For linear city, no subway, the bid rent is:

$$\max_{x, y} \quad W - t x - r$$
$$\quad (W - t x - r)^{\frac{1}{b}} \geq \bar{u}$$

$$\Rightarrow \quad W - t x - r = \bar{u}^{\frac{1}{b}}$$
$$\Rightarrow \quad R_0(x) = W - t x - \bar{u}^{\frac{1}{b}}$$

For linear city with subways, commute costs look like this:

![Graph showing cost comparison between subway and car]

The bid rent for subway commuters is:

$$R_s(x) = \begin{cases} W - \bar{u}^{\frac{1}{b}} & x = 1 \\ 0 & x > 1 \end{cases}$$

And for drivers:

$$R_d(x) = \begin{cases} W - \bar{u}^{\frac{1}{b}} - \frac{t}{2} x & x \leq 1 \\ \left( W - \bar{u}^{\frac{1}{b}} - \frac{t}{2} - t(x-1) \right) & x > 1 \end{cases}$$

The upper envelope, \( \max \{ R_s, R_d \} \), is:

$$R(x) = \begin{cases} W - \bar{u}^{\frac{1}{b}} & x = 1 \\ \left( W - \bar{u}^{\frac{1}{b}} - \frac{t}{2} - t(x-1) \right) & x > 1 \end{cases}$$
Note that $R_1$ is discontinuous at 1.

Equilibrium city size w/o subways satisfies

$$R_0(x_0) = 0 \implies 0 = \omega - \bar{x}_0 - \bar{u}b$$

$$\implies \bar{x}_0 = \frac{\omega - \bar{u}b}{t}$$

Equilibrium city size with subways is determined by marginal driver

$$R_1(x_1) = 0 \implies \omega - \bar{u}b - \frac{t}{2} - t(x_1 - \bar{x}) = 0$$

$$\implies \frac{\omega - \bar{u}b - \frac{t}{2}}{t} + 1 = \bar{x}_1$$

$$= \frac{\omega - \bar{u}b + \frac{t}{2}}{t} = \bar{x}_1 = \bar{x}_0 + \frac{t}{2}$$

So $\bar{x}_1 = \bar{x}_0 + \frac{t}{2}$, so subways increase city size a little bit.

---

\[ \begin{array}{c}
\omega \\
\bar{u}b \\
R_0 \\
-1 \quad 0 \quad 1 \quad \bar{x}_0 \quad \bar{x}_1
\end{array} \]
(3) To make this easy, assume one way commuting.

When everyone drives, the number of cars past $x$ is

$$D_0(x) = \begin{cases} \overline{x}_0 - x & x \in [0, \overline{x}_0] \\ -[\overline{x}_0 - x] & x \in [0, -\overline{x}_0] \\ 0 & \text{else} \end{cases}$$

With subways, none of the commuters in $x \in [-1, 1]$ drive, so we have

$$D_1(x) = \begin{cases} \overline{x}_1 - 1 & x \in [-1, 1] \\ (\overline{x}_1 - 1) - x & x \geq 1 \\ -(\overline{x}_1 + 1 - x) & x \leq -1 \\ 0 & \text{else} \end{cases}$$

OR

$$D_0(x) = D_1(x) \quad \text{when} \quad \overline{x}_0 - x = \overline{x}_1 - 1$$

$$\Rightarrow \overline{x}_0 - x = \overline{x}_0 + \frac{1}{2} - 1$$

$$\Rightarrow \quad x = \frac{1}{2}$$
Without subway, total driving in central city i.e. subway catchment is

\[ Z \int_{0}^{\frac{x_0}{2}} D_0(x) \, dx = Z \int_{0}^{\frac{x_0}{2}} x - x \, dx = Z \left[ \frac{x_0}{2} x - \frac{x^2}{4} \right]_{0}^{\frac{x_0}{2}} = Z \left[ \frac{x_0}{2} - \frac{x_0^2}{2} \right] \]

Total driving in the whole city is

\[ Z \int_{0}^{x_0} D_0(x) \, dx = Z \int_{0}^{x_0} x \, dx = \frac{x_0^2}{2} \]

With subway, total driving in the central city is,

\[ Z \int_{0}^{\frac{x_0}{2}} D_0(x) \, dx = Z \left[ \frac{x_0}{2} - \frac{x_0^2}{2} \right] \]

In the whole city:

\[ Z \left[ \frac{x_0}{2} - \frac{x_0^2}{2} \right] + \int_{x_0}^{x_0 + \frac{x_0}{2}} (x_0 - \frac{x_0}{2} - x) \, dx = \left( \text{Integral of } x_0 - \frac{x_0}{2} - x \text{ from } x_0 \text{ to } x_0 + \frac{x_0}{2} \right) \]

\[ = Z \left[ \frac{x_0}{2} \frac{x^2}{2} - \frac{x_0^2}{4} \right] = \left( x_0 + \frac{x_0}{2} \right)^2 - \left( \frac{x_0}{2} \right)^2 \]

So, as long as \( x_0 \) is big enough, total driving increases with subway, but driving in the central part decreases. As long as pollution is local, this rationalizes Duranton and Turner, AER 2011, and Chen and Whalley, AER 2012.
\[ \begin{align*}
\text{Maximize} & \quad h^x z^{1-x} \\
\text{subject to} & \quad ph + z = \omega - 2x \\
\Rightarrow & \quad z(p) = (1-x)(\omega - 2x) \\
& \quad h(p) = \frac{x}{p} (\omega - 2x) \\
\text{with free mobility} & \quad \begin{bmatrix} h(p) \end{bmatrix}^{1-x} \begin{bmatrix} z(p) \end{bmatrix} = \mu \\
\Rightarrow & \quad \mu(x) = \left[ \frac{(\omega - 2x)x^\alpha (1-x)^{1-x}}{\mu} \right]^{\frac{1}{\mu}}
\end{align*} \]

\[ \begin{align*}
\text{[b] SER DURAMIN + PUGA HANDBOOK P8}
\end{align*} \]

\[ \begin{align*}
\frac{d}{dx} E(p, \mu) = \frac{de}{dp} \cdot \frac{dp}{dx} = -2 \\
\text{since} \quad \frac{d}{dx} (\omega - 2x) = -2 \\
\text{but} \quad \frac{de}{dp} = h \quad \Rightarrow \quad \frac{dp}{dx} = \frac{-2}{h}
\end{align*} \]