ECONOMETRICA

JOURNAL OF THE ECONOMETRIC SOCIETY

An International Society for the Advancement of Economic Theory in its Relation to Statistics and Mathematics

http://www.econometricsociety.org/

Econometrica, Vol. 76, No. 6 (November, 2008), 1413–1436

MARGINAL CONTRIBUTIONS AND EXTERNALITIES IN THE VALUE

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MARGINAL CONTRIBUTIONS AND EXTERNALITIES IN THE VALUE

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Our concern is the extension of the theory of the Shapley value to problems involving externalities. Using the standard axiom systems behind the Shapley value leads to the identification of bounds on players' payoffs around an "externality-free" value. The approach determines the direction and maximum size of Pigouvian-like transfers among players, transfers based on the specific nature of externalities that are compatible with basic normative principles. Examples are provided to illustrate the approach and to draw comparisons with previous literature.

KEYWORDS: Externalities, marginal contributions, Shapley value, Pigouvian transfers.

1. INTRODUCTION

SINCE THE PATH-BREAKING WORK of Shapley (1953), much effort has been devoted to the problem of "fair" distribution of the surplus generated by a collection of people who are willing to cooperate with one another. More recently, the same question has been posed in the realistic case where externalities across coalitions are present. This is the general problem to which this paper contributes. The presence of such externalities is an important feature in many applications. In an oligopolistic market, the profit of a cartel depends on the level of cooperation among the competing firms. The power of a political alliance depends on the level of coordination among competing parties. The benefit of an agent who refuses to participate in the production of a public good depends on the level of cooperation of the other agents (free-riding effect) and so on.

In the absence of externalities, Shapley (1953) obtained a remarkable *uniqueness* result. He characterized a unique solution using the axioms of efficiency, anonymity, additivity, and null player. Today we refer to this solution as the Shapley value, which happens to be calculated as the average of marginal contributions of players to coalitions. This comes as a surprise at first glance: uniqueness is the consequence of four basic axioms, and nothing in those axioms hints at the marginality principle, of long tradition in economic theory. In the clarification of this puzzle, Young (1985) provided a key piece. He formulated the marginality principle as an axiom, that is, that the solution should be a function of players' marginal contributions to coalitions. He dropped additivity

[1]We thank Isa Emin Hafalir, Sergiu Hart, Eric Maskin, and David Pérez-Castrillo for comments and encouragement. The comments of David Levine and two anonymous referees also helped improve the exposition. Serrano gratefully acknowledges the hospitality of Universidad Carlos III and CEMFI in Madrid, and research support from Fundación Banco Herrero and Universidad Carlos III.

 DOI: 10.3982/ECTA7224

and null player as requirements. The result is that the only solution satisfying efficiency, anonymity, and marginality is again the Shapley value.

In our extension of the theory of the Shapley value to settings with externalities, we shall also pursue an axiomatic analysis (we leave strategic considerations and issues of coalition formation to a companion paper—de Clippel and Serrano (2008)). In our axiomatic analysis, we will find that appealing systems of basic axioms that were used in problems with no externalities do not suffice to yield a unique solution. Thus, *multiplicity* of solutions seems essential to the problem at hand (this is confirmed by previous studies, in which authors must resort to new additional axioms to get uniqueness). Despite the multiple solutions, the novelty of our approach is to provide refined predictions based on payoff bounds implied by the axioms.

To tackle the question in axiomatic terms, we sort out the effects of *intrinsic marginal contributions* of players to coalitions from those coming from *externalities*. The model we shall employ is that of partition functions, in which the worth of a coalition S may vary with how the players not in S cooperate. In the model, $v(S,\Pi)$ is the worth of S when the coalition structure is Π, S being an element of Π. To define player i's marginal contribution to coalition S—a trivial task in the absence of externalities—it is now crucial to describe what happens after i leaves S. Suppose i plans to join T, another coalition in Π. The *total effect* on S of i's move is the difference $v(S,\Pi) - v(S\setminus\{i\}, \{S\setminus\{i\}, T\cup\{i\}\}\cup\Pi_{-S,-T})$. This effect can be decomposed into two. First, there is an *intrinsic marginal contribution effect* associated with i leaving S but before joining T, that is, $v(S,\Pi) - v(S\setminus\{i\}, \{S\setminus\{i\}, \{i\}\}\cup\Pi_{-S})$. Second, there is an *externality effect*, which stems from the change in the worth of $S\setminus\{i\}$ when i, instead of remaining alone, joins T, that is, the difference $v(S\setminus\{i\}, \{S\setminus\{i\}, \{i\}\}\cup\Pi_{-S}) - v(S\setminus\{i\}, \{S\setminus\{i\}, T\cup\{i\}\}\cup\Pi_{-S,-T})$. (Note how this latter difference is *not* a "partial derivative," a marginal contribution of player i to coalition S.) Our results follow from exploiting this decomposition.[2]

Assuming that the grand coalition forms, we investigate the implications of efficiency and anonymity, together with a weak version of marginality.[3] According to this last property, the solution may depend on all the total effects—the sum of the intrinsic marginal contribution and the externality effects. We find the first noteworthy difference with respect to the case of no externalities, because in our larger domain these axioms are compatible with a wide class of linear and even nonlinear solutions (Examples 1 and 2). However, despite such a large multiplicity of solutions, our first result shows that if the partition function can be written as the sum of a symmetric partition function and a

[2]In this decomposition, we are focusing on the "simplest path," that is, that in which player i leaves S and stays as a singleton before joining T. As we shall see, the singletons coalition structure acts as a useful origin of coordinates from which externalities are measured.

[3]Similar results can be established for any exogenous coalition structure if the axioms are imposed atom by atom in the partition.

characteristic function, a unique payoff profile is the consequence of the three axioms (Proposition 1). As will be formally defined in the sequel, this class amounts to having *symmetric externalities.*

For general partition functions, as a result of our first finding, we seek the implications of strengthening the weak version of marginality, and we do so in two ways. First, we require monotonicity, that is, a player's payoff should be increasing in all the total effects—the sum of his intrinsic marginal contribution and externality effects. Then we are able to establish useful upper and lower bounds to each player's payoff (Proposition 2). Second, complementing this result, we require a marginality axiom, according to which a player's payoff should depend on the vector of intrinsic marginal contributions, not on the externality effect. The result is a characterization of an "externality-free" value on the basis of efficiency, anonymity, and marginality (Proposition 3). In a second characterization result (Proposition 4), this solution is obtained using a system of axioms much like the original one attributable to Shapley (with a strong version of the dummy axiom that also disregards externalities).[4]

The externality-free value thus appears to be a natural reference point. Obviously, an analysis based solely on the externality-free value is not desirable in a model of externalities. This is why we do not insist on uniqueness, and accept the multiplicity of solutions inherent to the problem. The combination of both kinds of results—the externality-free value benchmark and the obtention of bounds around it—is a way to understand how externalities might benefit or punish a player in a context where normative principles are in place. In effect, the two results together provide a range for acceptable Pigouvian-like transfers (externality-driven taxes or subsidies among players) when efficiency is accompanied by our other normative desiderata.

2. DEFINITIONS

Let N be the finite set of *players*. *Coalitions* are subsets of N. $P(N)$ denotes the set of all coalitions and lowercase letters denote the cardinality of coalitions ($s = \#S$, $n = \#N$, etc.). A *partition* of N is a set $\Pi = \{(S_k)_{k=1}^K\}$ $(1 \leq K \leq n)$ of disjoint coalitions that cover N, that is, $S_i \cap S_j = \varnothing$ for each $1 \leq i < j \leq K$ and $N = \bigcup_{k=1}^K S_k$. By convention, $\{\varnothing\} \in \Pi$ for every partition Π. Elements of a partition are called *atoms*. A partition Π' is *finer* than a partition Π if each atom of Π' is included in an atom of Π: if $S' \in \Pi'$, then $S' \subseteq S$ for some $S \in \Pi$. We will say equivalently that Π is *coarser* than Π'. An *embedded coalition* is a pair (S, Π), where Π is a partition and S is an atom of Π. EC denotes the set of embedded coalitions. If S is a coalition and i is a member of S, then S_{-i}

[4]Straightforward adaptations of the principle of balanced contributions (Myerson (1980)), of the concept of potential (Hart and Mas-Colell (1989)), and of some bargaining procedures (Hart and Mas-Colell (1996); Pérez-Castrillo and Wettstein (2001)), once applied to the larger domain of partition functions, lead to the externality-free value as well.

(resp. S_{+i}) denotes the set $S \setminus \{i\}$ (resp. $S \cup \{i\}$). Similarly, if Π is a partition and S is an atom of Π, then Π_{-S} denotes the partition $\Pi \setminus \{S\}$ of the set $N \setminus S$.

A *partition function* (Thrall and Lucas (1963)) is a function v that assigns to every embedded coalition (S, Π) a real number $v(S, \Pi)$, with the convention $v(\{\varnothing\}, \Pi) = 0$, for all Π. Externalities are *positive* (resp. negative) if $v(S, \Pi) \geq v(S, \Pi')$ for each embedded coalition (S, Π) and (S, Π') such that Π is coarser (resp. finer) than Π'. There are *no externalities* if $v(S, \Pi) = v(S, \Pi')$ for all embedded coalitions (S, Π) and (S, Π'). In the latter case, a partition function is also called a *characteristic function*.

A partition function is *superadditive* if each coalition can achieve as much as the sum of what its parts can, that is, $\sum_{k=1}^{K} v(S_k, \Pi') \leq v(S, \Pi)$ for every embedded coalition (S, Π) and every partition $\{(S_k)_{k=1}^{K}\}$ of S $(1 \leq K \leq s)$ with $\Pi' = \Pi_{-S} \cup \{(S_k)_{k=1}^{K}\}$.[5]

A *value* is a function σ that assigns to every partition function v a unique utility vector $\sigma(v) \in \mathbb{R}^N$. Shapley (1953) defined and axiomatized a value for characteristic functions:

$$\mathrm{Sh}_i(v) := \sum_{S \subseteq N \text{ s.t. } i \in S} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S_{-i})]$$

for each player $i \in N$ and each characteristic function v. In the next sections, we are concerned with the extension of the theory of the Shapley value to partition functions.

3. WEAK MARGINALITY AND MULTIPLICITY OF SOLUTIONS

Based on the marginality principle, Young (1985) proposed a beautiful axiomatization of the Shapley value for characteristic functions. We shall explore the implications of marginality, together with other basic axioms, on the class of partition functions. The first two axioms that we shall impose are hardly controversial.

ANONYMITY: *Let π be a permutation of N and let v be a partition function. Then $\sigma(\pi(v)) = \pi(\sigma(v))$, where $\pi(v)(S, \Pi) = v(\pi(S), \{\pi(T) | T \in \Pi\})$ for each embedded coalition (S, Π) and $\pi(x)_i = x_{\pi(i)}$ for each $x \in \mathbb{R}^N$ and each $i \in N$.*

EFFICIENCY: $\sum_{i \in N} \sigma_i(v) = v(N)$ *for each partition function v.*

[5]It is well known that a characteristic function v is superadditive if and only if $v(R) + v(T) \leq v(R \cup T)$ for every *pair* (R, T) of disjoint coalitions. In other words, it is sufficient to check the case $K = 2$ in the absence of externalities. This observation does not extend to partition functions and one needs to consider all the possible K's (see Hafalir (2007, Example 1)). Notice that Hafalir (2007) used the term "full cohesiveness" to describe superadditivity, while reserving the term "superadditivity" for the property with $K = 2$.

Anonymity means that players' payoffs do not depend on their names. We assume that the grand coalition forms and we leave issues of coalition formation out of this paper. Efficiency then simply means that the value must be feasible and exhaust all the benefits from cooperation, given that everyone cooperates.

Next, we turn to our discussion of marginality, central in our work. The marginal contribution of a player i within a coalition S is defined, for characteristic functions, as the loss incurred by the other members of S if i leaves the group. This number could depend on the organization of the players not in S when there are externalities. It is natural, therefore, to define the marginal contribution of a player within each embedded coalition.

To begin, one may consider the general case where a player may join another coalition after leaving S. Since many numbers qualify as marginal contributions, the resulting marginality axiom is rather weak. Still, it coincides with Young's concept of marginal contributions in the absence of externalities.

WEAK MARGINALITY: *Let v and v' be two partition functions. If*

$$\begin{aligned} &v(S,\Pi) - v(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T}) \\ &\quad = v'(S,\Pi) - v'(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T}) \end{aligned}$$

for each embedded coalition (S,Π) such that $i \in S$ and each atom T of Π that is different from S, then $\sigma_i(v) = \sigma_i(v')$.

Suppose for instance that $N = \{i, j, k\}$. Then player i's payoff should depend only on the seven real numbers

$$\begin{aligned} A_i(v) &= v(N, \{N\}) - v\big(\{j,k\}, \{\{i\}, \{j,k\}\}\big), \\ B_i(v) &= v\big(\{i,j\}, \{\{i,j\}, \{k\}\}\big) - v\big(\{j\}, \{\{j\}, \{i,k\}\}\big), \\ C_i(v) &= v\big(\{i,j\}, \{\{i,j\}, \{k\}\}\big) - v\big(\{j\}, \{\{i\}, \{j\}, \{k\}\}\big), \\ D_i(v) &= v\big(\{i,k\}, \{\{i,k\}, \{j\}\}\big) - v\big(\{k\}, \{\{k\}, \{i,j\}\}\big), \\ E_i(v) &= v\big(\{i,k\}, \{\{i,k\}, \{j\}\}\big) - v\big(\{k\}, \{\{i\}, \{j\}, \{k\}\}\big), \\ F_i(v) &= v\big(\{i\}, \{\{i\}, \{j,k\}\}\big), \\ G_i(v) &= v\big(\{i\}, \{\{i\}, \{j\}, \{k\}\}\big). \end{aligned}$$

For general partition functions, there is no hope to get a characterization result of a value with Efficiency, Anonymity, and this weak notion of marginality. To see this, consider the following examples:

EXAMPLE 1: The value σ^α, defined by $\sigma_i^\alpha(v) := \frac{1}{3}A_i(v) + \frac{1}{6}(\alpha B_i(v) + (1-\alpha)C_i(v)) + \frac{1}{6}(\alpha D_i(v) + (1-\alpha)E_i(v)) + \frac{1}{3}(\alpha F_i(v) + (1-\alpha)G_i(v))$, satisfies Anonymity, Efficiency, and Weak Marginality for every $\alpha \in \mathbb{R}$. The values σ^α

are instances of the average approach characterized by Macho-Stadler, Pérez-Castrillo, and Wettstein (2007), as they coincide with the Shapley value of a fictitious characteristic function v^α, where $v^\alpha(\{i\}) = \alpha v(\{i\}, \{\{i\}, \{j, k\}\}) + (1 - \alpha)v(\{i\}, \{\{i\}, \{j\}, \{k\}\})$.

In addition, and what is perhaps more surprising, a large class of nonlinear values satisfy the three axioms. (Recall that they imply linearity in the domain of characteristic functions.) In this sense, our approach differs substantially from Fujinaka's (2004). He proposed several versions of marginality, whereby a marginal contribution is constructed as a weighted linear average of the marginal contributions over different coalition structures. This assumption already builds linearity in Fujinaka's result.

EXAMPLE 2: Let $m : \mathbb{R} \to \mathbb{R}$ be any function. Then the value $\sigma^{\alpha,m}$, defined by $\sigma_i^{\alpha,m}(v) := \sigma_i^\alpha(v) + m(F_i(v) - G_i(v)) - (m(C_i(v) - B_i(v)) + m(E_i(v) - D_i(v)))/2$, also satisfies the three axioms. Observe that the differences $C_i(v) - B_i(v)$, $E_i(v) - D_i(v)$, and $F_i(v) - G_i(v)$ measure the externality that the agents face. The function m transforms the externality that a player faces into a transfer paid equally by the two other players. The value $\sigma^{\alpha,m}$ is then obtained by adding to σ^α the net transfer that each player receives.

We shall say that a partition function v is *symmetric* if $\pi(v)(S, \Pi) = v(S, \Pi)$, for each embedded coalition (S, Π) and each permutation π of the players. That is, the worth of an embedded coalition is a function only of its cardinality and of the cardinality of the other atoms of the partition.

The earlier examples illustrate the large class of linear and nonlinear solutions compatible with the three axioms. Yet, one can obtain a surprisingly sharp prediction on an interesting subclass of partition functions. This is the content of our first result:

PROPOSITION 1: *Let σ be a value that satisfies Anonymity, Efficiency, and Weak Marginality. Let u be a symmetric partition function and let v be a characteristic function. Then $\sigma_i(u + v) = \frac{u(N)}{n} + \mathrm{Sh}_i(v)$ for each $i \in N$.*

PROOF: The proof is a variant of Young's (1985) argument for characteristic functions. To prove uniqueness, Young started by observing that any value that satisfies the axioms must give a zero payoff to all the players when the game is null. He then proved that any such value must coincide with the Shapley value for any characteristic function v, proceeding by induction on the number of nonzero terms appearing in Shapley's (1953) decomposition of v in terms of the vectors in the basis of the space of characteristic functions. The key to prove our result is to start the induction with the symmetric partition function u instead of the null game. Details are found in de Clippel and Serrano (2005). *Q.E.D.*

Proposition 1 is straightforward if v is the null characteristic function. It coincides with Young's (1985) result if u is the null partition function. The important conclusion we can draw from the proposition is that the three axioms together imply additivity on the class of partition functions that can be decomposed as the sum of a symmetric partition function and a characteristic function.[6] This result is not trivial, in view of Example 2. As an illustration of the power of Proposition 1, consider the following example:

EXAMPLE 3: This example features prominently in Maskin (2003). A similar example was first proposed by Ray and Vohra (1999, Example 1.2). It describes a simple "free rider" problem created by a public good that can be produced by each two-player coalition. The set of agents is $N = \{1, 2, 3\}$ and the partition function is

$$
\begin{aligned}
&v(N) = 24,\\
&v(\{1,2\}) = 12, \quad v(\{1,3\}) = 13, \quad v(\{2,3\}) = 14,\\
&v\big(\{i\}, \{\{i\}, \{j,k\}\}\big) = 9 \quad \text{for all } i, j, k,\\
&v\big(\{i\}, \{\{i\}, \{j\}, \{k\}\}\big) = 0 \quad \text{for all } i, j, k.
\end{aligned}
$$

Observe that this partition function is not symmetric and it is not a characteristic function, yet it can be decomposed as the sum of a characteristic function v' (where each coalition's worth is zero, except $v'(\{1,3\}) = 1$ and $v'(\{2,3\}) = 2$) and a symmetric partition function u. We conclude that any value σ that satisfies Anonymity, Efficiency, and Weak Marginality (and we remark that this is a large class) must be such that

$$\sigma(v) = \sigma(u + v') = (8, 8, 8) + (-0.5, 0, 0.5) = (7.5, 8, 8.5)$$

in this example.

Next, we wish to consider general partition functions again. Given the large class of solutions identified in Examples 1 and 2 above, we propose to follow two alternative paths. First, we shall strengthen Weak Marginality into a monotonicity property. Second, we shall look more closely at the notion of "marginal contributions" to propose an alternative marginality axiom. We undertake each of the alternatives in the next two sections.

[6] We argue at the end of Section 5 that a partition function v can be decomposed as the sum of a symmetric partition function and a characteristic function if and only if externalities are symmetric for v.

4. MONOTONICITY AND BOUNDS ON PAYOFFS

This section investigates what happens when, in addition to requiring Efficiency and Anonymity, Weak Marginality is strengthened to the following monotonicity axiom. The result will be the derivation of useful bounds to the payoff of each player.

MONOTONICITY[7]: *Let v and v' be two partition functions. If*

$$\begin{aligned} &v(S,\Pi) - v(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T}) \\ &\quad \geq v'(S,\Pi) - v'(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T}) \end{aligned}$$

for each embedded coalition (S,Π) such that $i \in S$ and each atom T of Π is different from S, then $\sigma_i(v) \geq \sigma_i(v')$.

In words, if for a partition function the vector of marginal contributions of a player to the different coalitions, for any organization of the complement, dominates coordinate by coordinate that of a second partition function, the value must pay this player more in the first partition function. For instance, the value σ^α of Example 1 is monotonic for each $\alpha \in [0,1]$.

First, we point out in the following example that Monotonicity, combined with Anonymity and Efficiency, does not imply additivity either:

EXAMPLE 4: The value $\sigma^{\alpha,m}$ from Example 2 satisfies Monotonicity if $\alpha = 1/2$, $m(x) = x^2$ if $|x| \leq 1/12$, and $m(x) = (1/12)^2$ if $|x| \geq 1/12$.

We may nevertheless bound each player's payoff from below and from above. The approach we follow seeks to obtain bounds that rely on decompositions of certain partition functions into the sum of a symmetric partition function and a characteristic function, the class of partition functions uncovered in Proposition 1.

Let $\mathcal{S}$ be the set of symmetric partition functions and let $\mathcal{C}$ be the set of characteristic functions. Let $i \in N$. For each partition function v, let $M_i(v)$ be the set of pairs $(u, v') \in \mathcal{S} \times \mathcal{C}$ such that

$$\begin{aligned} &v(S,\Pi) - v(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T}) \\ &\quad \geq \left[u(S,\Pi) - u(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T})\right] + [v'(S) - v'(S_{-i})] \end{aligned}$$

[7]Perhaps the term "Weak Monotonicity" would be more appropriate to emphasize the link with Weak Marginality, hence comparing all the total effects instead of limiting ourselves to the intrinsic marginal contributions (see Section 5). Yet, since weak monotonicity has other meanings and since it will not be necessary to introduce other monotonicity properties, we opted for the term "monotonicity."

for each embedded coalition (S, Π) such that $i \in S$ and each atom T of Π that is different from S. Monotonicity and Proposition 1 imply that $\sigma_i(v) \geq \frac{u(N)}{n} + \mathrm{Sh}_i(v')$ for each $(u, v') \in M_i(v)$. Therefore, the best lower bound following this approach is obtained by solving the following linear programming problem, which always has a unique optimal objective value:

$$\mu_i(v) = \max_{(u,v')\in M_i(v)} \left[\frac{u(N)}{n} + \mathrm{Sh}_i(v')\right].$$

Similarly, for each partition function v, let $N_i(v)$ be the set of pairs $(u, v') \in \mathcal{S} \times \mathcal{C}$ such that

$$\begin{aligned} &v(S, \Pi) - v(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T}) \\ &\quad \leq \left[u(S, \Pi) - u(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T})\right] + [v'(S) - v'(S_{-i})] \end{aligned}$$

for each embedded coalition (S, Π) such that $i \in S$ and each atom T of Π that is different from S. Again, Monotonicity and Proposition 1 imply that $\sigma_i(v) \leq \frac{u(N)}{n} + \mathrm{Sh}_i(v')$, for each $(u, v') \in N_i(v)$. The best upper bound following this approach is obtained by solving the linear programming problem

$$\nu_i(v) = \min_{(u,v')\in N_i(v)} \left[\frac{u(N)}{n} + \mathrm{Sh}_i(v')\right].$$

That is, we have shown the following proposition:

PROPOSITION 2: *If σ is a value that satisfies Anonymity, Efficiency, and Monotonicity, then*

$$\sigma_i(v) \in [\mu_i(v), \nu_i(v)]$$

for each $i \in N$.

These bounds in general improve upon those that one could obtain by using only symmetric partition functions or only characteristic functions. (In particular, the "most pessimistic" and "most optimistic" characteristic functions that each player i can construct from the partition function. Player i's most pessimistic characteristic function minimizes over Π the worth $v(S, \Pi)$ of each coalition, S where $i \in S$, and maximizes it for coalitions that do not include i. The opposite happens for his most optimistic characteristic function.)

One important question is whether the bounds provided by Proposition 2 are "tight," in the sense that for each partition function one can always find solutions to fill up the entire identified cube of payoffs. Although we do not know the answer to this question in general, at least for the case of three players, the bounds are tight. We will elaborate on this point in the next section after we prove its result.

5. MARGINALITY AND THE EXTERNALITY-FREE VALUE

An alternative route to Monotonicity is to strengthen Weak Marginality into another marginality axiom. To do this, it will be instructive to look closer at the concept of marginal contribution in contexts with externalities.

Consider player i and an embedded coalition (S, Π) with $i \in S$. Suppose player i leaves coalition S and joins coalition $T \in \Pi$, $T \neq S$. One can view this as a two-step process. In the first instance, player i simply leaves S and, at least for a while, he is alone, which means that for the moment the coalition structure is $\{S_{-i}, \{i\}\} \cup \Pi_{-S}$. At this point, coalition S_{-i} feels the loss of player i's *marginal contribution*, that is,

$$v(S, \Pi) - v\big(S_{-i}, \{S_{-i}, \{i\}\} \cup \Pi_{-S}\big).$$

In the second step, player i joins coalition $T \in \Pi_{-S}$, and then S_{-i} is further affected, but not because of a marginal contribution from player i. Rather, it is affected because of the corresponding *externalities* created by this merger, that is,

$$v\big(S_{-i}, \{S_{-i}, \{i\}\} \cup \Pi_{-S}\big) - v(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T}).$$

If one views this as an important distinction, one should reserve the term (intrinsic) *marginal contribution* for the former difference. We shall do this in the sequel.

Formally, let i be a player and let (S, Π) be an embedded coalition such that $i \in S$. Then the (intrinsic) *marginal contribution of i to* (S, Π) is given by

$$\mathrm{mc}_{(i,S,\Pi)}(v) = v(S, \Pi) - v\big(S_{-i}, \{S_{-i}, \{i\}\} \cup \Pi_{-S}\big)$$

for each partition function v. Player i's *vector of intrinsic marginal contributions* is obtained by varying (S, Π): $\mathrm{mc}_i(v) = (\mathrm{mc}_{(i,S,\Pi)})_{(S,\Pi)\in \mathrm{EC} \wedge i \in S}$. Here is the formal statement of the new marginality axiom for partition functions (note that it also reduces to Young's if applied to characteristic functions).

MARGINALITY: *Let i be a player and let v and v' be two partition functions. If* $\mathrm{mc}_i(v) = \mathrm{mc}_i(v')$, *then* $\sigma_i(v) = \sigma_i(v')$.

Consider now the following extension σ^* of the Shapley value to partition functions:

$$\sigma_i^*(v) := \mathrm{Sh}_i(v^*)$$

for each player $i \in N$ and each partition function v, where v^* is the fictitious characteristic function defined as

$$v^*(S) := v\big(S, \{S, \{j\}_{j\in N\setminus S}\}\big)$$

for each coalition S.

We call this the *externality-free value* and we shall discuss it below. Our next result follows.

PROPOSITION 3: *σ^* is the unique value satisfying Anonymity, Efficiency, and Marginality.*

PROOF: The set of partition functions forms a vector space. We prove the proposition by defining an adequate basis. Let (S, Π) be an embedded coalition, where S is nonempty. Then $e_{(S,\Pi)}$ is the partition function defined as

$$e_{(S,\Pi)}(S', \Pi') = \begin{cases} 1, & \text{if } S \subseteq S' \text{ and} \\ & (\forall T' \in \Pi' \setminus \{S'\})(\exists T \in \Pi) : T' \subseteq T, \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA 1: *The collection of vectors $(e_{(S,\Pi)})_{(S,\Pi)\in\mathrm{EC}}$ constitutes a basis of the space of partition functions.*

PROOF: The number of vectors in the collection equals the dimension of the space. It remains to show that they are linearly independent, that is,

$$\sum_{(S,\Pi)\in\mathrm{EC}} \alpha(S, \Pi) e_{(S,\Pi)} = 0 \tag{1}$$

implies $\alpha(S, \Pi) = 0$ for each $(S, \Pi) \in \mathrm{EC}$.

Suppose on the contrary that there exists a collection $(\alpha_{(S,\Pi)})_{(S,\Pi)\in\mathrm{EC}}$ of real numbers satisfying (1) and such that $\alpha(S, \Pi) \neq 0$, for some $(S, \Pi) \in \mathrm{EC}$. Let (S^*, Π^*) be an embedded coalition such that:

1. $\alpha(S^*, \Pi^*) \neq 0$;
2. $(\forall (S, \Pi) \in \mathrm{EC}) : S \subsetneq S^* \Rightarrow \alpha(S, \Pi) = 0$;
3. $(\forall (S^*, \Pi) \in \mathrm{EC}$ s.t. $\Pi \neq \Pi^*) : \Pi$ coarser than $\Pi^* \Rightarrow \alpha(S^*, \Pi) = 0$.

By definition, $e_{(S,\Pi)}(S^*, \Pi^*) = 0$ if S is not included in S^*. The second property of (S^*, Π^*) then implies that

$$\left[\sum_{(S,\Pi)\in\mathrm{EC}} \alpha(S, \Pi) e_{(S,\Pi)}\right](S^*, \Pi^*) = \left[\sum_{(S^*,\Pi)\in\mathrm{EC}} \alpha(S^*, \Pi) e_{(S^*,\Pi)}\right](S^*, \Pi^*).$$

By definition, $e_{(S^*,\Pi)}(S^*, \Pi^*) = 0$ if Π is not coarser than Π^*. The third property of (S^*, Π^*) then implies that

$$\left[\sum_{(S^*,\Pi)\in\mathrm{EC}} \alpha(S^*, \Pi) e_{(S^*,\Pi)}\right](S^*, \Pi^*) = \alpha(S^*, \Pi^*).$$

Equation (1) thus implies that $\alpha(S^*, \Pi^*) = 0$, a contradiction. *Q.E.D.*

Now we continue with the proof of Proposition 3. The properties of the Shapley value imply that σ^* satisfies the three axioms.

For uniqueness, let σ be a value satisfying the three axioms. We show that $\sigma(v) = \sigma^*(v)$ for each partition function v by induction on the number of nonzero terms appearing in the basis decomposition of v.

Suppose first that $v = \alpha e_{(S,\Pi)}$ for some $\alpha \in \mathbb{R}$ and some $(S, \Pi) \in \mathrm{EC}$. Let $i \in N \setminus S$ and let (S', Π') be any embedded coalition. The two following statements are equivalent:

1. $S \subseteq S'$ and $(\forall T' \in \Pi'_{-S'})(\exists T \in \Pi) : T' \subseteq T$.
2. $S \subseteq S' \setminus \{i\}$ and $(\forall T' \in \{\{i\}, \Pi'_{-S'}\})(\exists T \in \Pi) : T' \subseteq T$.

Hence $e_{(S,\Pi)}(S', \Pi') = 1$ if and only if $e_{(S,\Pi)}(S'_{-i}, \{S'_{-i}, \{i\}, \Pi'_{-S'}\}) = 1$. So, $\mathrm{mc}_i(v) = \mathrm{mc}_i(v^0)$, where v^0 is the null partition function (i.e., $v^0(S, \Pi) = 0$ for each $(S, \Pi) \in \mathrm{EC}$). Marginality implies that $\sigma_i(v) = \sigma_i(v^0)$. Anonymity and Efficiency imply that $\sigma_i(v^0) = 0$. Hence $\sigma_i(v) = 0 = \sigma_i^*(v)$. Anonymity implies in addition that $\sigma_j(v) = \sigma_k(v)$ and $\sigma_j^*(v) = \sigma_k^*(v)$ for all j, k in S. Hence $\sigma(v) = \sigma^*(v)$, since both σ and σ^* satisfy Efficiency.

Suppose now that we have proved the result for all the partition functions that have at most k nonzero terms when decomposed in the basis and let

$$v = \sum_{(S,\Pi)\in\mathrm{EC}} \alpha(S, \Pi) e_{(S,\Pi)}$$

be a partition function with exactly $k + 1$ nonzero coefficients. Let S^* be the intersection of the coalitions S for which there exists a partition Π such that $\alpha(S, \Pi)$ is different from zero. If $i \in N \setminus S^*$, then player i's marginal contribution vector in v coincides with his marginal contribution vector in the partition function

$$v' = \sum_{(S,\Pi)\in\mathrm{EC}\ \text{s.t. } i\in S} \alpha(S, \Pi) e_{(S,\Pi)}.$$

Marginality implies that $\sigma_i(v) = \sigma_i(v')$. Note that the number of nonzero terms in the basis decomposition of v' is at most k. Then, by the induction hypothesis, $\sigma_i(v') = \sigma_i^*(v')$. Since σ^* satisfies Marginality as well, we conclude that $\sigma_i(v) = \sigma_i^*(v)$. Anonymity implies in addition that $\sigma_j(v) = \sigma_k(v)$ and $\sigma_j^*(v) = \sigma_k^*(v)$ for all j, k in S^*. Hence $\sigma(v) = \sigma^*(v)$, since both σ and σ^* satisfy Efficiency. The proof of Proposition 3 is now complete. *Q.E.D.*

Marginality does not imply on its own that externalities play no role in the computation of payoffs. Indeed, a player's intrinsic marginal contribution to an embedded coalition (S, Π) depends on the composition of Π_{-S}. Instead, it is the combination of Marginality, Efficiency, and Anonymity that leads to the externality-free value. Hence, Proposition 3 is definitely not a trivial variation on Young's (1985) original theorem: some information present in the partition function has to be discarded as a consequence of the combination of the three axioms. To gain an intuition for this, consider first a three-player partition function. The result tells us that player 1's payoff does not depend on $x = v(\{1\}, \{\{1\}, \{2, 3\}\})$. This does not follow from any of the three axioms taken separately (in particular, by Marginality it could depend on x if $S = \{1\}$ and $\Pi_{-S} = \{\{2, 3\}\}$ in the definition of the axiom). Instead, the reasoning goes as follows. Player 2's and 3's payoffs do not depend on x according to Marginality. Efficiency then implies that player 1's payoff cannot depend on x either.

To strengthen one's intuition, let us pursue this heuristic argument with four players. In principle, player 1's payoff could depend on fifteen numbers according to Marginality. Only eight of them are actually relevant to compute σ^*. Let us show, for instance, why player 1's payoff cannot depend on $y = v(\{1\}, \{\{1\}, \{2, 3\}, \{4\}\})$. Marginality implies that, apart from player 1's payoff, only the payoff of player 4 could depend on y or, more precisely, on $z - y$, where $z = v(\{1, 4\}, \{\{1, 4\}, \{2, 3\}\})$. Marginality implies also that the payoffs of players 2 and 3 do not depend on z. On the other hand, as we know from Proposition 3, the three axioms together imply that the solution must be an anonymous and additive function. Thus, the payoffs of players 1 and 4 depend identically on z (if z increases, payoffs to both players 1 and 4 also increase). Hence, Efficiency implies that player 4's payoff cannot depend on z and therefore not on y either. Player 2's and 3's payoffs do not depend on y by Marginality. Hence player 1's payoff cannot depend on y, by Efficiency.

We regard the externality-free value σ^* as a fair compromise that takes into account the pure or intrinsic *marginal contributions* of players to coalitions, stripped down from externality components. Note also that σ^* satisfies Monotonicity. Then the range of payoffs identified in Proposition 2 for each player captures how externalities affect his payoff, when one still requires Efficiency, Anonymity, and Monotonicity. Thus, the size of the difference $\nu_i(v) - \sigma_i^*(v)$ expresses the maximum "subsidy" or benefit to player i associated with externalities that favor him, and the difference $\sigma_i^*(v) - \mu_i(v)$ represents how much i can be "taxed" or suffer, due to harmful externalities, in a value that obeys these three axioms.

Consider the case of three players, $N = \{i, j, k\}$. By solving the linear programs described to prove Proposition 2, one can show that

$$\nu_i(v) - \sigma_i^*(v) = \frac{\max\{0, \varepsilon_i(v) - \varepsilon_j(v)\} + \max\{0, \varepsilon_i(v) - \varepsilon_k(v)\}}{6},$$

$$\sigma_i^*(v) - \mu_i(v) = \frac{\max\{0, \varepsilon_j(v) - \varepsilon_i(v)\} + \max\{0, \varepsilon_k(v) - \varepsilon_i(v)\}}{6},$$

where $\varepsilon_i(v) = v(\{i\}, \{\{i\}, \{j, k\}\}) - v(\{i\}, \{\{i\}, \{j\}, \{k\}\})$ is the externality index associated to player i. Notice that it is not the sign or the magnitude of $\varepsilon_i(v)$

alone that determines the bounds on player i's payoff, but instead how far $\varepsilon_i(v)$ is from $\varepsilon_j(v)$ and $\varepsilon_k(v)$. To fix ideas, suppose that $\varepsilon_i(v) \le \varepsilon_j(v) \le \varepsilon_k(v)$. Then

$$\sigma_i(v) \in [\sigma_i^1(v), \sigma_i^0(v)] \quad \text{and} \quad \sigma_k(v) \in [\sigma_k^0(v), \sigma_k^1(v)].$$

Here, σ^0 and σ^1 are the solutions σ^α of Example 1 for the cases of $\alpha = 0$ and $\alpha = 1$, respectively ($\sigma^0 = \sigma^*$). The bounds are intuitive. For the player who benefits the least from externalities (player i), the lowest possible payoff compatible with the axioms happens at the Shapley value of the average characteristic function that puts all the weight on his worth when players j and k cooperate. His highest possible payoff is obtained at the Shapley value of the average characteristic function where his worth corresponds to the situation in which players j and k do not cooperate. Exactly the opposite happens for player k, who benefits the most from externalities. Therefore, the bounds we obtained in Proposition 2 are tight in the following sense. There exists a value (e.g., σ^0 or σ^1) that satisfies Anonymity, Efficiency, and Monotonicity, and such that, for each partition function v, some player gets the lower bound, while another player gets the upper bound, as defined before Proposition 2. Another observation in support of the tightness of our bounds is that $\nu_i(v) \le v(N) - \mu_j(v) - \mu_k(v)$ and that $\mu_i(v) \ge v(N) - \nu_j(v) - \nu_k(v)$ (for each i, j, k). The easy proof is left to the interested reader. The next example further illustrates the tightness of the bounds and the relative position of σ^* within those bounds.

EXAMPLE 5: Consider a variant of Example 3 in which player 1 is the only agent capable of free-riding from a two-player coalition, receiving a worth of 9, as before, when coalition $\{2, 3\}$ gets together. However,

$$v\big(\{2\}, \{\{2\}, \{1, 3\}\}\big) = v\big(\{3\}, \{\{3\}, \{1, 2\}\}\big) = 0.$$

One can easily check that $\sigma^0(v) = \sigma^*(v) = (7.5, 8, 8.5)$, $\sigma^1(v) = (10.5, 6.5, 7)$, and that the bounds from Proposition 2 are

$$\mu_1(v) = 7.5, \quad \nu_1(v) = 10.5;$$
$$\mu_2(v) = 6.5, \quad \nu_2(v) = 8;$$
$$\mu_3(v) = 7, \quad \nu_3(v) = 8.5.$$

That is, no monotonic solution σ ever punishes player 1 or rewards 2 and 3 with respect to the payoffs in σ^*. Given the nature of externalities in this example (player 1 is the only one who benefits from externalities), it is clear what direction externality-driven transfers should take for each player.

For more than three players, we do not know how tight the bounds of Proposition 2 are. However, in many examples, they certainly provide an insightful refinement of the set of feasible payoffs.

In the same way that σ^* serves as an interesting reference point in the space of payoffs, the characteristic function v^* can serve as an interesting reference point to measure the size of externalities. Let us define the *externality index* $\varepsilon_v(S, \Pi)$ associated to (S, Π) as the difference $v(S, \Pi) - v^*(S)$ (as in Example 2 or in the discussion preceding Example 5).[8] We can now better understand the class of partition functions that was uncovered in Proposition 1 and that plays a key role in many parts of our paper. Externalities are *symmetric* for v if the associated externality indices $(\varepsilon_v(S, \Pi))_{(S,\Pi)\in\mathrm{EC}}$ form a symmetric partition function, that is, the level of externality associated to any embedded coalition (S, Π) depends only on the cardinality of S and on the cardinality of the other atoms of Π. It is then not difficult to check that a partition function can be decomposed as the sum of a symmetric partition function and a characteristic function if and only if externalities are symmetric for v. Example 3 provides an illustration of this equivalence.

6. ALTERNATIVE APPROACHES TO THE EXTERNALITY-FREE VALUE

A natural adaptation of Shapley's original axiomatic system also leads to the externality-free value σ^*. Let i be a player and let v be a partition function. We say that player i is *null* in v if his marginal contribution to any embedded coalition is *nil*: $\mathrm{mc}_i(v) = 0$. Note how this notion of a null player disregards externalities, as only intrinsic marginal contributions matter.

NULL PLAYER: *Let v be a partition function. If player i is null in v, then $\sigma_i(v) = 0$.*

ADDITIVITY: *Let v and w be two partition functions. Then $\sigma(v + w) = \sigma(v) + \sigma(w)$.*

A null player must receive a zero payoff, according to the first axiom. Additivity essentially amounts to the linearity of the value.[9] It expresses a form of mathematical simplicity by requiring a strong specific functional form.

PROPOSITION 4: *σ^* is the unique value satisfying Anonymity, Efficiency, Null Player, and Additivity.*

PROOF: The properties of the Shapley value imply that σ^* satisfies the four axioms.

[8]Any difference $v(S, \Pi) - v(S, \Pi')$ can be recovered from our externality indices, since $v(S, \Pi) - v(S, \Pi') = \varepsilon_v(S, \Pi) - \varepsilon_v(S, \Pi')$.

[9]To be precise, it implies linearity only with respect to linear combinations involving rational numbers; see Macho-Stadler, Pérez-Castrillo, and Wettstein (2007).

Let σ be a value satisfying the four axioms and let v be a partition function. Remember that v can be decomposed in the basis described in Lemma 1:

$$v = \sum_{(S,\Pi)\in \mathrm{EC}} \alpha(S,\Pi)e_{(S,\Pi)} \tag{2}$$

for some real numbers $\alpha(S,\Pi)$. Anonymity, Null Player, and Efficiency imply that $\sigma(\alpha(S,\Pi)e_{(S,\Pi)}) = \sigma^*(\alpha(S,\Pi)e_{(S,\Pi)})$ for each $(S,\Pi)\in \mathrm{EC}$ (a similar argument was made in the proof of Proposition 3). Since both σ and σ^* are additive, we conclude that $\sigma(v) = \sigma^*(v)$. *Q.E.D.*

Proposition 4 is equivalent to Theorem 2 of Pham Do and Norde (2007). However, their proof relies on the canonical basis instead of using the basis identified in Lemma 1. Our basis has the advantage of allowing us to prove a similar result on the smaller class of superadditive partition functions.

A player is null if his vector of marginal contributions is *nihil*. The other notion of marginal contributions, which contained the externality effects and was used to define Weak Marginality, leads in turn to a weaker version of the null player property. Player i is *null in the strong sense* if

$$v(S,\Pi) - v(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T}) = 0$$

for each embedded coalition (S,Π) such that $i\in S$ and each atom T of Π that is different from S. Clearly, if player i is null in the strong sense, then he is null. One can use this definition to propose a different null player axiom:

WEAK VERSION OF THE NULL PLAYER AXIOM: *Let $i\in N$ and let v be a partition function. If player i is null in the strong sense, then $\sigma_i(v) = 0$.*

This is equivalent to the dummy player axiom of Bolger (1989) and Macho-Stadler, Pérez-Castrillo, and Wettstein (2007).[10] Macho-Stadler, Pérez-Castrillo, and Wettstein (2007, Theorem 1) showed that any solution that satisfies this version of the Null Player axiom, as well as the axioms of Efficiency, linearity and (a strong version of) Anonymity is a Shapley value of a characteristic function that is obtained by performing averages of the partition function. Our externality-free value σ^* belongs to this class of solutions. Macho-Stadler, Pérez-Castrillo, and Wettstein also characterized a unique solution by adding an axiom of similar influence that σ^* does not satisfy.

[10]Bolger and Macho-Stadler, Pérez-Castrillo, and Wettstein said that player i is dummy if $v(S,\Pi) = v(S',\Pi')$ for each (S,Π) and each (S',Π') that can be deduced from (S,Π) by changing player i's affiliation. This clearly implies that player i is null in our strong sense. The converse is straightforward after proving that $v(S,\Pi) = v(S,\Pi')$ for each pair of embedded coalitions (S,Π) and (S,Π') such that $i\notin S$ and (S,Π') can be deduced from (S,Π) by changing only player i's affiliation. Indeed, if i is null in our strong sense, then $v(S,\Pi) = v(S_{+i}, \{S_{+i}\}\cup\{T_{-i} | T\in \Pi_{-S}\}) = v(S_{+i}, \{S_{+i}\}\cup\{T_{-i} | T\in \Pi'_{-S}\}) = v(S,\Pi')$.

In light of our first discussion concerning even nonlinear solutions (recall Example 2), we prefer Proposition 3 to Proposition 4 (even though both lead to the same value), because we find Marginality more compelling than Additivity. It is easier to interpret and justify a restriction on the set of variables required to compute the players' payoffs, than to impose a specific functional form. It is interesting to note in that respect that the nonadditive solution defined in Example 4 satisfies the strong symmetry and the similar influence axioms of Macho-Stadler, Pérez-Castrillo, and Wettstein (2007), in addition to satisfying Anonymity, Efficiency, and Weak Marginality. Once again, Additivity cannot be justified by the marginality principle that underlies their dummy player axiom (the weak version of marginality), even if one imposes their other requirements.

One can take a bargaining approach to understand the Shapley value. This was done, for instance, in Hart and Mas-Colell (1996) or Pérez-Castrillo and Wettstein (2001) for characteristic functions. It is not difficult to see that we obtain σ^* if we apply these procedures to superadditive partition functions. Indeed, it is assumed in these two papers that when a proposal is rejected, the proposer goes off by himself and does not form a coalition with anyone else. We leave the details to the interested reader. Other new rules concerning the rejected proposers would lead to values that treat externalities differently (see Macho-Stadler, Pérez-Castrillo, and Wettstein (2006)).

Myerson's (1980) principle of balanced contributions (or the related concept of potential proposed by Hart and Mas-Colell (1989)) offers another elegant justification for the Shapley value. Instead of characterizing a value (i.e., a function that determines a payoff vector for each characteristic function), Myerson characterizes a payoff configuration (i.e., a vector that determines how the members of each coalition would share the surplus that is created should they cooperate). A payoff configuration satisfies the principle of balanced contributions if, for any two members i, j of any coalition S, the payoff loss that player i suffers if player j leaves S equals the payoff loss that player j suffers if player i leaves S. This principle, combined with efficiency within each coalition, implies that the players agree on the Shapley value within the grand coalition. This methodology can be easily extended to partition functions. As was the case with our analysis of Young's marginal contributions, the payoff loss that a player i suffers if a player j leaves S usually depends on what i does after leaving S (stay on his own or join another atom, and, if so, which one?). It may be worthwhile to study different scenarios. Here we simply observe that if a player is assumed to stay on his own after leaving a coalition (as was the case for our concept of intrinsic marginal contribution), then the extended principle of balanced contributions à la Myerson leads to a unique payoff configuration, and the resulting payoff for the grand coalition coincides once again with the payoff associated to the externality-free value. We again leave the details to the interested reader.

Section 5 and the current section together lead to an apparent conclusion. Suppose one simply wishes to find an extension of the Shapley value to environments with externalities by using any of the different approaches that have provided its foundations on the domain of characteristic functions. Then the result is the externality-free value, using our *intrinsic* marginal contributions as the parallel concept to the standard marginal contributions of the reduced domain. However, in dealing with externalities, any such approach must be complemented by a treatment of the externalities themselves, and that is what has led to the bounds around the externality-free value.

7. COMPARISONS WITH THE LITERATURE

In this section we present some examples to draw comparisons with previous axiomatic approaches in the literature.

Myerson (1977) proposed the first value for partition functions. His key axiom is a version of the carrier axiom, used also in the domain of characteristic functions. This value violates Monotonicity and sometimes yields unintuitive predictions.

EXAMPLE 6: Consider the three-player partition function v, where $v(\{1\}, \{\{1\}, \{2, 3\}\}) = 1 = v(N)$ and $v(S, \Pi) = 0$ otherwise. Myerson's value, $(1, 0, 0)$, falls outside of our cube of payoffs ($[1/3, 2/3]$ for player 1 and $[1/6, 1/3]$ for players 2 and 3). Hence Myerson's value is not monotonic.

In a second three-player partition function, where $v(N) = v(\{1\}, \{\{1\}, \{2\}, \{3\}\}) = 1$ and $v(S, \Pi) = 0$ otherwise, Myerson's value assigns the payoffs $(0, 1/2, 1/2)$. Again, this falls outside of our range of payoffs, which is the same as before.

The cube of payoffs compatible with Monotonicity is the same for both partition functions. However, the externality-free value and the direction of the externality-based transfers do vary. The externality-free value yields $(1/3, 1/3, 1/3)$ in the first partition function, and taking into account externality-based transfers in monotonic values can only help player 1 at the expense of either player 2 or 3—positive externalities on 1 when 2 and 3 cooperate. In contrast, σ^* yields $(2/3, 1/6, 1/6)$ in the second partition function, and taking externalities into account can only hurt player 1 to favor 2 and 3—negative externalities for 1 when 2 and 3 get together.

Bolger's (1989) work is based on the additivity axiom, yet he characterizes a unique value, while using the weak dummy axiom as in Macho-Stadler, Pérez-Castrillo, and Wettstein (2007), by introducing an axiom of (expected) marginality. Bolger applies this axiom only on the class of "simple games." The expectation is computed by assuming that a player has an equal chance of joining any of the other atoms when he is leaving a group. Macho-Stadler, Pérez-Castrillo, and Wettstein (2007) showed that Bolger's value cannot be obtained through the "average approach" and that it violates their axioms of strong symmetry and similar influence. Clearly, it also violates our Null Player axiom.

EXAMPLE 7: Consider the following three-player partition function, where

$$\begin{aligned}
&v(N) = v(\{2,3\}) = 1,\\
&v(\{1,2\}) = v(\{1,3\}) = 1/2,\\
&v(\{2\}, \{\{1\},\{2\},\{3\}\}) = a, \quad v(\{3\}, \{\{1\},\{2\},\{3\}\}) = a,\\
&v(\{2\}, \{\{2\},\{1,3\}\}) = 1-a, \quad v(\{3\}, \{\{3\},\{1,2\}\}) = 1-a,\\
&v(\{1\}, \{\{1\},\{2\},\{3\}\}) = v(\{1\}, \{\{1\},\{2,3\}\}) = 0.
\end{aligned}$$

Because of the expected marginality axiom, player 1 is null "in average": for example, if he abandons coalition $\{1,2\}$ and there is equal probability of him being alone or joining player 3, his expected marginal contribution (as defined by Bolger) to player 2 is 0. Bolger's value assigns to this partition function the payoffs $(0, 1/2, 1/2)$. This is true with independence of the size of the parameter a. In contrast,

$$\sigma^*(v) = \left(\frac{0.5-a}{3}, \frac{2.5+a}{6}, \frac{2.5+a}{6}\right).$$

That is, as a increases, the externality-free value transfers surplus from player 1 to players 2 and 3.

The bounds depend on whether the externality is positive or negative. If $a < 1/2$, then

$$\begin{aligned}
&\mu_1(v) = \sigma_1^*(v) - \frac{1-2a}{3}, \quad \nu_1(v) = \sigma_1^*(v),\\
&\mu_2(v) = \sigma_2^*(v), \quad \nu_2(v) = \sigma_2^*(v) + \frac{1-2a}{6},\\
&\mu_3(v) = \sigma_3^*(v), \quad \nu_3(v) = \sigma_3^*(v) + \frac{1-2a}{6}.
\end{aligned}$$

So, if externalities are to be taken into account in the determination of the solution, players 2 and 3 should expect positive transfers from player 1 with respect to the externality-free value.

However, the opposite happens if $a > 1/2$:

$$\begin{aligned}
&\mu_1(v) = \sigma_1^*(v), \quad \nu_1(v) = \sigma_1^*(v) + \frac{2a-1}{3},\\
&\mu_2(v) = \sigma_2^*(v) - \frac{2a-1}{6}, \quad \nu_2(v) = \sigma_2^*(v),
\end{aligned}$$

$$\mu_3(v) = \sigma_3^*(v) - \frac{2a-1}{6}, \quad \nu_3(v) = \sigma_3^*(v).$$

Macho-Stadler, Pérez-Castrillo, and Wettstein (2007) used a "similar influence" axiom to uniquely characterize a value within the class of solutions that they term the average approach. In the previous example, the value identified by Macho-Stadler, Pérez-Castrillo, and Wettstein coincides with Bolger's and, hence, it is insensitive to the parameter a.[11] To better understand the differences between Macho-Stadler, Pérez-Castrillo, and Wettstein and our approach, the next two examples are illustrative.

EXAMPLE 8: Suppose there are 101 players. Let $v(N) = v(\{1\}, \{\{i\}_{i\in N}\}) = 1$ and $v(S, \Pi) = 0$ otherwise. The Macho-Stadler, Pérez-Castrillo, and Wettstein value coincides with the Shapley value of the average characteristic function $\bar{v}$, where $\bar{v}(\{1\}) = 1/100!$, a very small positive number. Their value thus prescribes something extremely close to equal split, paying about 1/101 per player (player 1 getting a tiny bit more than the others). According to our approach, the externality-free value pays 2/101 to player 1 and 99/10,100 to each of the others. Player 1 gets more than double of the share the others if one ignores externalities. Our cube allows the range of payoffs for player 1 to vary between 1/101 and 2/101, underlining the fact that he will have to worry about making transfers to the others to bribe them not to cooperate. For the others, the range of payoffs places the externality-free value at the bottom, from which they can only improve through transfers from player 1. This seems to capture better what is going on in the problem. Indeed, the partition function is "strongly asymmetric" from player 1's point of view, and so it is counterintuitive to prescribe essentially the equal split, as the Macho-Stadler, Pérez-Castrillo, and Wettstein value does. Our cube of payoffs and the position of σ^* in it seem to better capture the "strong asymmetry" of the partition function.

In the next example, we show that the cube of payoffs identified by our bounds contains payoffs associated with nonlinear solutions, which cannot be obtained through Monotonicity in the average approach of Macho-Stadler, Pérez-Castrillo, and Wettstein (2007). The example also shows that the axioms may imply restrictions on the players' payoffs that are not captured by the bounds.

EXAMPLE 9: Consider the three-player partition function

$$v(N) = 1,$$

[11] The same payoff is prescribed for that partition function by the value proposed by Albizuri, Arin, and Rubio (2005), because of their embedded coalition anonymity axiom. This value is one of the average approach values of Macho-Stadler, Pérez-Castrillo, and Wettstein (2007).

$$v(\{i,j\}) = 0 \quad \text{for all } i, j,$$
$$v\big(\{1\}, \{\{1\}, \{2,3\}\}\big) = 1/10,$$
$$v\big(\{2\}, \{\{2\}, \{1,3\}\}\big) = 1/20,$$
$$v\big(\{3\}, \{\{3\}, \{1,2\}\}\big) = 0,$$
$$v\big(\{i\}, \{\{i\}, \{j\}, \{k\}\}\big) = 0 \quad \text{for all } i, j, k.$$

The cube of payoffs associated with monotonic values is

$$\mu_1(v) = \frac{1}{3}, \quad \nu_1(v) = \frac{1}{3} + \frac{1}{40},$$
$$\mu_2(v) = \frac{1}{3} - \frac{1}{120}, \quad \nu_2(v) = \frac{1}{3} + \frac{1}{120},$$
$$\mu_3(v) = \frac{1}{3} - \frac{1}{40}, \quad \nu_3(v) = \frac{1}{3}.$$

Using $\alpha = 0$ in the average approach gives the externality-free value, which yields an equal split of surplus: $\sigma^*(v) = (1/3, 1/3, 1/3)$. On the other hand, if $\alpha = 1$, the corresponding average approach value gives $\sigma^1(v) = (\frac{1}{3} + \frac{1}{40}, \frac{1}{3}, \frac{1}{3} - \frac{1}{40})$. Player 2 actually receives $1/3$ according to every monotonic value obtained from the average approach.

However, the nonlinear monotonic value of Example 4 gives

$$\sigma^{\alpha,m}(v) = \left(\frac{1}{3} + \frac{131}{7200}, \frac{1}{3} - \frac{7}{7200}, \frac{1}{3} - \frac{124}{7200}\right).$$

Here, the transfers driven by externalities do not cancel out for player 2, because they enter nonlinearly in the solution. Nonlinear solutions are far more complex than their linear counterparts, as the next paragraph will suggest.

Let us rescale up the partition function of this example by multiplying it by 60. Call the resulting partition function v'. The new bounds are

$$\mu_1(v') = 20, \quad \nu_1(v') = 21.5,$$
$$\mu_2(v') = 19.5, \quad \nu_2(v') = 20.5,$$
$$\mu_3(v') = 18.5, \quad \nu_3(v') = 20.$$

The average approach, under $\alpha = 0$, yields $\sigma^*(v') = (20, 20, 20)$. With $\alpha = 1$, it yields $\sigma^1(v') = (21.5, 20, 18.5)$. Again, every monotonic average approach value will pay player 2 exactly 20. That is, so far everything has been rescaled up by the same factor of 60. However, consider our monotonic solution of Example 4. It yields

$$\sigma^{\alpha,m}(v') = \left(20 + \frac{217}{288}, 20 + \frac{1}{288}, 20 - \frac{109}{144}\right),$$

which does not preserve the rescaling and which, more interestingly, changes the nature of transfers across agents vis-à-vis the externality-free payoff.

The partition function v also illustrates another point. Let v'' be the partition function obtained from v by increasing the surplus of agent 2 to $1/10$ when he free-rides. If the value is monotonic, then player 2's payoff is larger in v'' than in v, while player 1's payoff is larger in v than in v''. On the other hand, if the value is anonymous, then players 1 and 2 must receive the same payoffs in v''. Hence player 1's payoff is larger than player 2's payoff in v. This conclusion cannot be reached by comparing the bounds, since $\nu_2(v) > \mu_1(v)$. Hence it is possible, for some partition functions, to reach conclusions regarding the players' payoffs that are not captured by the bounds.

Fujinaka (2004) introduced different notions of marginalism, using exogenous weights to aggregate the different scenarios that could follow the departure of a coalitional member. Particularly, his axiom boils down to Marginality if one puts all the weight on the scenario where a player stays on his own after leaving a coalition. Hence it appears that Fujinaka obtained independently a result similar to our third proposition. His proof differs substantially from ours and does not make use of the basis we uncovered in Lemma 1. It is not clear whether his proof can be adapted to apply on the important subclass of superadditive partition functions as ours does. Our decomposition of the total effect into the intrinsic marginal contribution and the externality effect at the beginning of Section 5 shows that if one has to choose one specific scenario within the class of scenarios considered by Fujinaka, the one where a player stays on his own after leaving a coalition is well motivated and appealing. Actually, many of Fujinaka's values are not monotonic.

EXAMPLE 10: Consider the partition function from Example 5. Recall that Efficiency, Anonymity, and Monotonicity of σ determine the payoff intervals

$$\sigma_1(v) \in [7.5, 10.5], \quad \sigma_2(v) \in [6.5, 8], \quad \sigma_3(v) \in [7, 8.5].$$

Using Fujinaka's notation, consider the case where $\alpha_i(\{j\}, \{\{j\}, \{i\}, \{k\}\}) = -1$ and $\alpha_i(\{j\}, \{\{j\}, \{i, k\}\}) = 2$. The associated value yields the payoff $(13.5, 5, 5.5)$, which falls outside the cube.

Maskin (2003) also provided an axiomatic treatment of coalitional problems with externalities, although his axioms are best understood in the context of his specific strategic model. The best comparison between his work and ours is drawn from our coalition formation analysis in de Clippel and Serrano (2008).

8. CONCLUSION

This paper has explored partition functions. Our basic approach is rooted in the concept of marginal contributions of players to coalitions. In problems involving externalities, we have argued how important it is to separate the concept of intrinsic marginal contributions from that of externalities.

The paper follows an axiomatic methodology and presumes that the grand coalition has exogenously formed. Then the implications of Anonymity, Monotonicity, and Marginality are explored, leading to two main results: the first one establishes bounds to players' payoffs if they are to be derived from solutions that are monotonic with respect to the (weak version of) marginal contributions. The second result provides a sharp characterization of a solution that captures value-like principles, if one abstracts from the externalities. The combination of both results provides insights to the size of the Pigouvian-like transfers compatible with our normative principles. Based on this analysis, one can extend the results to arbitrary coalition structures. This extension and its application to coalition formation are found in de Clippel and Serrano (2008).

REFERENCES

ALBIZURI, M. J., J. ARIN, AND J. RUBIO (2005): "An Axiom System for a Value for Games in Partition Function Form," *International Game Theory Review*, 7, 63–73. [1432]

BOLGER, E. M. (1989): "A Set of Axioms for a Value for Partition Function Games," *International Journal of Game Theory*, 18, 37–44. [1428,1430]

DE CLIPPEL, G., AND R. SERRANO (2005): "Marginal Contributions and Externalities in the Value," Working Paper 2005-11, Department of Economics, Brown University. [1418]

——— (2008): "Bargaining, Coalitions, and Externalities: A Comment on Maskin," Unpublished Manuscript, Department of Economics, Brown University. [1414,1434,1435]

FUJINAKA, Y. (2004): "On the Marginality Principle in Partition Function Form Games," Unpublished Manuscript, Graduate School of Economics, Kobe University. [1418,1434]

HAFALIR, I. E. (2007): "Efficiency in Coalition Games With Externalities," *Games and Economic Behavior*, 61, 242–258. [1416]

HART, S., AND A. MAS-COLELL (1989): "Potential, Value and Consistency," *Econometrica*, 57, 589–614. [1415,1429]

——— (1996): "Bargaining and Value," *Econometrica*, 64, 357–380. [1415,1429]

MACHO-STADLER, I., D. PÉREZ-CASTRILLO, AND D. WETTSTEIN (2006): "Efficient Bidding With Externalities," *Games and Economic Behavior*, 57, 304–320. [1429]

——— (2007): "Sharing the Surplus: An Extension of the Shapley Value for Environments With Externalities," *Journal of Economic Theory*, 135, 339–356. [1418,1427-1430,1432]

MASKIN, E. (2003): "Bargaining, Coalitions and Externalities," Presidential Address to the Econometric Society, Institute for Advanced Study, Princeton University. [1419,1434]

MYERSON, R. B. (1977): "Value of Games in Partition Function Form," *International Journal of Game Theory*, 6, 23–31. [1430]

——— (1980): "Conference Structures and Fair Allocation Rules," *International Journal of Game Theory*, 9, 169–182. [1415,1429]

PÉREZ-CASTRILLO, D., AND D. WETTSTEIN (2001): "Bidding for the Surplus: A Non-Cooperative Approach to the Shapley Value," *Journal of Economic Theory*, 100, 274–294. [1415,1429]

PHAM DO, K. H., AND H. NORDE (2007): "The Shapley Value for Partition Function Form Games," *International Game Theory Review*, 9, 353–360. [1428]

RAY, D., AND R. VOHRA (1999): "A Theory of Endogenous Coalition Structures," *Games and Economic Behavior*, 26, 286–336. [1419]

SHAPLEY, L. S. (1953): "A Value for *n*-Person Games," in *Contributions to the Theory of Games II*, ed. by A. W. Tucker and R. D. Luce. Princeton, NJ: Princeton University Press, 307–317. [1413,1416,1418]

THRALL, R. M., AND W. F. LUCAS (1963): "*n*-Person Games in Partition Function Form," *Naval Research Logistics Quarterly*, 10, 281–298. [1416]

YOUNG, H. P. (1985): "Monotonic Solutions of Cooperative Games," *International Journal of Game Theory*, 14, 65–72. [1413,1416,1418,1419,1424]

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Manuscript received June, 2007; final revision received April, 2008.