Conflict Leads to Cooperation in Demand Bargaining*

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Abstract

We consider a multilateral Nash demand game in which short-sighted players come to the bargaining table with requests for both coalition partners and the potentially generated resource. We prove that the resulting process converges with probability one to a state in which all players agree on a strictly self-enforcing division of resources (i.e., a strict core allocation). Highlighting group dynamics, we show how the myopic actions of players may lead to the break up of groups in the short run, but can ultimately bring about a situation from which a strictly self-enforcing allocation can be reached.

Keywords: Demand bargaining, strictly self-enforcing allocations, strict core, best response dynamics.

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Consider a situation repeated over time, with \( n \) players who may form coalitions. Different groups of players vary in their ability to generate surplus, but there are synergies and increasing returns to scale in cooperation (that is, the characteristic function describing the surplus each group generates is convex and strictly superadditive). To determine which coalitions will form, and how the surplus will be divided among members, players come to the bargaining table in each period with a demand for the surplus and a list of acceptable coalition partners. The players in this multilateral demand game are boundedly rational, in the sense that they find calculating optimal strategies difficult. Players may not always adjust their behavior from previous rounds, and when they do, they may play only a short-sighted best response (Kandori, Mailath & Rob (1993), Young (1993\(^a\)), Samuelson (1997)). How will the process of coalition formation evolve over time? Will players eventually agree on some division of resources? This paper shows that players ultimately agree on a strictly self-enforcing division of surplus (a \textit{strict core} allocation), highlighting the process of coalition formation by which this may occur. The repeated break up and formation of groups along the path of convergence we exhibit cannot be sustained in endless cycles, and leads to a situation from which a stable, strictly self-enforcing agreement can be reached.

This paper is closely related to Agastya (1997), who studies an evolutionary setting in which myopic players incompletely sample past plays and submit demand requests, but cannot choose their partners.\(^1\) Whether those demand requests are satisfied is determined (probabilistically) according to a convex characteristic function. Agastya (1997) proves that convergence to core allocations, or weakly self-enforcing divisions of resources, is achieved. In this paper, the possibility of excluding players from one’s partner request lays bare a convergent process in which the possibility of exclusion, and the disintegration and reformation of coalitions, lead to a strict core outcome. The process of convergence we exhibit has less exogenous randomness than that in Agastya (1997). This is enabled by the richer strategy space that arises from the direct modeling of partner selection, and less exogenous randomness in the learning dynamic and modeling of coalition formation.\(^2\) The introduction of partner selection gives an intuitive and understandable way to represent characteristic function form games in a noncooperative setting. In Agastya (1997), a player will surely

\(^1\)His approach and our own are different generalizations of Nash (1953)’s original two-player game.

\(^2\)A limited form of coalition selection is permitted by Arnold & Schwalbe (2002), who allow players to switch only among existing coalitions (hence groups cannot split, and new coalitions cannot be formed); but they assume that non-core allocations are unstable by having players in blocking coalitions play randomly.
receive his demand request if it is feasible for every largest coalition, but players cannot strategize over their partners. In our setting, only groups which are demand-feasible and mutually compatible in terms of partner requests may form. Specifically, coalition structures that form with positive probability are maximal structures satisfying the additional efficiency requirement that as many player demands as possible are satisfied. The possibility of strategizing over partners allows, for example, an intuitive proof for why strict Nash equilibria must have demand requests in the strict core. Strategic choice over partners also aids in showing how convergence to a strict core allocation can be achieved through the breaking up and reformation of groups.

This paper belongs to the literature on evolutionary game theory, which is interested in how stable states may emerge given minimal assumptions on rationality. As Young (1993a) notes, “society can ‘learn’ even when its members do not.” Milgrom & Roberts (1991) formally define an adaptive learning process as a sequence of moves in which a player eventually chooses strategies which are close to best replies to some probability distribution on others’ behavior, assigning near zero probability to strategies which have not been played for a sufficiently long time. The mechanism determining what information players have, and when they myopically best respond to it, takes several forms in the literature. Common dynamics include the best response dynamic with inertia (used here), and adaptive play (i.e., incomplete sampling of histories, as used in Agastya (1997)). In addition to modeling different forms of imperfect rationality, assumptions such as inertia or incomplete sampling also serve a technical purpose, ensuring that the dynamic process can exit potentially suboptimal cycles. As discussed in Samuelson (1997), different dynamics may lead to different predictions. A standard assumption is that players myopically best respond to their estimate of current behavior. Saez-Marti & Weibull (1999) introduce “clever” players who instead best respond to the anticipated best response. In this paper, so long as there is some probability that agents do not always update and that they best respond to current conditions, introducing any probability of being clever in the sense of Saez-Marti & Weibull (1999) would not change the results: the path of convergence exhibited still leads to strictly-self enforcing allocations, and those allocations remain absorbing states.

The results in this paper depend on the assumption of shortsightedness. Another literature studying games of bargaining and coalition formation is interested in the implications of perfect rationality. Chwe (1994), for instance, extends the notion of a possible coalitional
deviation to include actions which are unprofitable given their immediate consequence, but which will surely lead to an improvement given subsequent behavior; his idea is an equilibrium concept rather than an explicit description of equilibrium dynamics. Perry & Reny (1994) offer a bargaining procedure of successive proposals and counterproposals under which a coalition does not form until an offer is accepted (see also Chatterjee, Dutta, Ray & Sengupta (1993) for a game of sequential offers). Their procedure implements the core in a continuous time setting (but not with discrete time) when agents are farsighted. In a setting with potential externalities, Gomes & Jehiel (2005) study Markov perfect equilibria of a dynamic game in which change in coalitions requires the consent of certain members, a process which they show does not support core outcomes. In their model, the dynamics may exhibit cycles and inefficiencies. Konishi & Ray (2003) abstract from cycles, modeling players’ decisions using value functions derived from dynamic programming. Given that continuation values may depend on what other coalitions do, externalities also arise in their setting. In a game in which consent is not required to break up coalitions, they show (when players are sufficiently patient) that if a process of coalitional bargaining has a unique limit, then it belongs to the core, but that when the limit is not unique, noncore allocations may be supported.

The rest of this paper is organized as follows. Section 2 formalizes the multilateral Nash demand bargaining game studied here. Section 3 shows that the set of strict Nash equilibrium outcomes of this game corresponds to the set of allocations which are strictly self-enforcing (those in the strict core). Section 4 considers how play evolves when the demand game is played repeatedly over time by myopic players, showing that the resulting resource allocation converges with probability one to a strict core allocation.

2 A multilateral Nash demand game

There is a set \( N = \{1, 2, \ldots, n\} \) with \( n \geq 3 \) players. Letting \( 2^N \) denote the set of all possible coalitions, the resources a particular group may obtain are described by a convex and strictly superadditive characteristic function \( v : 2^N \to \mathbb{R} \). Convexity means that there are increasing returns to scale: for all \( S, T \subseteq 2^N \), \( v(S \cup T) - v(S) \geq v(T) - v(S \cap T) \). Strict superadditivity means that there are strictly positive synergies: if \( S \cap T = \emptyset \), \( v(S \cup T) > v(S) + v(T) \).

Players come to the bargaining table with two requests. First, as in the standard bilateral demand game, player \( i \) requests some amount \( d_i \in [v(i), v(N)] \) of the resource for herself.
Second, player $i$ specifies a list of players $P_i \in 2^N$ with whom she is willing to form a coalition. For notational simplicity, we assume that player $i$’s list always includes herself. The list of all resource and partner requests submitted is given by $(d, P)$, where $d = (d_1, d_2, \ldots, d_n)$ and $P = (P_1, P_2, \ldots, P_n)$. Player $i$ is said to exclude player $j$ if $j \notin P_i$.

Not every combination of resource and partner requests is feasible. Letting $\Pi(N)$ denote the set of all coalition structures (i.e., partitions of $N$), a particular coalition structure $\pi \in \Pi(N)$ will be feasible given the requests $(d, P)$ if all of its coalitions are mutually compatible and demand feasible. Mutual compatibility of a coalition structure $\pi$ given $(d, P)$ means that for each group $S \in \pi$, no member $j \in S$ is excluded from the partner list of some other player in that group. More formally, for all $S \in \pi$ and for all $i, j \in S$, it must be that $i \in P_j$. Demand feasibility of a coalition structure $\pi$ given $(d, P)$ is the condition that for each coalition $S \in \pi$ with $|S| \geq 2$, $\sum_{i \in S} d_i \leq v(S)$. A coalition $S$ with $|S| \geq 2$ is (strictly) demand feasible given $d$ if $\sum_{i \in S} d_i(<) \leq v(S)$.

Each player has a strictly increasing utility $u_i : \mathbb{R}_+ \to \mathbb{R}$ over the resource. Player $i$ is unpartnered in a coalition structure $\pi$ if $(i) \notin \pi$. A player $i$ who is unpartnered in $\pi$ receives $v(i)$ regardless of her resource request. The utility of player $i$, under the requests $(d, P)$ and the coalition structure $\pi$, is $u_i(d_i)$ if $\pi$ specifies a nontrivial coalition for $i$, and $u_i(v(i))$ otherwise. The request $d_i$ is (strictly) individually rational for player $i$ if $d_i(>) \geq v(i)$.

We now define how feasible coalition structures emerge as a function of players’ resource and partner requests. Notice that, trivially, the coalition structure $\{(1), (2), \ldots, (n)\}$, in which every player is unpartnered, is feasible given any requests $(d, P)$. In general, requests may result in multiple feasible coalition structures, some of which may be more natural than others. Given $(d, P)$ and any two feasible coalition structures $\pi, \pi' \in \Pi(N)$, we say that $\pi$ is coarser than $\pi'$ if for every $S \in \pi$, there exist $S', S'' \in \pi'$ (possibly equal) such that $S = S' \cup S''$. We assume that a coalition structure does not form unless it is a coarsest feasible coalition structure. To illustrate why this criterion might still be too permissive, consider the case $N = 4$, with $P_1 = \{1, 2\}$, $P_4 = \{3, 4\}$, and $P_i = \{1, 2, 3, 4\}$ for $i = 2, 3$. Suppose that the players’ requests are strictly individually rational and the groups $(12), (34), \text{ and } (23)$ are all demand-feasible. Then, the coarsest feasible coalition structures would be $\{(1), (23), (4)\}$ and $\{(12), (34)\}$. Because they are unpartnered, players 1 and 4 cannot receive a strictly individually rational amount of resource in the coalition structure $\{(1), (23), (4)\}$; whereas in $\{(12), (34)\}$, all players receive their strictly individually rational request. Refining the previous criterion, we assume that the feasible coalition structures that could form with positive probability given $(d, P)$ are those which minimize the number of
unpartnered individuals, subject to being in the set of coarsest feasible coalition structures given \((d, P)\); we denote this set of coalition structures by \(\mathcal{CS}(d, P)\). Formally, the probability of a feasible coalition structure \(\pi\) is determined according to a probability distribution \(F\) with full support on \(\Pi(N)\), conditioning on the set \(\mathcal{CS}(d, P)\). Prior to knowing which \(\pi\) forms, a player maximizes his expected utility.

3 Enforceability through exclusion

The core of a cooperative game with characteristic function \(v\), defined over the set of players \(N\), is the set of self-enforcing allocations \(\text{Core}(v, N) = \{ d \mid \sum_{i \in N} d_i = v(N) \text{ and } \sum_{i \in S'} d_i \geq v(S') \forall S' \subset N \}\). We will be interested in the set of all strictly self-enforcing allocations (i.e., the strict core, obtained by using strict inequalities above), which we denote \(\text{Core}^*(v, N)\). The set of such allocations is nonempty: convexity implies that the core is nonempty, and strict superadditivity implies that the inequalities can be satisfied strictly. In fact, each strict core allocation corresponds to a strict Nash equilibrium outcome of the demand game.

**Theorem 1** (Core equivalence). The requests \((d, P)\) are a strict Nash equilibrium outcome of the demand game if and only if \(d \in \text{Core}^*(v, N)\) and \(P_i = N\) for all \(i\).

We sketch the intuition behind the proof, which is given in the appendix. Instead of concentrating on demand requests, our proof concentrates on when the players have disincentives to exclude others. A player should never exclude a player who can steal away members of his coalition, leaving him unpartnered. Excluding such a player increases his probability of remaining unpartnered and receiving only \(v(i)\), thereby lowering his expected utility. If some player \(i\) has a feasible group, then players outside \(i\)’s group must be able to form a group with some of \(i\)’s partners to prevent \(i\) from excluding them. That is, \(i\) must have some chance of ending up alone in the resulting coalition structure in order to ensure that he strictly prefers not to restrict his set of acceptable partners. But then, the same

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4As noted in Hart & Kurz (1983), which considers coalition formation more generally, it is not evident how to predict which coalition structure forms, if, say, a player leaves a group. Since then, no consensus has emerged in the literature. The assumption here is meant to refine the prediction; without making a judgement on a player’s identity or the magnitude of her demand, the criterion ensures that as many strictly individually rational requests are satisfied as possible. The notions of coalitional compatibility suggested in Hart & Kurz (1983) are related to but differ from the definition here. While we have not studied all the possible coalition selection rules, minor changes in the proof show that the result is robust to conditioning the probability distribution on the set of feasible coalition structures which have the smallest number of coalitions, instead of \(\mathcal{CS}(d, P)\).
disincentive to exclude must exist for that other feasible coalition, and so and so forth. Continuing in this manner, more and more groups must be feasible until finally, by convexity, the grand coalition will itself be strictly feasible. However, if the grand coalition is strictly feasible, some player has an incentive to raise his resource request, which contradicts being at a strict Nash equilibrium.

We note that Theorem 1 would not hold without the assumption of convexity. Indeed, one may find a nonconvex characteristic function under which demand requests which are not in the core correspond to strict Nash equilibria.\(^5\)

### 4 Conflict leads to cooperation

We now consider the \(n\)-player demand bargaining game played over time \(t = 1, 2, \ldots\). The players can be interpreted as either successive generations of short-lived players or as a fixed set of myopic players.\(^6\) The players respond only to the list of resource and partner requests \((d, P)\) submitted in the previous period. Let \(\nu \in (0, 1)\) be a parameter describing inertia. With probability \(1 - \nu\), and independently of other players, a player chooses a myopic best response to the previous period’s demands. When there are multiple best responses, the player may choose any one of the strategies among which she is indifferent. With probability \(\nu\), however, a player is inert: she does not update her request, leaving the previous period’s demand in effect. Inertia is typically interpreted as a form of bounded rationality (e.g., attentional issues, computational costs, or simply slow updating of suboptimal strategies), but can also capture exogenous constraints on the ability to actively bargain, difficulties in synchronizing the timing of demands, and behaviors that have been passed down in the case of successive generations.

At each point in time, the previous period’s requests \((d, P)\) serve as the state of the game. Each player \(i\)'s resource requests are restricted to the discretized set \([v(i), v(N)]_K\) of \(K\)-place decimal fractions in \([v(i), v(N)]\).\(^7\) The evolution of the game then defines a

\(^5\)To see this, consider Example 2 of Agastya (1997), where \(N = \{1, 2, 3, 4\}, v(i) = 0\) for all \(i\), \(v(ij) = v(ijk) = 7\) for \(i \neq j \neq k\), \(v(N) = 16\), players have linear utility functions, and coalitions which could occur with positive probability are equally likely. Then the strict core is nonempty (e.g., it contains \((4, 4, 4, 4)\)). However, \(P = (N, N, N, N)\) and \(d = (3, 3, 3, 7)\) (which is not in the core) is a strict Nash equilibrium.

\(^6\)The arguments also extend immediately to the case of multiple parallel populations that each population samples or more general matching technologies.

\(^7\)We assume there is \(K^*\) such that the values of \(v\) are \(K^*\)-place decimal fractions and that \(K \geq K^*\). It will be the case that best responses are always in \([v(i), v(N)]_K\).
finite-state Markov chain over the state space of players’ partner and resource requests. We are interested in how play evolves over time. Observe that any strict Nash equilibrium corresponds to an absorbing state of the dynamic process. We will show that starting from any state, there is also positive probability of reaching a strict core allocation, where the dynamic process then remains. We describe below the main ideas behind the argument.

We prove that the following sequence of events, which ends with the grand coalition agreeing on a strictly self-enforcing allocation, has positive probability. If players in a group do not agree on a strict core allocation of their group, then some members of that group may choose to form a group of their own. Consequently, players may reach a situation in which they are partitioned into mutually exclusive groups, each of which agrees on a strict core allocation of their group. If any of these groups consists of an unpartnered player $j$, then that player would prefer to receive any strictly individually rational amount. In particular, $j$ is willing to include any existing group $S$ in her partner request, and is willing to accept strictly less than her marginal contribution to that group.\footnote{Accepting less than one’s marginal contribution to a group corresponds to a notion of exploitation discussed in Howe & Roemer (1981).} If $S$ includes $j$ into its partner request, then a strict core allocation of the enlarged group can be created. With only nonsingleton groups remaining, each agreeing on a strict core allocation for themselves, one group $S$ may include another group $S'$ in its partner request. If the members of $S'$ best respond by including $S$ and requesting the marginal contribution of their group to $S \cup S'$, then under the resulting demand profile, members of $S$ would not be able to form demand-feasible groups with members of $S'$. Members of $S$ could then best respond by again excluding $S'$. With the excluded group $S'$ requesting an infeasible allocation, one member best responds by lowering her resource request. If these events occur repeatedly, eventually it will no longer be strictly individually rational for that member of $S'$ to continue lowering her request, and she will leave the group. As an unpartnered player she can join $S$, creating a strict core allocation of the enlarged group. This process can repeat itself, with those leaving other groups joining $S$ until it becomes the grand coalition. On the surface, these events take the appearance of a “divide and conquer” process, although the players involved are myopic.

To formalize this argument, let $d|_S$ and $v|_S$ denote the restrictions of the allocation and characteristic function, respectively, to $S \subseteq N$. The resource allocation $d|_S$ is said to be in the strict core of $S$ if $d|_S \in \text{Core}^*(v|_S, S)$. We begin by stating three lemmas which are proved in the appendix. The first two lemmas describe transitions of the dynamic process from one state to another, while the third studies the demand feasibility of certain requests. 
**Lemma 1** (Factionization). Suppose the current state \((d, P)\) is such that \(d|_S \not\in \text{Core}^*(v|_S, S)\) for some \(S \subseteq N\), and for all \(j \not\in S\), \(P_j \cap S = \emptyset\). Then there is positive probability that the game moves to a state \((d', P')\) in which either (1) \(d'|_S \in \text{Core}^*(v|_S, S)\) or (2) for some \(T \subset S\), \(P'_i = T\) for all \(i \in T\).

In words, Lemma 1 says that if members of a group \(S\) are excluded by others and their demands fall outside the strict core of \(S\), then the process can transition to a state in which either their demands do belong to the strict core of \(S\), or the group \(S\) breaks up. Like Theorem 1, the proof of Lemma 1 studies when players have disincentives to exclude others.

**Lemma 2** (Enlarging a strictly self-enforcing agreement). Suppose the current state \((d, P)\) is such that for some \(S \subseteq N\), \(d|_S \in \text{Core}^*(v|_S, S)\), \(P_i = S\) for every \(i \in S\), and \(P_i \cap S = \emptyset\) for any \(i \not\in S\). If there exists \(j \in N\) such that \(j \not\in P_i\) for all \(i \in N \setminus \{j\}\), then for large enough \(K\), there is positive probability the game reaches a state \((\tilde{d}, \tilde{P})\) with \(\tilde{d}|_{S \cup \{j\}} \in \text{Core}^*(v|_{S \cup \{j\}}, S \cup \{j\})\) and \(\tilde{P}_i = S \cup \{j\}\) for all \(i \in S \cup \{j\}\); and for \(i \not\in S \cup \{j\}\), \(\tilde{d}_i = d_i\) and \(\tilde{P}_i = P_i\).

Lemma 2 thus provides conditions under which the dynamic process can transition from a state in which a player \(j\) is unpartnered, to an analogous state in which she is partnered with an existing group \(S\), and the respective resource requests are in the strict core of \(S \cup \{j\}\).

**Lemma 3** (Disposability). Suppose that the current state \((d, P)\) is such that there exist two disjoint groups \(S\) and \(S'\) with \(d|_S \in \text{Core}^*(v|_S, S)\) and \(d|_{S'} \in \text{Core}^*(v|_{S'}, S')\), and \(d\) is a strictly individually rational request vector. Let \(\tilde{d}\) be the resource request vector where \(\tilde{d}_i = d_i\) for every \(i \not\in S'\), and \(\tilde{d}_i = d_i + v(S \cup S') - v(S) - v(S')\) for all \(i \in S'\). Then given \(\tilde{d}\), there does not exist a demand feasible group \(\tilde{S} \subseteq S \cup S'\) such that \(\tilde{S} \cap S' \not= \emptyset\).

This final lemma considers two groups, \(S\) and \(S'\), whose respective demands belong to the strict cores of their group. If the demands of the members of one group \(S'\) are increased by its marginal contribution to \(S \cup S'\), then no subset of \(S \cup S'\) which includes members of \(S'\) can be demand feasible.

We are now ready to present our main result.

**Theorem 2** (Convergence). For sufficiently large \(K\), the bargaining game converges with probability one to a state with \(d \in \text{Core}^*(v, N)\) and \(P_i = N\) for all \(i \in N\).

*Proof.* As a preliminary step, suppose that the game is at an initial state \((d^0, P^0)\) with \(d^0 \not\in \text{Core}^*(v)\) and there exists a group \(S\) such that \(P_i \cap S = \emptyset\) for all \(i \not\in S\), and \(P_i \subseteq S\)
for all \( i \in S \) (it may be that \( S = N \)). Iterated application of Lemma 1 implies that from any initial state, the game can transition within finite time to a state in which the coalition structure is composed of mutually exclusive groups, each agreeing on a strict core allocation of their group. That is, there is positive probability that the game reaches a state \( (d, P) \) under which the resulting coalition structure \( \pi \) is such that \( P_i = S' \) for all \( S' \in \pi \) and all \( i \in S' \); and \( d|_{S'} \in \text{Core}^*(v|_{S'}, S') \) for every nonsingleton \( S' \in \pi \).

If this coalition structure \( \pi \) is \( \{(1), (2), \ldots, (n)\} \), then the game can transition to a strict core allocation within one period. This is because the players are indifferent among all resource requests. Suppose then that \( \pi \) is a nontrivial coalition structure. By Lemma 2, we know that if some player \( j \) is unpartnered in \( \pi \), then player \( j \) can join a nonsingleton group \( S \in \pi \) - and create an allocation for \( S \cup \{j\} \) which is in the strict core of that group - by requesting strictly less than her marginal contribution \( v(S \cup \{j\}) - v(\{j\}) \). Hence we may assume that all groups in \( \pi \) contains at least two players.

Suppose there exist two distinct groups \( S, S' \in \pi \) (otherwise the conclusion is immediate). With positive probability, each \( i \in S \) may set \( P_i = S \cup S' \), without changing her resource request. A best response for each \( j \in S' \) is to set \( P_j = S \cup S' \) as well; assume all others are inert. Suppose that in the next period, each \( j \in S' \) best responds with the requests \( (d_j + v(S \cup S') - v(S) - v(S'), S \cup S') \), optimally requesting all the remaining surplus, while the players in \( S \) are inert. By Lemma 3, players in \( S \) can no longer form any demand-feasible groups that contain players in \( S' \). Hence each player \( i \in S \) may best respond with \( (d_i, S) \), excluding \( S' \) from their partner request, while the members of \( S' \) are inert.

Since \( S' \) was at a strict core allocation under \( d \), its members are unable to form any feasible coalitions with each other. In fact, if there is any player \( k \in S' \) who is unable to obtain a payoff bigger than \( v(k) \) by lowering her request, she is indifferent between setting \( P_k = S' \) or \( P_k = \{k\} \). If she does the latter then in the next period, \( k \) is unpartnered and may join \( S \) à la Lemma 2.

If a strictly individually rational payoff for each of the players in \( S' \) would be possible by lowering some request, then there is positive probability that at least one of the members of \( S' \) will do so. Fix a \( j \in S' \) and suppose she is the only player in \( S' \) to lower her request in the next period. If \( j \) can obtain her best-response payoff by creating a coalition with just a subgroup of \( S' \), then \( j \) is willing to set \( P_j = S'' \), where \( S'' \subset S' \) is the smallest subgroup of \( S' \) with which \( j \) may obtain her best payoff. Note that the resulting allocation for \( S'' \) would be in \( \text{Core}^*(v|_{S''}, S'') \). In the next period, all \( i \in S'' \) can set \( P_i = S'' \). The remaining group
If $j$ can only obtain a payoff larger than $v(j)$ by creating a coalition with the entire group $S'$, then the resulting allocation will be in Core$^*(v|_{S'}, S')$. But now suppose the process repeats itself with the same player $j$: $S$ includes $S'$, the members of $S'$ respond by requesting all of the surplus, $S$ removes $S'$ from its partner request, and $j$ is the only player in $S'$ to lower her request. This need only be repeated a finite number of times before $j$’s best response includes setting $P_j = \{j\}$, at which point she may join $S$ à la Lemma 2. Furthermore, $S$ can repeat this process against other groups until it grows to become the grand coalition and a strict core allocation is reached. Because it corresponds to a strict Nash equilibrium by Theorem 1, this is an absorbing state of the dynamic process.

The sequence of events described above to prove Theorem 2 requires very little “rationality” on the part of the players. Indeed, it is the combination of inertia and myopic best response to the current state that leads to the repeated break up and reformation of coalitions, which in turn, leads to a situation from which a strictly self-enforcing agreement can be reached. While there may be different possible ways to converge to a strict core allocation, the proof of Theorem 2 sheds light on a plausible channel. The result holds for the best response dynamic with inertia, which is a commonly applied learning process. Whether a similar path of convergence (if any exists) can be demonstrated for other learning processes, which would then model other forms of imperfect rationality, is left for future research.

5 Conclusion

In this paper, we have shown that strictly self-enforcing allocations arise with probability one when short-sighted players repeatedly interact in a multilateral demand game. We allow a player’s strategy to include a request for both coalition partners and an amount of resource. The strict core then coincides with demand requests associated with strict Nash equilibria of the multilateral demand game. We have shown how this follows from an intuitive argument that capitalizes on the possibility of excluding players from one’s group. Moreover, in a learning process given by the best response dynamic with inertia, the possibility of strategizing over partners sheds light on a process of convergence to strict core outcomes. We have highlighted that while shortsighted actions may lead to the repeated break up of groups in the short run, in the long run they can bring about a situation from which the grand coalition and a strictly self-enforcing outcome emerge.
Appendix

Proof of Theorem 1. That a strict core allocation and $P_i = N$ must be a strict Nash equilibrium outcome is clear: any deviation surely yields a player strictly less of the resource.

To understand why being at a strict core allocation is necessary, it is helpful to make the following observations. In any strict Nash equilibrium, it must be that $d_i > v(i)$ for all $i \in N$, that $P_i = N$ for all $i \in N$, and that the grand coalition is not strictly demand-feasible. Next, notice that switching from $P_i$ to any $P_i'$ with $P_i \subset P_i'$ does at least as well: either the set of coarsest feasible coalition structures in which the greatest number of individuals receive an individually rational amount of the resource is unchanged, or some coalition structures in which $i$ and some individuals in $P_i' \setminus P_i$ were in different groups are now replaced with analogous coalition structures in which those groups are merged. In either case, there is a weak increase in the probability that $i$ will receive his resource request. Finally, even though the grand coalition must be mutually compatible, it cannot be strictly feasible because if it were, then players would want to increase their resource requests. Consider a vector of resource requests $d$ and suppose $(d, P)$, with $P_i = N$ for all $i \in N$, is a strict Nash equilibrium.

We now show that given $d$, $N$ cannot contain any demand-feasible subgroup. This would complete the proof, since being at a strict Nash equilibrium would then require that $\sum_{i \in N} d_i = v(N)$ (if the sum is strictly greater than $v(N)$, no individual would receive a strictly individually rational amount of the resource, and if it is strictly smaller, then each individual has a strict incentive to raise their resource request). Let $s$ denote the size of the largest demand-feasible subgroup of $N$ (that is, $s < |N|$) and suppose $S_1$ is a feasible group with $|S_1| = s$. To ensure that a player $i \in S_1$ strictly prefers to include a player $j \not\in S_1$ in her partner request (i.e. $j \in P_i$ is strictly optimal), it must be possible for $i$ to remain alone under the coalition selection rule if she excludes $j$. This means $j$ has a feasible coalition $S_2$ of size $s$ which does not contain $i$. To see this, first note if $S_1$ occurs with positive probability, then it can be the only nonsingleton coalition in the corresponding coalition structure, otherwise strict superadditivity would imply a larger strictly demand-feasible coalition exists. Recall that $P_k = N$ for all $k \neq i$, so that if $i$ is unpartnered in another coalition structure in which $S_2$ occurs, superadditivity again implies there are no other nonsingleton groups. If $|S_2| < |S_1|$, then the coalition structure in which $S_2$ forms and $i$ remains unpartnered would have fewer individuals receiving a strictly individual rational payoff and so it would not form. The same holds for players in this next feasible group $S_2$, and so on and so forth. Let $\{S_k\}_{1 \leq k \leq \hat{N}}$
denote the collection of all the demand-feasible groups of size $s$, given $d$. This collection satisfies: (1) no player can be in every largest feasible group (i.e., $\bigcap_{k \in \{1,\ldots,N\}} S_k = \emptyset$) and (2) if $S_k \neq S_{k'}$, then $S_k \cap S_{k'}$ is a feasible group. Property (1) follows from no player being indifferent to excluding anyone in a strict Nash equilibrium. Property (2) follows from the definition of convexity, else $S_k \cup S_{k'}$ would be strictly feasible and have size greater than $s$. By property (2), $S_1 \cap S_2$ must be a feasible group, else the group $S_1 \cup S_2$ would be strictly feasible and of size larger than $s$. Inductively, for every $k \leq \hat{N}$, $\bigcap_{j=1}^k S_j$ must be a feasible group. If $\bigcap_{j=1}^{k-1} S_j \subset S_k$ then $\bigcap_{j=1}^k S_j$ is feasible because $\bigcap_{j=1}^{k-1} S_j$ is feasible. If $\bigcap_{j=1}^{k-1} S_j \not\subset S_k$ but $\bigcap_{j=1}^k S_j$ was infeasible, then $S_k \cup (\bigcap_{j=1}^{k-1} S_j)$ would be strictly feasible and of size larger than $s$, a contradiction. But then $\bigcap_{j=1}^{N-1} S_j$ must be nonempty, contradicting property (1) and completing the proof.

**Proof of Lemma 1.** Throughout, assume without loss that $d_i > v(i)$, $P_i = S$ for all $i \in S$ unless stated otherwise, $\sum_{i \in S} d_i \geq v(S)$, and $S$ and $N \setminus S$ mutually exclude each other.

If $\sum_{j \in S} d_j = v(S)$ and no subgroup is feasible, (1) is satisfied. If $\sum_{j \in S} d_j > v(S)$ and no subgroup is feasible, there is positive probability one $k$ best responds and the rest remain inert. Either (a) this results in a strict core allocation for $S$, satisfying (1); (b) under the resulting allocation some $j \in S$ receives $v(j)$ with probability one, so $P_j = \{j\}$ is a best response for $j$ and next period, $P_i = S \setminus \{j\}$ becomes a best response for $i \in S \setminus \{j\}$, satisfying (2); or (c) each $k \in S$ receives $d_j > v(j)$ with positive probability, $\sum_{j \in S} d_j \geq v(S)$, and some $T \subset S$ is feasible. Consider case (c). For any $d$, define $s_d = \max\{|T \subset S : \sum_{j \in T} d_j \leq v(T)\} | T |$ and $T_d = \{ T \subset S \mid \sum_{i \in T} d_i \leq v(T) \text{ and } |T| = s_d \}$.

**Case (c-i).** There is $T \in T_d$ such that for all $i \in T$, $d_i$ is a best response to $(d, P)$. If a state satisfying (2) cannot be reached, $(d_j, S)$ must be strictly preferred to $(d_j, T)$ for all $j \in T$, so for each $j \in T$ there is a feasible group of size $s_d$ containing some members in $T \setminus \{j\}$. No player in $T$ can be in every such feasible group of size $s_d$, and their intersection must be feasible, else a bigger group is feasible. A contradiction arises as in the proof of Theorem 1.

**Case (c-ii).** For all $T \in T_d$, there is $i \in T$ such that $d_i$ is not a best response to $(d, P)$. For each $(d, P)$ we may partition the members of $S$ into the following three groups:

$$
T_{(d,P)} = \{ i \in S \mid d_i \text{ is a best response to } (d, P) \},
$$

$$
T_{(d,P)}^+ = \{ i \in S \setminus T_{(d,P)} \mid \text{there is a best response } d_i^* \text{ to } (d, P) \text{ with } d_i^* > d_i \}, \text{ and}
$$

$$
T_{(d,P)}^- = \{ i \in S \setminus (T_{(d,P)} \cup T_{(d,P)}^+) \mid \text{there is a best response } d_i^* \text{ to } (d, P) \text{ with } d_i^* < d_i \}.
$$
Beginning at state \((d, P)\), let all players in \(T_{(d, P)} \cup T_{(d, P)}^-\) be inert and let all players in \(T_{(d, P)}^+\) raise their requests. Call the resulting state \((d', P')\). If \(T_{(d', P')}^+ = \emptyset\), stop; otherwise this can be repeated a finite number of times until \(T_{(d, P)}^+ = \emptyset\) in the resulting state \((\tilde{d}, \tilde{P})\).

Suppose a state satisfying (2) cannot be reached. The expected utility of every \(i \in S\) given their best response to \((\tilde{d}, \tilde{P})\) must be strictly greater than \(u(v(i))\), else some \(k \in S\) could best respond by setting \(P_k = \{k\}\). Also, \(T \in \mathcal{T}_{\tilde{d}} \Rightarrow T \not\subseteq T_{(\tilde{d}, \tilde{P})}\), otherwise one returns to case (c-i). Therefore, \(T_{(\tilde{d}, \tilde{P})}^- \neq \emptyset\). Under \((\tilde{d}, \tilde{P})\), if a state satisfying (2) cannot be reached then \(S\) cannot have any feasible subgroups. Indeed, suppose that there is at least one feasible subgroup of \(S\), and let \(s_{\tilde{d}}\) be the size of the largest such subgroup. \(T_{(\tilde{d}, \tilde{P})}^+ = \emptyset\) and \(T \in \mathcal{T}_{\tilde{d}} \Rightarrow T \not\subseteq T_{(\tilde{d}, \tilde{P})}\), so some \(i \in T_{(\tilde{d}, \tilde{P})}^-\) must be both included and excluded from feasible subgroups of \(S\) of size \(s_{\tilde{d}}\). If she were never excluded, lowering her request would not be a best response. No player can be in every feasible subgroup of size \(s_{\tilde{d}}\) (otherwise (2) would hold), and the intersection of any two such subgroups must be a feasible subgroup (because no subgroup of size larger than \(s_{\tilde{d}}\) is feasible and \(S\) is not strictly feasible). As in the previous case, a contradiction arises. Hence, \(S\) lacks feasible subgroups under \((\tilde{d}, \tilde{P})\).

Let \(k \in T_{(\tilde{d}, \tilde{P})}^-\) best respond and let all others be inert. Note the best response of \(k\) has \(d_{k}^* = \max_{T \subseteq S, k \in T} v(T) - \sum_{j \in T \setminus \{k\}} \tilde{d}_j\). If \(T^* \in \arg \max_{T \subseteq S, k \in T} v(T) - \sum_{j \in T \setminus \{k\}} \tilde{d}_j\) for some \(T^* \neq S\), then the lemma holds because \((d_{k}^*, T^*)\) is optimal for \(k\) and \((d_j, T^*)\) is a best response next period for all \(j \in T^* \setminus \{k\}\); else \(S\) forms with no feasible subgroups, and (1) holds. \(\square\)

**Proof of Lemma 2.** Any request is a best response for \(j\); and those in \(S\) are indifferent about inviting players who exclude them. Suppose that in the same period, player \(j\) requests \((v(S \cup \{j\}) - v(S) - 10^{-K}, S \cup \{j\})\) and each \(i \in S\) requests \((d_i, S \cup \{j\})\), with \(\ell \notin S \cup \{j\}\) making the same requests as before. Next period, some \(k \in S\) best responds by requesting \((d_k + 10^{-K}, S \cup \{j\})\) and players in \(N \setminus \{k\}\) have the same requests as last period. It remains to verify that the resulting resource requests, \(\tilde{d}\), satisfy \(\tilde{d}|_{S \cup \{j\}} \in \text{Core}^*(v|_{S \cup \{j\}}, S \cup \{j\})\). Define \(\varepsilon^* = \min_{T,T': T \cap T' = \emptyset} v(T \cup T') - v(T) - v(T')\) and assume \(2^n \cdot 10^{-K} < \varepsilon^*\). In particular, this ensures \(\tilde{d}_j > v(j)\).

Since \(d|_S \in \text{Core}^*(v|_S, S)\), \(\sum_{i \in S} d_i = v(S)\). For any \(S' \subset S\), we must show \(S' \cup \{j\}\) is infeasible given \(\tilde{d}\). If \(k \in S'\), this is trivial by convexity and \(d|_S \in \text{Core}^*(v|_S, S)\):

\[
\sum_{i \in S' \cup \{j\}} \tilde{d}_i = \sum_{i \in S'} d_i + v(S \cup \{j\}) - v(S) > v(S') + v(S \cup \{j\}) - v(S) \geq v(S') + v(S' \cup \{j\}) - v(S').
\]
Since $d|_S \in \text{Core}^*(v|_S, S)$, for $S' \subset S$, $\sum_{i \in S'} d_i \geq v(S') + 10^{-K}$ (the RHS is the smallest possible group request bigger than $v(S')$). If $\sum_{i \in S'} d_i > v(S') + 10^{-K}$ and $k \notin S'$, convexity and $\tilde{d}_i = d_i$ for $i \in S'$ ensure $v(S' \cup \{j\}) < \tilde{d}_j + \sum_{i \in S'} d_i$, since

$$\tilde{d}_j + \sum_{i \in S'} d_i > v(S \cup \{j\}) - v(S) - 10^{-K} + v(S') + 10^{-K} \geq v(S' \cup \{j\}) - v(S') + v(S') = v(S' \cup \{j\}).$$

Let $\hat{S} = \{S' \subset S, S' \neq \emptyset \mid \sum_{i \in S'} d_i \leq v(S') + 10^{-K}\}$. If $\hat{S} = \emptyset$, the proof is complete by the arguments above. Otherwise, we show that $\cap_{S' \in \hat{S}} S' \neq \emptyset$ if $2^n \cdot 10^{-K} < \epsilon^*$. The proof is then completed by taking the best-responding player $k \in \cap_{S' \in \hat{S}} S'$.

Let $T_1 := S_1 \cap S_2 \neq \emptyset$, otherwise we would have $\sum_{i \in T_1 \cup S_3} d_i = v(T_1) + v(S_3) + 3 \cdot 10^{-K} < v(T_1 \cup S_3)$. But then, since $S_1' \cup S_2'$ cannot be strictly feasible,

$$\sum_{i \in T_1} d_i = \sum_{i \in S_1'} d_i + \sum_{i \in S_2'} d_i - \sum_{i \in S_1' \cup S_2'} d_i \leq v(S_1') + v(S_2') + 2 \cdot 10^{-K} - v(S_1' \cup S_2') \leq v(T_1) + 2 \cdot 10^{-K},$$

where the last inequality follows from the definition of convexity. Similarly, note that $T_2 := T_1 \cap S_3' \neq \emptyset$, otherwise we would have $\sum_{i \in T_1 \cup S_3} d_i = v(T_1) + v(S_3') + 3 \cdot 10^{-K} < v(T_1 \cup S_3')$. Again, since $T_1 \cup S_3'$ cannot be strictly feasible, and using the previous bound on $\sum_{i \in T_1} d_i$,

$$\sum_{i \in T_2} d_i = \sum_{i \in T_1} d_i + \sum_{i \in S_3'} d_i - \sum_{i \in T_1 \cup S_3'} d_i \leq v(T_1) + v(S_3') + 3 \cdot 10^{-K} - v(T_1 \cup S_3') \leq v(T_2) + 3 \cdot 10^{-K}.$$

Iterating this procedure over the enumerated elements of $\hat{S}$, and initializing $T_0 := S_1'$, we know that for each $i \in \{1, \ldots, \ell - 1\}$, $T_i := T_{i-1} \cap S_{i+1}'$ cannot be empty (or else $\sum_{i \in T_{i-1} \cup S_{i+1}'} d_i \leq v(T_{i-1}) + v(S_{i+1}') + (i + 1) \cdot 10^{-K}$ means $T_{i-1} \cup S_{i+1}'$ would be strictly feasible) and that $\sum_{i \in T_i} d_i \leq v(T_i) + (i+1) \cdot 10^{-K}$. But then $T_{\ell-1} = T_{\ell-2} \cap S_{\ell}' \neq \emptyset$, as desired. \qed

**Proof of Lemma 3.** Define $\bar{d}$ by $\bar{d}_i = d_i + [v(S \cup S') - v(S) - v(S')] \cdot 1_{i \in S'}$, where $1_X$ is an indicator function. First, we prove the following intermediate result using convexity: *take nonempty $A, A', B \subset N$ with $A \cap B = \emptyset$ and $A' \subset A$; and let $d$ be such that $d|_A \in \text{Core}^*(v|_A, A)$. If $A' \cup B$ is a feasible coalition under $d$, then $A \cup B$ is strictly feasible under $d$. To see this, note that by convexity, $v(A \cup B) - v(A) \geq v(A' \cup B) - v(A')$. By assumption, both $\sum_{i \in A'} d_i > v(A')$ and $\sum_{i \in A' \cup B} d_i \leq v(A' \cup B)$. Hence $v(A \cup B) - v(A) > \sum_{i \in B} d_i$. Noting that $\sum_{i \in A} d_i = v(A)$ completes the proof of the claim.*
We claim that for any $\emptyset \neq S'' \subseteq S'$, $\sum_{i \in S''} \tilde{d}_i > v(S \cup S'') - v(S)$. To see this, note that by convexity, strict superadditivity, and $d|_{S'} \in \text{Core}^*(v|_{S'}, S')$,

$$\sum_{i \in S''} d_i + |S''|[v(S \cup S') - v(S) - v(S')] - v(S \cup S'') + v(S) \geq \sum_{i \in S''} d_i + |S''|[v(S \cup S'') - v(S) - v(S'')] - v(S \cup S'') + v(S)$$

$$\geq (|S''| - 1)[v(S \cup S'') - v(S) - v(S'')] .$$

If $S''$ is non-singleton the last term is strictly positive, and if $S''$ is singleton the intermediate term is strictly positive by strict individual rationality of the request. The lemma then follows from the contrapositive of the intermediate claim.

**References**


