^i(x^i))_{i\in S}$ belongs to $V_x(S)$ so that $(x^i)_{i\in S}$ maximizes $\sum_{i\in S} \lambda_S^i u^i(y^i)$ over the set of $(y^i)_{i\in S}$ such that $\sum_{i\in S} y^i \leqslant \sum_{i\in S} x^i$, giving

$$\lambda_S^i/\lambda_S^j = \frac{\partial u^j(x^j)}{\partial x_1^j} \bigg/ \frac{\partial u^i(x^i)}{\partial x_1^i} = \lambda^i/\lambda^j. \qquad \square$$

The next example illustrates how the use of lotteries allows some coalition to L-improve on allocations that are in the core, but not in the inner core.

Example 2. Figure 1 represents a possible configuration in the utility space, focusing on payoffs of individuals 1 and 2. We are testing a Pareto-optimal allocation x with associated utilities $u = (u^1(x^1), u^2(x^2))$. As in the proof of the previous proposition, $V(12)$ (respectively $V_x(12)$) represents the set of utility pairs that are achievable by some reallocation of the initial endowments (respectively the tested allocation) between individuals 1 and 2. Coalition $\{1,2\}$ cannot improve on x, as $u \notin V(12)$. Nevertheless, x is not in the inner core. Indeed, if it were, it should be supported by a vector λ of utility weights that is proportional to $(1/\frac{\partial u^1}{\partial x_1^1}(x^1), \ldots, 1/\frac{\partial u^I}{\partial x_1^I}(x^I))$. The restriction of such a λ onto $\mathbb{R}^{\{1,2\}}$ is proportional to λ_{12}, the vector that is orthogonal to $V_x(12)$ at u, and, as one can see in the

[1] For a given subset X of $\mathbb{R}^I$, $Con(X)$ denotes the convex hull of X.

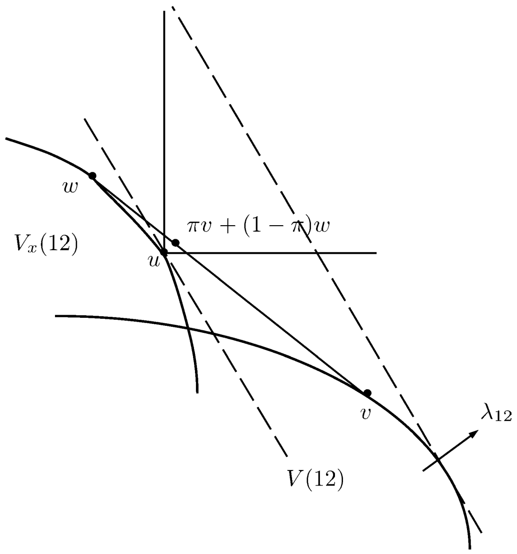

Fig. 1.

figure, coalition $\{1, 2\}$ can λ-improve on x. The construction of an L-improvement can be seen as follows. Individuals 1 and 2 agree on the following random plan: with probability π, they reject x and implement an allocation (y^1, y^2) that is such that $y^1 + y^2 \leq e^1 + e^2$ and $v = (u^1(y^1), u^2(y^2))$. With probability $(1 - \pi)$, they accept x, but agree to reallocate it once it is realized, in order to achieve the payoff pair w. This plan would generate expected payoffs $\pi v + (1 - \pi)w$, which dominate u. The smoothness of $V_x(12)$ guarantees the feasibility of such a construction: locally, around u, individuals 1 and 2 can approximate λ-weighted utility transfers.

2. Compactly generated games

Let $I = \{1, 2, \ldots, I\}$ be the set of players. Let $P(I)$ be the set of coalitions, that is the set of non empty subsets of I. A payoff allocation for coalition $S \in P(I)$ is a vector in $\mathbb{R}^S = \{u \in \mathbb{R}^I \mid (\forall i \in I \setminus S)\colon\ u^i = 0\}$. For a coalition S, let Δ_+^S (respectively Δ_{++}^S) be the set of vectors in $\mathbb{R}^S$ with nonnegative (respectively strictly positive for $i \in S$) components that sum up to one. Similarly, we denote by $\Delta(P(I))$ the set of probability distributions over coalitions.

A *game* is a function V which associates to each coalition S a nonempty, closed, comprehensive, convex subset of $\mathbb{R}^S$, the set of feasible payoff allocations for S. We will focus on *compactly generated* games, that is on games such that, for each S, the set $V(S)$ is the comprehensive closure of a compact set.

Let V be a compactly generated game. For each $\lambda \in \Delta_+^I$, let $v^\lambda : P(I) \to \mathbb{R}$ be the function defined as

$$v^\lambda(S) := \max_{u \in V(S)} \sum_{i \in S} \lambda^i u^i$$

for each $S \in P(I)$. For a given vector of utility weights $\lambda \in \mathbb{R}_{++}^I$, a coalition S can λ-improve on a payoff allocation u if there exists $w \in V(S)$ such that $\sum_{i \in S} \lambda^i w^i > \sum_{i \in S} \lambda^i u^i$. A payoff allocation u is an *inner core payoff allocation* if it is feasible and there exists $\lambda \in \mathbb{R}_{++}^I$ such that no coalitions can λ-improve on it. The inner core of the game V is the set $IC(V)$ of its inner core payoff allocations.[2]

The definition requires strictly positive utility weights. On the class of games we are considering here, this restriction implies technical difficulties for existence and characterization results (see Example 1 of Qin, 1993). One possible way out of these difficulties is to restrict the class of games, as in Qin (1993, 1994b). Another approach, proposed by Myerson (1991, 1992), is to slightly enlarge the solution by taking some topological closure of the set of allocations generated by the Shapley procedure.

Let us define the *Myerson's inner core* as the set of feasible payoff allocations u such that $u \in cl(\{w \in \mathbb{R}^I \mid (\exists \lambda \in \Delta_{++}^I)(\forall S \in P(I))\colon v^\lambda(S) \leqslant \sum_{i \in S} \lambda^i w^i\})$.

In the sequel we elaborate on Myerson's ideas. We first review his characterization in terms of a notion of random improvements, Proposition 3, and discuss its relationship with our Proposition 2. In Proposition 5 we provide a new characterization of Myerson's inner core, this time focusing on the structure of supporting utility weights.

A payoff allocation u can be *M-improved on* (respectively *weakly M-improved on*) if there exist $\mu \in \Delta(P(I))$ and $w \in \times_{S \in P(I)} V(S)$ such that

$$u^i < \sum_{S \in P(I) \mid i \in S} \mu(S) w^i(S) \Big/ \sum_{S \in P(I) \mid i \in S} \mu(S) \quad \text{(respectively } \leq\text{)},$$

for each $i \in \bigcup_{S \in \text{support}(\mu)} S$.

Proposition 3. *Let u be a payoff allocation. Then u cannot be M-improved on if and only if $u \in cl(\{w \in \mathbb{R}^I \mid (\exists \lambda \in \Delta_{++}^I)(\forall S \in P(I))\colon v^\lambda(S) \leqslant \sum_{i \in S} \lambda^i w^i\})$.*

Myerson (1991) calls a payoff allocation strongly inhibitive if it cannot be weakly M-improved on, and an allocation inhibitive if there exists a sequence of strongly inhibitive allocations that converge to it. Proposition 3 is then a consequence of Theorem 9.5 of Myerson (1991) if one notices that an allocation u cannot be M-improved on if and only if it is inhibitive. Indeed, if u is an allocation that cannot be M-improved on, then, for each positive integer k, the allocation $u_k := u + (1/k, \ldots, 1/k)$ is strongly inhibitive, and u is inhibitive. On the other hand, notice that the set of allocations that cannot be M-improved on is closed and that any allocation that is strongly inhibitive cannot be M-improved on.

[2] The inner core of the exchange economy E, as defined in Section 1, corresponds to the inner core of the associated game V, where, as in the proof of Proposition 2, for each S, $V(S) = \{u \in \mathbb{R}^S \mid \exists y \in \mathbb{R}_+^{LS}\colon \sum_{i \in S} y^i \leqslant \sum_{i \in S} e^i \wedge u \leqslant (u^i(y^i))_{i \in S}\}$.

The notion of M-improvement may be interpreted by imagining a mediator who can propose a plan that specifies which coalition will form, with what probability, and which payoff allocation will be implemented if the coalition forms. When being asked to participate to the plan, each individual knows only the probability distribution over coalitions but not the particular realization, and computes his conditional expected payoff. If all individuals who might be called to participate in the plan are better off than at the status quo allocation, the improvement is effective. The definition of L-improvement that we introduced in the previous section was motivated by this idea of Myerson, but aimed at staying closer to the usual interpretation of the core, in which coalitional improvements only involve the members of the deviating coalition. This turned out to work well for regular concave exchange economies. One could ask what are the properties of games generated from regular concave exchange economies which allows for a characterization in terms of this more intuitive notion of improvement. The need for centralized plans in Proposition 3 is due to the possible lack of smoothness of the Pareto frontier, as illustrated by the following example.

Example 3. Let $I = \{1, 2, 3\}$,

$$\begin{aligned}
&V(\{i\}) = \{u \in \mathbb{R}^{\{i\}} \mid u^i \leqslant 0\} \quad \text{for } i \in I,\\
&V(\{1,2\}) = \{u \in \mathbb{R}^{\{1,2\}} \mid u^1 + 9u^2 \leqslant 9,\ u^1 \leq 9,\ u^2 \leq 1\},\\
&V(\{2,3\}) = \{u \in \mathbb{R}^{\{2,3\}} \mid u^2 + 9u^3 \leqslant 9,\ u^2 \leq 9,\ u^3 \leq 1\},\\
&V(\{1,3\}) = \{u \in \mathbb{R}^{\{1,3\}} \mid u^3 + 9u^1 \leqslant 9,\ u^3 \leq 9,\ u^1 \leq 1\},\\
&V(I) = \{u \in \mathbb{R}^I \mid (u^1, u^2, u^3) \leqslant (3,3,3)\}.
\end{aligned}$$

The point $(3, 3, 3)$ does not belong to the inner core. Indeed it can be randomly improved upon by a plan which puts probability $1/3$ on each coalition of size two, and gives, in each of these coalitions, 9 units of utility to one individual. Each individual expects a conditional payoff of 4.5, which is better than 3. We now argue that $(3, 3, 3)$ cannot be randomly improved on by a plan that puts positive probability only on the grand coalition and one of the two-person coalitions, let's say $\{1, 2\}$. First, no such plan can guarantee to individual 3 a conditional expected payoff bigger than 3 as required by the definition of random improvement. Second, even if we do not require individual 3 to be strictly better off, it is not possible to guarantee an expected payoff greater than 3 to both player 1 and player 2. The argument in the proof of Proposition 2 cannot be adapted to this example, due the lack of smoothness of the Pareto frontier of $V(I)$. Indeed there is no way for players 1 and 2, to implement transfers of weighted utility contingent on the formation of the big coalition.

Let us say that a compactly generated game is *regular* if there exists a differentiable, weakly increasing function[3] $F : \mathbb{R}^I \to \mathbb{R}$ such that $V(I) = \{u \in \mathbb{R}^I \mid F(u) \leqslant 0\}$, and the boundary of $V(I)$ has a strictly positive gradient at each individually rational point. The argument used for Proposition 2 may be adapted to prove the following result.

[3] A function $F : \mathbb{R}^I \to \mathbb{R}$ is weakly increasing if $F(x') > F(x)$ whenever $x' \gg x$ and $x' \neq x$.

Proposition 4. *Let V be a regular compactly generated game. Then u belongs to Myerson's inner core if and only if u is feasible for the grand coalition and there does not exist $(\pi, S, v, w) \in [0,1] \times P(I) \times V(S) \times V(I)$ such that $w^i > u^i$ for all $i \in I \setminus S$ and $\pi v^i + (1-\pi)w^i > u^i$ for all $i \in S$.*

Proof. ($\Rightarrow$) If u belongs to Myerson's inner core then it is feasible and there exists $\lambda \in \mathbb{R}^I_{++}$ such that

$$v^\lambda(S) \leqslant \sum_{i \in S} \lambda^i u^i \tag{1}$$

for each S. Suppose that there exists $(\pi, S, v, w) \in [0,1] \times P(I) \times V(S) \times V(I)$ such that $w^i > u^i$ for all $i \in I \setminus S$ and $\pi v^i + (1-\pi)w^i > u^i$ for all $i \in S$. If we premultiply each term of the last expression by λ^i and sum over all $i \in S$, we obtain:

$$\pi \sum_{i \in S} \lambda^i v^i + (1-\pi) \sum_{i \in S} \lambda^i w^i > \sum_{i \in S} \lambda^i u^i.$$

We also know that $(1-\pi)\sum_{i \in I \setminus S} \lambda^i w^i > (1-\pi)\sum_{i \in I \setminus S} \lambda^i u^i$. Adding the two inequalities and rearranging terms, we have

$$\pi \sum_{i \in S} \lambda^i v^i + (1-\pi) \sum_{i \in I} \lambda^i w^i > \pi \sum_{i \in S} \lambda^i u^i + (1-\pi) \sum_{i \in I} \lambda^i u^i,$$

contradicting Eq. (1).

($\Leftarrow$) Let $u \in V(I)$. Under the hypothesis, u must be Pareto-optimal; by convexity and regularity, there exists a unique (normalized) strictly positive vector $\lambda \in \mathbb{R}^I$ of utility weights such that for all i and $j \in I$

$$\lambda^i / \lambda^j = \frac{\partial F(u)}{\partial u^j} \Big/ \frac{\partial F(u)}{\partial u^i}.$$

Fix a coalition S and define the following set of utility payoffs:

$$V_u(S) = \{ z \in \mathbb{R}^S \mid (z, u_{-S}) \in V(I) \}.$$

If there does not exist (π, S, v, w) with $w^i > u^i$ for all $i \in I \setminus S$ and $\pi v^i + (1-\pi)w^i > u^i$ for all $i \in S$ then (using the fact that the boundary of $V(I)$ has a strictly positive gradient at each individually rational point) there does not exist $z \in Con(V(S) \cup V_u(S))$ such that $z^i > u^i$ for $i \in S$. By a standard separation argument there exists a non zero vector $\lambda_S \in \mathbb{R}^S_+$ of utility weights for the coalition such that $\sum_{i \in S} \lambda^i_S u^i \geqslant \sum_{i \in S} \lambda^i_S z^i$ for all $z \in Con(V(S) \cup V_u(S))$. In particular, $\sum_{i \in S} \lambda^i_S u^i \geq \sum_{i \in S} \lambda^i_S z^i$ for all $z \in V(S)$. To finish the proof, we argue that λ_S is proportional to the restriction of λ to S. Indeed, u_S belongs to $V_u(S)$ so that it maximizes $\sum_{i \in S} \lambda^i_S z^i$ over the set of $z = (z^i)_{i \in S}$ such that $F(z, u_{-S}) \leqslant 0$. The first order conditions for this maximization give

$$\lambda^i_S / \lambda^j_S = \frac{\partial F(u)}{\partial u^j} \Big/ \frac{\partial F(u)}{\partial u^i} = \lambda^i / \lambda^j. \qquad \square$$

Thus, for regular compactly generated games an M-improvement can be achieved by putting positive probability only on the grand coalition and one sub-coalition S. Nevertheless, it is in general not possible to express the improvement only in terms of decisions of members of S. The characterization in Proposition 4 requires that members of S obtain the acceptance of non-members by proposing a new allocation w for the grand coalition as a part of their deviation plan.

In the next proposition we reinterpret the closure appearing in the definition of Myerson's inner core in terms of lexicographic system of utility weights. This allows a better understanding of the way in which the extension of Shapley's fictitious-transfer procedure is achieved, as illustrated in Example 4.

Proposition 5. *Let u be a payoff allocation. The two following properties are equivalent*:

(a) $u \in cl\ (\{w \in \mathbb{R}^I \mid (\exists \lambda \in \Delta^I_{++})\ (\forall S \in P(I)) \colon\ v^\lambda(S) \leqslant \sum_{i \in S} \lambda^i w^i\})$.

(b) *There exists an ordered partition $\{S(k)\}_{k=1}^K$ of I and a collection of vectors $\{\lambda(k)\}_{k=1}^K \in \times_{k=1}^K \Delta^{S(k)}_{++}$ such that*:

$$\bigl(\forall k \in \{1,\dots,K\}\bigr) \left(\forall S \in P\left(\bigcup_{j=k}^K S(j)\right)\right)\colon \quad v^{\lambda(k)}(S) \leq \sum_{i \in S} \lambda^i(k) u^i.$$

Proof. (a) → (b) Condition (a) implies that there exists a sequence $(w_k)_{k \in \mathbb{N}^*}$ and a sequence $(\lambda_k)_{k \in \mathbb{N}}$ in Δ^I_{++} such that $w_k \to u$ and, for all k and all S

$$v^{\lambda_k}(S) \leqslant \sum_{i \in S} \lambda^i_k w^i_k. \tag{2}$$

Without loss of generality, $\lambda_k \to \lambda$, for some $\lambda \in \Delta^I_+$. Let $S(1) := \text{support}(\lambda)$ and $\lambda(1) := \lambda\ (\in \Delta^{S(1)}_{++})$.

Taking the limit of expression (2), we obtain, for all S, $v^{\lambda(1)}(S) \leqslant \sum_{i \in S} \lambda^i(1) u^i$. If $S(1) = I$, then we are done.

If $S(1)$ is strictly included in I, then $I \setminus S(1) \neq \emptyset$. Then let us define $\lambda'_k \in \Delta^{I \setminus S(1)}_{++}$ as the (normalized) projection of λ_k on $\mathbb{R}^{I \setminus S(1)}$. Without loss of generality, the sequence $(\lambda'_k)_{k \in \mathbb{N}}$ converges to some $\lambda' \in \Delta^{I \setminus S(1)}_{+}$. Let $S(2) := \text{support}(\lambda')$ and $\lambda(2) := \lambda'\ (\in \Delta^{S(2)}_{++})$.

Expression (2) being homogeneous of degree zero, it holds true for λ'_k, for each $k \in \mathbb{N}$ and each $S \in P(I \setminus S(1))$. Taking the limit, we obtain, for all $S \in P(I \setminus S(1))$, $v^{\lambda(2)}(S) \leqslant \sum_{i \in S} \lambda^i(2) u^i$. If $S(1) \cup S(2) = I$, then we are done.

If $S(1) \cup S(2)$ is strictly included in I, then $I \setminus (S(1) \cup S(2)) \neq \emptyset$, and we can repeat the argument. Since I is a finite set, property (b) is proved.

(b) → (a) We prove that any payoff allocation w such that $w \gg u$, satisfy the following property:

$$\bigl(\exists \lambda \in \mathbb{R}^I_{++}\bigr)\ \bigl(\forall S \in P(I)\bigr)\colon \quad v^\lambda(S) \leqslant \sum_{i \in S} \lambda^i w^i.$$

By property (b), we may consider an ordered partition $\{S_k\}_{k=1}^K$ of I and a sequence $(\lambda(k))_{k=1}^K \in \times_{k=1}^K \Delta_{++}^{S_k}$ such that

$$\big(\forall k \in \{1, \dots, K\}\big) \left(\forall S \in P\left(\bigcup_{j=k}^{K} S_j\right)\right): \quad v^{\lambda(k)}(S) \leqslant \sum_{i \in S} \lambda^i(k) u^i. \tag{3}$$

Let $\lambda_l \in \mathbb{R}_{++}^I$ be defined by

$$\lambda_l^i := \left(\frac{1}{l}\right)^{k(i)-1} \lambda^i\big(k(i)\big),$$

where $k(i)$ is the k in $\{1, \dots, K\}$ such that $i \in S_k$, for each $i \in I$ and each $l \in \mathbb{N}^*$.

We now prove that for l large enough,

$$v^{\lambda_l}(S) \leqslant \sum_{i \in S} \lambda_l^i w^i$$

for each $S \in P(I)$.

(Reductio ad absurdum) Suppose, on the contrary, that

$$(\forall l \in \mathbb{N}^*)\ \big(\exists S \in P(I)\big): \quad v^{\lambda_l}(S) > \sum_{i \in S} \lambda_l^i w^i.$$

Given that $P(I)$ is a finite set, this implies that

$$\big(\exists S \in P(I)\big)\ (\exists f : \mathbb{N}^* \to \mathbb{N}^* \text{ increasing})(\forall l \in \mathbb{N}^*): \quad v^{\lambda_{f(l)}}(S) > \sum_{i \in S} \lambda_{f(l)}^i w^i.$$

For this coalition S, using the fact that $V(S)$ is compactly generated, there exists an increasing function $g : \mathbb{N}^* \to \mathbb{N}^*$, and a sequence $(z_l)_{l \in \mathbb{N}^*}$ of elements of $V(S)$ that converges to some $z \in V(S)$ such that

$$(\forall l \in \mathbb{N}^*): \quad \sum_{i \in S} \lambda_{g(l)}^i z_l^i > \sum_{i \in S} \lambda_{g(l)}^i w^i. \tag{4}$$

Let $k^* := \min\{k \in \{1, \dots, K\} \mid S_k \cap S \neq \emptyset\}$.

Multiplying both sides of inequality (4) by $g(l)^{k^*-1}$, we obtain:

$$(\forall l \in \mathbb{N}^*): \quad g(l)^{k^*-1} \sum_{i \in S} \lambda_{g(l)}^i z_l^i > g(l)^{k^*-1} \sum_{i \in S} \lambda_{g(l)}^i w^i.$$

Using the definition of $\lambda_{g(l)}$ we conclude that

$$(\forall l \in \mathbb{N}^*): \quad \sum_{k=k^*}^{K} \sum_{i \in S \cap S_k} \frac{\lambda^i(k)}{g(l)^{k-k^*}} z_l^i > \sum_{k=k^*}^{K} \sum_{i \in S \cap S_k} \frac{\lambda^i(k)}{g(l)^{k-k^*}} w^i.$$

Letting l tend to infinity in the previous expression, we have:

$$\sum_{i \in S \cap S_{k^*}} \lambda^i(k^*) z^i \geqslant \sum_{i \in S \cap S_{k^*}} \lambda^i(k^*) w^i.$$

Now notice that, by expression (3), $\sum_{i\in S\cap S_{k^*}} \lambda^i(k^*)u^i \geq \sum_{i\in S\cap S_{k^*}} \lambda^i(k^*)z^i$. Indeed, $z \in V(S)$ and $S \in P(\bigcup_{j=k^*}^{K} S_j)$ (by definition of k^*). On the other hand, $\sum_{i\in S\cap S_{k^*}} \lambda^i(k^*)w^i > \sum_{i\in S\cap S_{k^*}} \lambda^i(k^*)u^i$, since $w \gg u$ and support$(\lambda(k^*)) \cap S \neq \emptyset$.

Putting everything together, we conclude that

$$\sum_{i\in S\cap S_{k^*}} \lambda^i(k^*)u^i > \sum_{i\in S\cap S_{k^*}} \lambda^i(k^*)u^i,$$

which is absurd. □

A feasible payoff allocation is in Shapley's inner core if and only if it satisfies condition (b) for the trivial partition whose only element is the set of all players. The following example shows that Myerson's inner core can be strictly larger than Shapley's, and illustrates how the former provides a reasonable way to treat zero transfer weights.

Example 4. Let $I = \{1, 2\}$,

$$V(1) = \{u \in \mathbb{R}^{\{1\}} \mid u_1 \leqslant -1\}, \qquad V(2) = \{u \in \mathbb{R}^{\{2\}} \mid u_2 \leqslant -1\},$$
$$V(1,2) = \{u \in \mathbb{R}^{\{1,2\}} \mid u_1 \leq 1,\ u_2 \leq 1,\ u_1 + u_2 \leq 1\}.$$

The point $(-1, 1)$ belongs to Myerson's inner core but not to Shapley's inner core. The partition in (2) of Proposition 2 is in this case $S(1) = \{2\}$, $S(2) = \{1\}$, with associated vectors $\lambda(1) = (0, 1)$, $\lambda(2) = (1, 0)$, and (2) reduces to the requirement that the allocation u maximizes player 2's utility, while maintaining individual rationality for player 1.

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