Chapter 5: The Schumpeterian Model*

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5.1 Introduction

This chapter develops an alternative model of endogenous growth, in which growth is generated by a random sequence of quality-improving (or "vertical") innovations. The model grew out of modern industrial organization theory[1], which portrays innovation as an important dimension of industrial competition. This model is *Schumpeterian* in that: (*i*) it is about growth generated by innovations; (*ii*) innovations result from entrepreneurial investments that are themselves motivated by the prospects of monopoly rents; and (*iii*) new innovations replace old technologies: in other words, growth involves *creative destruction*.

Over the past 25 years,[2] Schumpeterian growth theory has developed into an integrated framework for understanding not only the macroeconomic structure of growth but also the many microeconomic issues regarding incentives, policies and organizations that interact with growth: who gains and who loses from innovations, and what the net rents from innovation are; these ultimately depend on characteristics such as property right protection, competition and openness, education, democracy and so forth and to a different extent in countries or sectors at different stages of development. Moreover, the recent years have witnessed a new generation of Schumpeterian growth

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[1] See Tirole (1988).

[2] The approach was initiated in the fall of 1987 at MIT, where Philippe Aghion was a first-year assistant professor and Peter Howitt a visiting professor on sabbatical from the University of Western Ontario. During that year they wrote their "model of growth through creative destruction" (see Section 5.3 below); which was published as Aghion and Howitt (1992). Parallel attempts at developing Schumpeterian growth models include Segerstrom, Anant and Dinopoulos (1990) and Corriveau (1991).

models focusing on firm dynamics and reallocation of resources among incumbents and new entrants.[3] These models are easily estimable using micro firm-level datasets, which also bring the rich set of tools from other empirical fields into macroeconomics and endogenous growth. Subsequent chapters will describe each of these applications in great detail.

This model of growth with vertical innovations has the natural property that new inventions make old technologies or products *obsolete*. This obsolescence (or "creative destruction") feature in turn has both *positive* and *normative* consequences. On the *positive* side it implies a negative relationship between current and future research, which results in the existence of a unique steady-state (or balanced growth) equilibrium. On the *normative* side, although current innovations have positive externalities for future research and development, they also exert a negative externality on incumbent producers. This *business-stealing effect* in turn introduces the possibility that growth be *excessive* under laissez-faire, a possibility that did not arise in the endogenous growth models surveyed in the previous chapter.

In this chapter, we describe the basics of the Schumpeterian framework. In particular, Section 5.2 presents a simple discrete time version of the Schumpeterian growth model. Section 5.3 then presents a continuous time version of the one-sector Schumpeterian model. Section 5.4 then extends the continuous time model to multiple sectors.

5.2 A toy (myopic-agents) version of the Schumpeterian model

In this section we develop a simple version of the Schumpeterian growth model with discrete time and where individuals and firms live for one period. The basic model abstracts from capital accumulation completely.[4] There is a unique "final" good in this economy, Y_t, which is used for consumption C_t, intermediate good production X_t, and R&D R_t. Therefore the resources constraint of this economy is simply

$$Y_t = C_t + X_t + R_t.$$

[3] See Klette and Kortum (2004), Lentz and Mortensen (2008), Akcigit and Kerr (2010), and Acemoglu, Akcigit, Bloom and Kerr (2013)

[4] The implications of introducing human and physical capital accumulation are explored in Chapter XXX.

5.2.1 The production technology

There is a sequence of discrete time periods $t = 1, 2, ...$ Each period there is a fixed number L of individuals, each of whom lives for just that period and is endowed with one unit of labor services which she supplies inelastically. Her utility depends only on her consumption and she is risk-neutral, so she has the single objective of maximizing expected consumption.

People consume only one good, called the "final" good, which is produced by perfectly competitive firms using two inputs - labor and a single intermediate product - according to the Cobb-Douglas production function:

$$Y_t = (A_t L_t)^{1-\alpha} y_t^{\alpha} \tag{5.1}$$

where Y_t is output of the final good in period t, A_t is a parameter that reflects the productivity of the intermediate input that period and y_t is the amount of intermediate product used. The coefficient α lies between zero and one. The economy's entire labor supply L is used in final-good production. As in the neoclassical model, we refer to the product $A_t L$ as the economy's effective labor supply. We normalize the price of the final good to unity without loss of any generality.

The intermediate product is produced by a monopolist each period, using the final good as an input, one for one. Let us denote the amount of final good used for intermediate-good production by X_t. Then the production function is simply

$$y_t = X_t. \tag{5.2}$$

That is, for each unit of intermediate product, the monopolist must use one unit of final good as input.

5.2.2 Innovation

Growth results from innovations that raise the productivity parameter A_t by improving the quality of the intermediate product. Each period there is one person (the "entrepreneur") who has an opportunity to attempt an innovation. If she succeeds, the innovation will create a new version of the intermediate product, which is more productive than previous versions. Let us denote last period's productivity as A_{t-1}. Specifically, the productivity of the intermediate good in use will go from last period's value A_{t-1} up to $A_t = \gamma A_{t-1}$, where $\gamma > 1$. On the other hand, if she fails then there will be no innovation at t and, in this case, another randomly chosen monopolist will produce the intermediate good with the old productivity that was used in $t-1$, so $A_t = A_{t-1}$. Hence

$$A_t = \begin{cases} \gamma A_{t-1} \text{ if entrepreneur is successful,} \\ A_{t-1} \text{ if entrepreneur fails.} \end{cases} \tag{5.3}$$

In order to innovate, the entrepreneur must conduct research, a costly activity that uses the final good as its only input. As indicated above, research is uncertain, for it may fail to generate any innovation. But the more the entrepreneur spends on research the more likely she is to innovate. Specifically, we assume to innovate from A_{t-1} up to $A_t = \gamma A_{t-1}$ with probability z_t, one must spend the amount

$$R_t = c(z_t)A_{t-1} \tag{5.4}$$

final good on research.

For simplicity, assume a quadratic R&D cost:

$$c(z_t) = \delta z_t^2/2, \tag{5.5}$$

where δ is a parameter which inversely measures the productivity of the research sector.

Timing of events Now we can summarize the timing of events in this model:

• **step 0** Period t begins with the initial productivity A_{t-1} which is inherited from the previous period (cohort),

• **step 1** a randomly chosen entrepreneur invests in R&D by chosing (z_t, R_t),

• **step 2** innovation (success/failure) is realized, and productivity evolves according to (5.3),

• **step 3** production of the intermediate good (y_t) takes place,

• **step 4** production of the final good (Y_t) takes place,

• **step 5** consumption (C_t) takes place, and period t ends.

5.2.3 Solving the model

The model is solved by backward induction: in each period t, we first compute the equilibrium production and profit of a successful innovator; then, we move back one step and compute the optimal innovation intensity by the firm selected to be an innovator.

5.2.3.1 Equilibrium production and profits

We start from step 4. The final good producer maximizes the following objective function

$$\max_{y_t,\ L_t} \left\{ (A_t L)^{1-\alpha} y_t^{\alpha} - w_t L_t - p_t y_t \right\}.$$

Therefore we can express the inverse demand for intermediate good y_t and labor as

$$p_t = \partial Y_t / \partial y_t = \alpha \left(A_t L \right)^{1-\alpha} y_t^{\alpha-1}, \tag{5.6}$$

and

$$w_t = (1-\alpha) A_t^{1-\alpha} L^{-\alpha} y_t^{\alpha},$$

where we already imposed that $L_t = L$.

Now we move to step 3. The monopolist with productivity A_t, taking the demand (5.6) as given, maximizes her expected profit $\Pi(A_t)$, measured in units of the final good:

$$\Pi(A_t) = \max_{y_t, p_t} \{ p_t y_t - y_t \} \text{ subject to } (5.6)$$

where p_t is the price of the intermediate product relative to the final good. That is, her revenue is price times quantity $p_t y_t$ and her cost is her input of final good, which must equal her output y_t. Substituting the constraint (5.6) into the objective function we get

$$\Pi(A_t) = \max_{y_t} \left\{ \alpha \left(A_t L_t \right)^{1-\alpha} y_t^{\alpha} - y_t \right\}, \tag{5.7}$$

which implies an equilibrium quantity:[5]

$$y_t = \alpha^{\frac{2}{1-\alpha}} A_t L, \tag{5.8}$$

and equilibrium price

$$p_t = p = \frac{1}{\alpha}.$$

Note that the equilibrium price is above the marginal cost (which is equal to one) since $\alpha \in (0, 1)$. The additional margin on top of the marginal cost is called the price markup which is governed by the inverse of α. As $\alpha \to 1$, the markup vanishes the price becomes equal to the marginal cost.

Finally, the equilibrium profit is:

$$\Pi(A_t) = \pi A_t L, \text{ where } \pi \equiv (1-\alpha)\,\alpha^{\frac{1+\alpha}{1-\alpha}} \tag{5.9}$$

that are both proportional to the effective labor supply $A_t L$.

Substituting from (5.8) into the production function (5.1), we see that final output will be proportional to $A_t L$:

$$Y_t = \alpha^{\frac{2\alpha}{1-\alpha}} A_t L \tag{5.10}$$

Therefore the growth rate of this economy will be equal to the growth rate of the productivity A_t.

5.2.3.2 Equilibrium innovation intensity

In step 2, for any given innovation rate z_t,

$$A_t = \begin{cases} \gamma A_{t-1} \text{ with probability } z_t, \\ A_{t-1} \text{ with probability } 1 - z_t. \end{cases}$$

We now move back to step 1 and consider the innovation investment decision of the entrepreneur who has the opportunity to innovate at date t. If the entrepreneur at t successfully innovates, she

[5] The first-order condition for the maximization problem is:

$$\alpha^2 (A_t L)^{1-\alpha} y_t^{\alpha-1} - 1 = 0$$

from which (5.8) follows directly. Substituting from (5.8) into (5.7) yields (5.9).

will become the intermediate monopolist that period, because she will be able to produce a better product than anyone else. Otherwise the monopoly will pass to someone else chosen at random, who is able to produce last period's product. Thus the entrepreneur will choose the innovation intensity z_t to $\max_{z_t}\{z_t\Pi(\gamma A_{t-1}) - c(z_t)A_{t-1}\}$ or equivalently to

$$\max_{z_t}\{z_t\pi\gamma L - c(z_t)\}.$$

where we used (5.9). The first order condition is simply:

$$c'(z_t) = \pi\gamma L,$$

which, once combined with (5.5) yields the equilibrium innovation intensity

$$z_t = z = \pi\gamma L/\delta. \tag{5.11}$$

The following assumption ensures that the innovation rate z is between zero and one.

Assumption 1 *The parameters of the model satisfies the following condition*

$$(1-\alpha)\,\alpha^{\frac{1+\alpha}{1-\alpha}}\gamma L < \delta.$$

5.2.3.3 Growth

The rate of economic growth is the proportional growth rate of final good (Y_t/L), which according to equation (5.10) is also the proportional growth rate of the productivity parameter A_t:

$$g_t = \frac{A_t - A_{t-1}}{A_{t-1}}$$

It follows that growth will be random. Each period, with probability z the entrepreneur will innovate, resulting in $g_t = \frac{\gamma A_{t-1} - A_{t-1}}{A_{t-1}} = \gamma - 1$; and with probability $1-z$ she will fail, resulting in

$g_t = \frac{A_{t-1} - A_{t-1}}{A_{t-1}} = 0$. The growth rate will be governed by this probability distribution every period:

$$g = \mathbb{E}(g_t) = z \cdot (\gamma - 1)$$

will also be the economy's long-run average growth rate.

To interpret this formula, note that z is not just the probability of an innovation each period but also the long-run frequency of innovations; that is, the fraction of periods in which an innovation will occur. Also, $\gamma - 1$ is the proportional increase in productivity resulting from each innovation. Thus the formula expresses a simple but important result of Schumpeterian growth theory:

Proposition 1 *In the long run, the economy's average growth rate equals the frequency of innovations times the size of innovations.*

Using (5.11) to replace z in the above formula, we see that the average growth rate is

$$g = \frac{\pi \gamma L}{\delta} (\gamma - 1) \tag{G}$$

where $\pi \equiv (1 - \alpha) \alpha^{\frac{1+\alpha}{1-\alpha}}$.

5.2.3.4 Comparative statics

According to the growth equation (G), our analysis yields the following comparative statics implications for the average growth rate g:

1. Growth increases with the productivity of innovations inversely measured by δ. This result points to the importance of education, and particularly higher education, as a growth-enhancing device. Countries that invest more in higher education will achieve a higher productivity of research, and will also reduce the opportunity cost of research by increasing the aggregate supply of skilled labor.

2. Growth increases with the size of innovations, as measured by the productivity improvement factor γ. This follows directly from Proposition 1 above, together with the result (5.11) which shows that the frequency of innovation is increasing in γ. The result in turn points to a feature

that will become important when we discuss cross-country convergence. A country that lags behind the world technology frontier has what Gerschenkron (1962) called an advantage of backwardness. That is, the further it lags behind the frontier, the bigger the productivity improvement it will get if it can implement the frontier technology when it innovates, and hence the faster it can grow.

3. An increase in the size of population should also bring about an increase in growth by raising the supply of labor L. This "scale effect" is also present in the product variety model, and has been challenged in the literature. In Appendix A.4.2 below we will see how this questionable comparative statics result can be eliminated by considering a model with both horizontal and vertical innovations.

5.2.3.5 Welfare Analysis

In this simplified framework, we can now ask the following question: What is the socially optimal level of production and R&D investment in this economy? To answer this question, we first need to take a stand on the objective function of the social planner. To be inline with this section, we consider a myopic social planner that maximizes the period-by-period utility which is equivalent to maximizing the per-period consumption.

We again follow a backward induction argument. Recall that the level of consumption from the resource constraint is simply

$$C_t = Y_t - X_t - R_t.$$

In step 4, therefore, the social planner maximizes consumption subject to final good and intermediate good production technologies (5.1) and (5.2). Note that in step 4, the R&D investment is already made, hence R_t is taken as constant at this stage. Let us denote the household's maximum consumption, for any given producitivity A_t and the sunk R&D investment R_t, by $\tilde{C}(A_t, R_t)$. Then the planner's maximization can be rewritten as

$$\tilde{C}(A_t, R_t) \equiv \max_{y_t} \left\{ (A_t L)^{1-\alpha} y_t^{\alpha} - y_t - R_t \right\}.$$

From this maximization, the socially optimal level of intermediate good production is

$$y_t^{sp} = A_t L \alpha^{\frac{1}{1-\alpha}} \tag{5.12}$$

and the resulting maximum consumption is

$$\tilde{C}\left(A_t, R_t\right) \equiv A_t L \alpha^{\frac{\alpha}{1-\alpha}} \left(1-\alpha\right) - R_t. \tag{5.13}$$

Now we can combine (5.12) with (5.1) to express the socially optimal GDP as

$$Y_t^{sp} = A_t L \alpha^{\frac{\alpha}{1-\alpha}}. \tag{5.14}$$

Remark 2 *We can already see the first distortion in this economy. The comparison of* (*5.12*) *to* (*5.8*) *and* (*5.14*) *to* (*5.10*) *indicate*

$$y_t^{sp} > y_t \text{ and } Y_t^{sp} > Y_t$$

which implies that the decentralized economy underproduces output due to "monopoly distortions"*.*

Now we go one step back in social planner's problem and specify the maximization problem for the optimal innovation decision as: $C_t^{sp} \equiv \max_{z_t}\{z_t \tilde{C}\left(\gamma A_{t-1}, R_t\right) + \left(1-z_t\right)\tilde{C}\left(A_{t-1}, R_t\right)\}$ subject to the R&D technology (5.4). Substituting the constraint into the objective function we can express the planner's innovation problem as

$$C_t^{sp} \equiv \max_{z_t}\{z_t \gamma A_{t-1} L \alpha^{\frac{\alpha}{1-\alpha}} \left(1-\alpha\right) + \left(1-z_t\right) A_{t-1} L \alpha^{\frac{\alpha}{1-\alpha}} \left(1-\alpha\right) - \frac{\delta z_t^2}{2} A_{t-1}\}. \tag{5.15}$$

Note here the the social planner compares the consumption level upon a successful innovation to consumption in the case of a failure, which implies that the innovation size plays a crucial role for the social planner. However, in the decentralized economy, the entrepreneur cares only about the success since in the case of a failure, the market is served by another firm. Therefore the entrepreneur does not internalize the size of the innovation γ as the social planner does and therefore creates an *innovation externality*.

Now we can find the socially optimal innovation rate by taking the first-order condition of (5.15)

$$z_t^{sp} = (\gamma - 1) \frac{L\alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha)}{\delta} \tag{5.16}$$

Remark 3 *Comparing (5.16) to (5.11), we find underinvestment in R&D ($z_t < z_t^{sp}$) if and only if*

$$\alpha^{\frac{1}{1-\alpha}} < \frac{\gamma - 1}{\gamma}.$$

This result is very intuitive. In this economy, there are two types inefficiencies: (i) Monopoly distortion, and (ii) innovation externality.

The former is governed by the parameter α, which determines the equilibrium markups. The latter is governed by γ as described above. Therefore, if the innovation externalities are above a threshold (high γ), then the economy features underinvestment in R&D and vice versa.

5.2.4 Industrial Policy

In this section, we will consider the role of industrial policy in the simplified Schumpeterian framework. As Remarks 2 and 3 have shown, the decentralized economy is not efficient. This implies that the policymaker can improve the welfare in this economy by using standard policy tools such as production subsidy or R&D subsidy. This is what we are going to study next.

Assume that the government subsidizes production and R&D at the rates τ^P and τ^R, respectively. Now we will find the optimal (welfare maximizing) rates of τ^P and τ^R. For simplicity, we assume that the government finances these subsidies through lump-sum taxes on the household.

Production subsidy Let us rewrite equation (5.7), the production decision of the monopolist, this time with the production subsidy:

$$\Pi^\tau (A_t) = \max_{y_t} \left\{ \alpha (A_t L)^{1-\alpha} y_t^\alpha - \left(1 - \tau^P\right) y_t \right\}.$$

The resulting output is

$$y_t^\tau = \left[\frac{\alpha^2}{1 - \tau^P} \right]^{\frac{1}{1-\alpha}} A_t L. \tag{5.17}$$

Now we can find the optimal subsidy rate by equating (5.17) to (5.12)

$$\tau^P = 1 - \alpha.$$

Our first finding is that the production subsidy is decreasing in α. This should not be surprising since τ^P here corrects for monopoly distortions and recall that the monopoly markups are decreasing in α. Hence a higher α implies a lower distortion and hence a lower subsidy rate τ^P.

Under the optimal production subsidy, the equilibrium profit is

$$\Pi^\tau (A_t) = A_t L \alpha^{\frac{1}{1-\alpha}} (1-\alpha)$$

R&D subsidy Now assume that the government is already imposing the optimal production subsidy $\tau^P = 1 - \alpha$ and also subsidizes R&D at the rate τ^R. Then the innovation decision of the entrepreneur is

$$\max_{z_t} \{ z_t \gamma A_{t-1} L \alpha^{\frac{1}{1-\alpha}} (1-\alpha) - \left(1 - \tau^R\right) \frac{\delta z_t^2}{2} A_{t-1} \}.$$

Then the optimal choice of the entrepreneur is

$$z_t^\tau = \frac{\gamma L \alpha^{\frac{1}{1-\alpha}} (1-\alpha)}{(1-\tau^R)\,\delta}$$

Now equating this innovation rate to the socially optimal rate in (5.16)

$$\tau^R = 1 - \frac{\alpha\gamma}{(\gamma - 1)}$$

Note that

$$\frac{\partial \tau^R}{\partial \gamma} > 0 \text{ and } \frac{\partial \tau^R}{\partial \alpha} < 0.$$

This implies the the optimal R&D subsidy rate is increasing in the size of the innovation γ. This is intuitive because as we saw in Remark 3, one of the main inefficiencies in R&D spending is the uninternalized contribution of each innovation, which is measured by γ.

5.2.5 Multisector extension

In this section we allow for multiple innovating sectors in the economy. Suppose there is not one intermediate product but a continuum, indexed on the interval $[0, 1]$. The final-good production function is now:

$$Y_t = L^{1-\alpha} \int_0^1 A_{it}^{1-\alpha} y_{it}^{\alpha} di, \tag{5.18}$$

where each y_{it} is the flow of intermediate product i used at t, and the productivity parameter A_{it} reflects the quality of that product. In any period the productivity parameters will vary across intermediate products because of the randomness of the innovation process.

According to (5.18), the final output produced by each intermediate product is determined by the production function:

$$Y_{it} = (A_{it} L)^{1-\alpha} y_{it}^{\alpha} \tag{5.19}$$

which is identical to the production function (5.1) of the one-sector model. The final good producer's maximization problem is

$$\max_{L_t, \{y_{it}\}_{i\in[0,1]}} \left\{ L_t^{1-\alpha} \int_0^1 A_{it}^{1-\alpha} y_{it}^{\alpha} di - w_t L_t - \int_0^1 p_{it} y_{it} di \right\}.$$

Each intermediate product has its own monopoly, and its price equals its marginal product in the final sector, which according to (5.19) is

$$p_{it} = \partial Y_{it} / \partial y_{it} = \alpha (A_{it} L)^{1-\alpha} y_{it}^{\alpha-1}. \tag{5.20}$$

Similarly, the profit maximizing labor choice is

$$\begin{aligned} w_t &= \partial Y_{it} / \partial L_t \\ &= (1-\alpha) L^{-\alpha} \int_0^1 A_{it}^{1-\alpha} y_{it}^{\alpha} di \end{aligned}$$

where the second line uses the labor market clearing, $L_t = L$.

Now, the monopolist in sector i takes the demand for its product (5.20) as given and chooses the quantity y_{it} that maximizes her profit:

$$\Pi\left(A_{it}\right) = \max_{y_{it}}\left\{p_{it}y_{it} - y_{it}\right\} = \max_{y_{it}}\left\{\alpha\left(A_{it}L\right)^{1-\alpha}y_{it}^{\alpha} - y_{it}\right\}, \tag{5.21}$$

which implies an equilibrium quantity:[6]

$$y_{it} = \alpha^{\frac{2}{1-\alpha}}A_{it}L \tag{5.22}$$

and an equilibrium profit:

$$\Pi\left(A_{it}\right) = \pi A_{it}L. \tag{5.23}$$

where the parameter π is the same as in the analogous equation (5.9) of the one-sector model.

The aggregate behavior of the economy depends on the aggregate (which also corresponds to average in this case) productivity index:

$$A_t = \int_0^1 A_{it}di$$

which is just the unweighted numerical average of all the individual productivities. In particular, final output and GDP in this multisector economy are determined by exactly the same equations as in the one-sector economy of the previous section, but with A_t now being this average, instead of being the productivity of the economy's only intermediate product.

More specifically, using (5.22) to substitute for each y_{it} in the production function (5.18) yields the same formula as before for final output:

$$Y_t = \alpha^{\frac{2\alpha}{1-\alpha}}A_tL \tag{5.24}$$

[6] The first-order condition for the maximization problem is:

$$\alpha^2\left(A_{it}L\right)^{1-\alpha}y_{it}^{\alpha-1} - 1 = 0$$

from which (5.22) follows directly. Substituting from (5.22) into (5.21) yields (5.23).

5.2.5.1 Innovation and research arbitrage

Innovation in each sector takes place exactly as in the one-sector model. Specifically, there is a single entrepreneur in each sector who spends final output in research and to innovate from A_{it-1} up to $A_{it} = \gamma A_{it-1}$ with probability z_t. In order to achieve this, she needs to spend the amount of final good

$$R_{it} = c(z_t)A_{it-1}$$

in research.

The entrepreneur chooses the innovation intensity z_{it} that maximizes her net expected benefit:

$$\max_{z_t \in [0,1]} \{z_t \Pi(\gamma A_{it-1}) - c(z_t)A_{it-1}\}$$

which yields

$$z_{it} \equiv z = \pi\gamma L/\delta. \tag{5.25}$$

This exactly the same as the research effort in the one-sector model. One important feature of this model is that the probability of innovation z is the same in all sectors, no matter what the starting level of productivity $A_{i,t-1}$. This might seem surprising, because the reward $\Pi(\gamma A_{it-1}) = \pi\gamma A_{i,t-1}L$ to a successful innovation is higher in more advanced sectors. But this advantage is just offset by the fact that the cost of innovating at any given rate is also correspondingly higher because what matters is research expenditure relative to the current productivity level $A_{i,t-1}$. As we will see, this feature allows a simple characterization of the aggregate growth rate in the economy.

5.2.5.2 Growth

Since per-capita GDP is again proportional to the aggregate productivity A_t (see equation (5.24)), therefore the economy's growth rate is again the proportional growth rate of A_t:

$$g_t = \frac{A_t - A_{t-1}}{A_{t-1}} \tag{5.26}$$

In this case, however, the aggregate growth rate is no longer random, because bad luck in some sectors will be offset by good luck in others.

In each sector i we have:

$$A_{it} = \left\{ \begin{array}{ll} \gamma A_{i,t-1} & \text{with probability } z \\ A_{i,t-1} & \text{with probability } 1-z \end{array} \right\} \tag{5.27}$$

Recall that the average productivity is $A_t = \int_0^1 A_{it} di$. We can express next period's average productivity as

$$\begin{aligned} A_t &= \int_0^1 \left[z\gamma A_{it-1} + (1-z) A_{it-1} \right] di = \int_0^1 A_{it-1} di + z(\gamma - 1) \int_0^1 A_{it-1} di \\ &= A_{t-1} + z(\gamma - 1) A_{t-1} \end{aligned}$$

It follows from this and (5.26) that the growth rate each period is equal to the constant

$$g = z \cdot (\gamma - 1)$$

which is the same as the long-run average growth rate of the one-sector model. Substituting (5.25) into this formula produces the same expression (G) as before, implying the same comparative static results as before.

5.3 The Aghion-Howitt model

Creative destruction was somewhat mechanical in the above discrete-time model, in that by assumption firms were assumed to live for only one period and thus to be replaced by newly born entrepreneurs. In this section we allow incumbent firms to remain on the market as long as they have not been replaced by a new innovator. The expected average lifetime of a firm will thus be equal to the inverse of the aggregate rate of innovation. This rate of innovation will in turn be determined by a research arbitrage equation which factors in the rate of creative destruction.

5.3.1 The setup

Time is continuous and the economy is populated by a continuous mass L of infinitely lived individuals with linear preferences, that discount the future at rate ρ.[7] Each individual is endowed with one unit of labor per unit of time, which she can allocate between production and research: in equilibrium, individuals are indifferent between these two activities.

There is a final good, which is also the numeraire. The final good at time t is produced competitively using an intermediate input, namely:

$$Y_t = A_t y_t^{\alpha}$$

where α is between zero and one, y_t is the amount of the intermediate good currently used in the production of the final good, and A_t is the productivity -or quality- of the currently used intermediate input.[8]

The intermediate good y is in turn produced one for one with labor: that is, one unit flow of labor currently used in manufacturing the intermediate input produces one unit of intermediate input of frontier quality. Thus y_t denotes both the current production of the intermediate input and the flow amount of labor currently employed in manufacturing the intermediate good.

Growth in this model results from innovations that improve the quality of the intermediate input used in the production of the final good. More formally, if the previous state-of-the-art intermediate good was of quality A, then a new innovation will introduce a new intermediate input of quality γA, where $\gamma > 1$. This immediately implies that growth will involve creative destruction, in the sense that Bertrand competition will allow the new innovator to drive the firm producing the intermediate good of quality A out of the market, since at the same labor cost the innovator produces a better good than that of the incumbent firm.[9]

[7] The linear preferences (or risk neutrality) assumption implies that the equilibrium interest rate will always be equal to the rate of time preference: $r_t = \rho$ (see Aghion and Howitt (2009), Chapter 2).

[8] In what follows we will use the words "productivity" and "quality" interchangeably.

[9] Thus, overall, growth in the Schumpeterian model involves both positive and negative externalities. The positive externality is referred to by Aghion and Howitt (1992) as a "knowledge spillover effect": namely, any new innovation raises productivity A forever, i.e., the benchmark technology for any subsequent innovation; however, the current (private) innovator captures the rents from her innovation only during the time interval until the next innovation occurs. This effect is also featured in Romer (1990), where it is referred to instead as "non-rivalry plus limited excludability." But in addition, in the Schumpeterian model, any new innovation has a negative externality as it destroys the rents of the previous innovator: following the theoretical IO literature, Aghion and Howitt (1992)

The innovation technology is directly drawn from the theoretical IO and patent race literatures: namely, if z_t units of labor are currently used in R&D, then a new innovation arrives during the current unit of time at the (memoryless) Poisson rate λz_t.[10] Henceforth we will drop the time index t, when it causes no confusion.

5.3.2 Solving the model

5.3.2.1 The research arbitrage and labor market clearing equations

We shall concentrate our attention on balanced growth equilibria where the allocation of labor between production (y) and R&D (z) remains constant over time. The growth process is described by two basic equations.

The first is the *labor market clearing equation*:

$$L = y + z \tag{L}$$

reflecting the fact that the total flow of labor supply during any unit of time is fully absorbed between production and R&D activities (i.e., by the demand for manufacturing and R&D labor).

The second equation reflects individuals' indifference in equilibrium between engaging in R&D or working in the intermediate good sector. We call it the *research arbitrage equation.* The remaining part of the analysis consists of spelling out this research arbitrage equation.

More formally, let w_k denote the current wage rate conditional on there having already been $k \in \mathbb{Z}_{++}$ innovations from time 0 until current time t (since innovation is the only source of change in this model, all other economic variables remain constant during the time interval between two successive innovations). And let V_{k+1} denote the net present value of becoming the next $((k+1)$ -th) innovator.

During a small time interval dt, between the k-th and $(k+1)$ -th innovations, an individual faces the following choice. Either she employs her (flow) unit of labor for the current unit of time

refer to this as the "business-stealing effect" of innovation. The welfare analysis in that paper derives sufficient conditions under which the intertemporal spillover effect dominates or is dominated by the business-stealing effect. The equilibrium growth rate under laissez-faire is correspondingly suboptimal or excessive compared to the socially optimal growth rate.

[10] More generally, if z_t units of labor are invested in R&D during the time interval $[t, t+dt]$, the probability of innovation during this time interval is $\lambda z_t dt$.

in manufacturing at the current wage, in which case she gets $w_t dt$. Or she devotes her flow unit of labor to R&D, in which case she will innovate during the current time period with probability λdt and then get V_{k+1}, whereas she gets nothing if she does not innovate.[11] The research arbitrage equation is then simply expressed as:

$$w_k = \lambda V_{k+1}. \tag{R}$$

The value V_{k+1} is in turn determined by a Bellman equation. We will use Bellman equations repeatedly in this survey; thus, it is worth going slowly here. During a small time interval dt, a firm collects $\pi_{k+1} dt$ profits. At the end of this interval, it is replaced by a new entrant with probability $\lambda z dt$ through creative destruction; otherwise, it preserves the monopoly power and V_{k+1}. Hence the value function is written as

$$V_{k+1} = \pi_{k+1} dt + (1 - r dt) \begin{bmatrix} \lambda z dt \times 0 + \\ (1 - \lambda z dt) \times V_{k+1} \end{bmatrix}$$

Dividing both sides by dt, then taking the limit as $dt \to 0$ and using the fact that the equilibrium interest rate is equal to the time preference, the Bellman equation for V_{k+1} can be rewritten as:

$$\rho V_{k+1} = \pi_{k+1} - \lambda z V_{k+1}.$$

In other words, the annuity value of a new innovation (i.e., its flow value during a unit of time) is equal to the current profit flow π_{k+1} minus the expected capital loss $\lambda z V_{k+1}$ due to creative destruction, i.e., to the possible replacement by a subsequent innovator. If innovating gave the innovator access to a permanent profit flow π_{k+1}, then we know that the value of the corresponding perpetuity would be π_{k+1}/r.[12] However, there is creative destruction at rate λz. As a result, we

[11] Note that we are implicitly assuming that previous innovators are not candidates for being new innovators. This in fact results from a replacement effect pointed out by Arrow (1962). Namely, an outsider goes from zero to V_{k+1} if she innovates, whereas the previous innovator would go from V_k to V_{k+1}. Given that the R&D technology is linear, if outsiders are indifferent betwen innovating and working in manufacturing, then incumbent innovators will strictly prefer to work in manufacturing. Thus new innovations end up being made by outsiders in equilibrium in this model. This feature will be relaxed in the next section.

[12] Indeed, the value of the perpetuity is:

$$\int_0^\infty \pi_{k+1} e^{-rt} dt = \frac{\pi_{k+1}}{r}.$$

have:

$$V_{k+1} = \frac{\pi_{k+1}}{\rho + \lambda z}, \tag{5.28}$$

that is, the value of innovation is equal to the profit flow divided by the risk-adjusted interest rate $\rho + \lambda z$ where the risk is that of being displaced by a new innovator.

5.3.2.2 Equilibrium profits, aggregate R&D and growth

As in the previous section, we solve for equilibrium profits π_{k+1} and the equilibrium R&D rate z by backward induction. That is, first, for a given productivity of the current intermediate input, we solve for the equilibrium profit flow of the current innovator; then we move one step back and determine the equilibrium R&D using equations (L) and (R).

Equilibrium profits Suppose that k_t innovations have already occurred until time t, so that the current productivity of the state-of-the-art intermediate input is $A_{k_t} = \gamma^{k_t}$. Given that the final good production is competitive, the intermediate good monopolist will sell her input at a price equal to its marginal product, namely

$$p_k = \frac{\partial(A_k y^{\alpha})}{\partial y} = A_k \alpha y^{\alpha-1}. \tag{5.29}$$

This is the inverse demand curve faced by the intermediate good monopolist.

Given that inverse demand curve, the monopolist will choose y to

$$\pi_k = \max_y \{p_k y - w_k y\} \text{ subject to (5.29)} \tag{5.30}$$

since it costs $w_k y$ units of the numeraire to produce y units of the intermediate good. Given the Cobb-Douglas technology for the production of the final good, the equilibrium price is a constant markup over the marginal cost ($p_k = w_k/\alpha$) and the profit is simply equal to $\frac{1-\alpha}{\alpha}$ times the wage bill, namely:[13]

$$\pi_k = \frac{1-\alpha}{\alpha} w_k y \tag{5.31}$$

[13] To see that $p_k = w_k/\alpha$, simply combine the first-order condition of (5.30) with expression (5.29).

where y solves (5.30).

Equilibrium aggregate R&D Combining (5.28), (5.31) and (R), we can rewrite the research arbitrage equation as:

$$w_k = \lambda \frac{\frac{1-\alpha}{\alpha} w_{k+1} y}{\rho + \lambda z}. \tag{5.32}$$

Using the labor market clearing condition (L) and the fact that on a balanced growth path all aggregate variables (the final output flow, profits and wages) are multiplied by γ *each time a new innovation occurs*, we can solve (5.32) for the equilibrium aggregate R&D z as a function of the parameters of the economy. Equation (5.33) implies that the steady-state equilibrium level of research (z^*) satisfies the equation:

$$1 = \lambda \frac{\gamma \frac{1-\alpha}{\alpha} (L - z^*)}{\rho + \lambda z^*} \tag{5.33}$$

or

$$z^* = \frac{\frac{1-\alpha}{\alpha} \gamma L - \frac{\rho}{\lambda}}{1 + \frac{1-\alpha}{\alpha} \gamma}. \tag{5.34}$$

Clearly it is sufficient to assume that $\frac{1-\alpha}{\alpha} \gamma L > \frac{\rho}{\lambda}$ to ensure positive R&D in equilibrium. Inspection of (5.34) delivers a number of important comparative statics. In particular, a higher productivity of the R&D technology as measured by λ or a larger size of innovations γ or a larger size of the population L has a positive effect on equilibrium R&D z^*. On the other hand a higher α (which corresponds to the intermediate producer facing a more elastic inverse demand curve and therefore getting lower monopoly rents) or a higher discount rate ρ tends to discourage R&D.

Equilibrium expected growth Once we have determined the equilibrium aggregate R&D, it is easy to compute the expected growth rate. First note that during a small time interval $[t, t + dt]$, there will be a successful innovation with probability $\lambda z^* dt$. Second, the final output is multiplied by γ each time a new innovation occurs. Therefore the expected log-output is simply:

$$\mathbb{E} (\ln Y_{t+dt}) = \lambda z^* dt \ln \gamma Y_t + (1 - \lambda z^* dt) \ln Y_t.$$

Subtracting $\ln Y_t$ from both sides, dividing through dt and finally taking the limit leads to the following expected growth

$$\mathbb{E}(g_t) = \lim_{dt \to 0} \frac{\ln Y_{t+dt} - \ln Y_t}{dt} = \lambda z^* \ln \gamma = \widehat{g}$$

which inherits the comparative static properties of z^* with respect to the parameters $\lambda, \gamma, \alpha, \rho$, and L.

A distinct prediction of the model is that *the turnover rate λz^* is positively correlated with the expected growth rate $\widehat{g}$.*

5.3.3 Welfare analysis

We can compare the equilibrium R&D investment and growth rate under laissez-faire with the R&D investment and growth rate that would be generated by a social planner who maximizes the expected present value of consumption. Since every innovation raises final output Y_t by the same factor γ, the optimal policy consists of a fixed level of research. Expected welfare is expressed as:

$$U = \int_0^\infty e^{-\rho t} Y_t dt = \int_0^\infty e^{-\rho t} (\sum_{k=0}^{\infty} \Pi(k,t) A_k y^\alpha) dt,$$

where $\Pi(k,t)$ is the probability that there will be exactly k innovations up to time t. Given that the innovation process is Poisson with parameter λz, we have

$$\Pi(k,t) = \frac{(\lambda z t)^k}{k!} e^{-\lambda z t}.$$

The details of this derivation is expressed in Appendix A.4.1. The social planner then chooses (y,z) to maximize U subject to the labor resource constraint

$$L = y + z.$$

Using the fact that

$$A_k = \gamma^k A_0,$$

we can reexpress the expected welfare U as

$$U(z) = \frac{A_0(L-z)^\alpha}{\rho - \lambda z(\gamma - 1)}.$$

Then the socially optimal level of research z^{sp} will satisfy the first order condition

$$U'(z^{sp}) = 0,$$

which in turn is equivalent to:

$$1 = \lambda \frac{(\gamma-1)(\frac{1}{\alpha})(L - z^{sp})}{\rho - \lambda z^{sp}(\gamma - 1)}. \tag{5.35}$$

This level of research produces an average growth rate equal to:

$$g^{sp} = \lambda z^{sp} \ln \gamma.$$

Whether the average growth rate under laissez-faire g^* is higher or smaller than the optimal growth rate g^{sp} will depend upon whether the steady-state equilibrium level of research z^* is greater or smaller than the socially optimal level z^{sp}. The comparison between z^* and z^{sp} boils down to the comparison between the two equations (5.35) and (5.33). There are three differences between these two equations. The first difference is that, for given z, the social discount rate $\rho - \lambda z^{sp}(\gamma - 1)$ in (5.35) is less than the private discount rate $\rho + \lambda z^*$ in (5.33). This difference corresponds to the *intertemporal spillover effect:* namely, the social planner takes into account that the benefit to the next innovation will continue forever, whereas the private research firm attaches no weight to the benefits that accrue beyond the succeeding innovation. This effect tends to generate insufficient research under laissez-faire.

The second difference is the factor $(1-\alpha)$ which appears on the numerator of (5.33) but not in (5.35). This difference reflects an *appropriability effect,* namely the private monopolists' inability to appropriate the whole output flow, she can only appropriate the fraction $(1-\alpha)$ of it. This effect also tends to generate too little research under laissez-faire.

The third difference is the factor $(\gamma - 1)$ in the numerator of (5.35) instead of factor γ in the numerator of (5.33). This corresponds to a *business-stealing effect.* Namely, the private research firm does not internalize the loss to the previous innovator caused by her new innovation. In contrast, the social planner takes into account that a new innovation destroys the social return from the previous innovation. This effect will tend to generate *too much* research under laissez-faire.

Overall, both the intertemporal spillover and the appropriability effects tend to make aggregate R&D and the average growth rate under laissez faire less than their socially optimal counterparts, whereas the business-stealing effect tends to make aggregate R&D and average growth under laissez-faire greater than their socially optimal counterparts. The business-stealing effect will tend to dominate for γ sufficiently close to 1, whereas the appropriability effect will tend to dominate for α close to 1.

5.4 Continuous time with multiple sectors

In this section we develop a simple multi-sector version of the Schumpeterian growth model in continuous time. As in the previous section, individuals are infinitely lived and risk-neutral, and they discount the future at rate ρ. This will again imply that

$$r_t = \rho$$

at every instant. But the final good is now produced using a continuum of intermediate inputs, according to the logarithmic production function:[14]

$$\ln Y_t = \int_0^1 \ln y_{jt} dj. \tag{5.36}$$

Each firm takes the wage rate as given and produces using labor as the only input according to the following linear production function,

$$y_{it} = A_{it} l_{it}, \quad i \in \{A, B\}$$

[14] The logarithmic production function was first introduced into the Schumpeterian model by Grossman and Helpman (1991a).

where l_{jt} is the labor employed.

An innovation in sector i at date t will move productivity in sector i from A_{it-1} to $A_{it} = \gamma A_{it-1}$. To innovate with Poisson flow rate z, an intermediate firm in any sector i needs to spend δz flow units of R&D labor.

5.4.1 Solving the model

As before, our focus is on a balanced growth path, where all aggregate variables grow at the same rate g (to be determined). We will now proceed in two steps. First, we will solve for the static production decision and then turn to the dynamic innovation decision of firms, which will determine the equilibrium rate of productivity growth, as well as various firm moments along with the invariant firm size distribution.

5.4.1.1 Static production decision and equilibrium profits

Let w_t denote the wage rate at date t. The logarithmic technology in (5.36) implies that final good producer spends the same amount Y_t on each variety j. As a result, the final good production function in (5.36) generates a unit elastic demand with respect to each variety: $y_{jt} = Y_t/p_{jt}$. Combined with the fact that firms in a single product line compete à la Bertrand, this implies that a monopolist with marginal cost $MC_{jt} = w_t/A_{jt}$ will follow limit pricing by setting its price equal to the marginal cost of the previous innovator $p_{jt} = \gamma w_t/A_{jt}$. The resulting equilibrium quantity and profit in product line j are:

$$y_{jt} = \frac{A_{jt}Y_t}{\gamma w_t}, \tag{5.37}$$

and equilibrium profit will be

$$\pi_{jt} = \pi Y_t.$$

where $\pi \equiv \frac{\gamma-1}{\gamma}$. To prove the latter, just note that

$$\pi_{jt} = (p_{jt} - MC_{jt})y_{jt}$$

or equivalently

$$\pi_{jt} = (p_{jt} - p_{jt}/\gamma)Y_t/p_{jt} = (1 - 1/\gamma)Y_t.$$

Note that profits are constant across product lines, which will significantly simplify the aggregation up to the firm level. Note also that the demand for production workers in each line is simply $Y_t/(\gamma w_t)$. Substituting (5.37) back into the final good production function (5.36) we get

$$w_t = \gamma^{-1} A_t \tag{5.38}$$

where $A_t \equiv \exp\left(\int_0^1 \ln A_{it} di\right)$ is the productivity index.

5.4.1.2 Research arbitrage and equilibrium innovation

Next we turn to the innovation decision of the firms. The value V_t of a successful innovator at date t satisfies:

$$\rho V_t - \dot{V}_t = \pi_t - zV_t, \tag{5.39}$$

where z is the aggregate innovation rate in the sector. In steady-state equilibrium we have $\dot{V}_t = gV$, so that

$$V_t = \frac{\pi Y_t}{\rho - g + z} \tag{5.40}$$

Potential innovators will then choose the R&D intensity z through the free-entry condition to maximize

$$\max_{z_t} \{z_t V_t - z_t \delta w_t\}.$$

Therefore in equilibrium

$$V_t = \delta w_t = \frac{\delta}{\gamma} A_t \tag{5.41}$$

where the last equality used (5.38).

Recall that the amount of labor used in every product line is identical $l_i = l$. Hence $\ln Y_t = \int_0^1 \ln A_{it} + \ln l$ which implies

$$Y_t = A_t l. \tag{5.42}$$

Now combine (5.40) with (5.41) and (5.42) to get

$$\frac{\delta}{\gamma} = \frac{\pi l}{\rho + z\,(1 - \ln \gamma)}$$

Then using the labor market clearing condition

$$l + z\delta = 1$$

we find the equilibrium innovation rate as

$$z = \frac{\frac{\pi\gamma}{\delta} - \rho}{\pi\gamma + 1 - \ln \gamma}.$$

5.4.2 Growth

By the law of large numbers, the equilibrium fraction of sectors that innovate during any time interval $[t, t + dt]$ is equal to the probability zdt that any sector innovate in equilibrium during this time interval. Moreover, recall that an innovation in any sector j increases the corresponding y_{jt} by factor γ, thereby increasing $\ln y_{jt}$ by the amount $\ln \gamma$. It then follows that during this time interval aggregate output Y_t will increases deterministically by the amount $z \ln \gamma dt$. In other words, growth of final output is now deterministic, and occurs at constant rate

$$g = z \ln \gamma.$$

This rate is: (i) increasing in the size of innovations γ; (ii) increasing in the productivity of R&D which is inversely measured by δ; (iii) decreasing in the rate of time preference ρ.

5.5 Conclusion

It may be useful to contrast again the Schumpeterian growth paradigm to the two alternative models of endogenous growth analyzed previously. The first was the AK model of Chapter 2, according to which knowledge accumulation is a serendipitous by-product of capital accumulation. Here thrift and capital accumulation were the keys to growth, not creativity and innovation. The second endogenous growth model was the product-variety model of Chapter 3, in which innovation causes productivity growth by creating new, but not necessarily improved, varieties of products.

Compared to the AK model, both the Schumpeterian model and the product variety model have the advantage of presenting an explicit analysis of the innovation process underlying long-run growth. Compared to the product variety model, the Schumpeterian model assigns an important role to exit and turnover of firms and workers, which, as we argued at the end of the previous chapter, is consistent with an increasing number of recent studies demonstrating that labor and product market mobility are key elements of a growth-enhancing policy near the technological frontier.[15]

5.6 Literature Notes

The Schumpeterian growth approach was initiated in the fall of 1987 at MIT, where Philippe Aghion was a first-year assistant professor and Peter Howitt a visiting professor on sabbatical from the University of Western Ontario. During that year they wrote their "model of growth through creative destruction" (see Section 5.3 below); which was published as Aghion and Howitt (1992). Parallel attempts at developing Schumpeterian growth models include Segerstrom, Anant and Dinopoulos (1990) and Corriveau (1991).[16]

What we present here is a simplified version of the model we laid out in Aghion and Howitt (1988, 1992), using modeling techniques from industrial organization theory (Tirole, 1988, Chapter 10 and Reinganum, 1989). Grossman and Helpman (1991b, 1991c) built on this framework first by introducing the logarithmic multi-sector technology, and then by using the framework to analyze the relationship between trade and growth, and between growth and the product cycle.

On the Scale Effect debate, Jones (1995b) has developed a “semi-endogenous” model in which the scale effect is dissipated by the diminishing returns to ideas in research, with the implication that

[15] As we will see in subsequent chapters, the Schumpeterian model also has the advantage of allowing for entrepreneurs to make the choice between implementation and frontier innovation, and for this choice to vary with distance to the frontier, something that does not fit easily into the product-variety model. This allows the Schumpeterian model to generate context-specific policy implications and comparative-statics predictions, dependent particularly on a county's distance to the frontier.

[16] Segerstrom, Anant, and Dinopoulos (1990), modeled sustained growth as arising from a succession of product improvements in a fixed number of sectors, but with no uncertainty in the innovation process. Corriveau (1991) produced a discrete-time model with uncertainty about cost-reducing process innovations.

population growth is the only long run determinant of economic growth. The approach was further developed by Kortum (1997) and Segerstrom (1998). Our alternative "fully endogenous" approach was developed by wave of Schumpeterian endogenous models without scale effects, in particular see Howitt (1999), Aghion and Howitt (1998a, ch.12), Dinopoulos and Thompson (1998), Peretto (1998). See also the recent contribution of Dinopoulos and Syropoulos (2006) which argues that efforts to build increasing barriers to entry are what dissipate the scale effect. Segerstrom (2000) and Jones (2005) point out that small changes in the assumptions of the fully endogenous model can result in drastic changes in its conclusions with respect to scale effects. However, Ha and Howitt (2006) confront the two main varieties of R&D-based growth models without scale effects with US data, and conclude that the Schumpeterian model without scale effects is more consistent with the long-run trends in R&D and TFP than semi-endogenous growth theory. Other tests, such as Laincz and Peretto (2004), and Ulku (2005), point in the same direction, using US data as well.

Mathematical Appendix and Extensions

A.4.1 Poisson Processes

Throughout these chapters we will often assume that some random event X is governed by a "Poisson process," with a certain "arrival rate" μ. What this means mathematically is that the time T you will have to wait for X to occur is a random variable whose distribution is exponential with parameter μ :

$$F(T) \equiv \text{Prob}\{\text{Event occurs before } T\} = 1 - e^{\mu T}.$$

So the probability density of T is

$$f(T) = F'(T) = \mu e^{-\mu T}.$$

That is, the probability that the event will occur sometime within the short interval between T and $T + dt$ is approximately $\mu e^{-\mu T} dt$. In particular, the probability that it will occur within dt from now (when $T = 0$) is approximately μdt. In this sense μ is the probability per unit of time that the event will occur now, or the "flow probability" of the event.

For example, in the present chapter the event that an individual researcher discovers innovation number $t+1$ is governed by a Poisson process with the arrival rate λ. The expression λV_{t+1} on the right-hand side of the arbitrage equation (R) represents the expected income of an individual researcher, because over a short interval of length dt the researcher will make an innovation worth V_{t+1} with probability λdt.

If X_1 and X_2 are two distinct events governed by independent Poisson processes with respective arrival rates μ_1 and μ_2, then the flow probability that at least one of the events will occur is just the sum of the two independent flow probabilities $\mu_1 + \mu_2$, because the probability that both events will occur at once is negligible. In this sense, independent Poisson processes are "additive." This is why, in the present chapter, when z_t independent researchers each innovate with a Poisson arrival rate λ, the Poisson arrival rate of innovations to the economy as a whole is the sum λz_t of the individual arrival rates.

If a sequence of independent events takes place, each governed by the same independent process with the constant arrival rate μ, then the expected number of arrivals per unit of time is obviously the arrival rate μ. For example, in the present chapter the expected number of innovations per year in a balanced growth equilibrium is the arrival rate λz.

Moreover, the number of events x that will take place over any interval of length dt is distributed according to the "Poisson distribution" that you will find described in most statistics textbooks:

$$g(x) = \text{prob}\{x \text{ events occur}\} = \frac{(\mu dt)^x e^{\mu dt}}{x!},$$

whose expected value is the arrival rate times the length of the interval μdt.

A.4.2 Scale effects

Both of the innovation-based growth theories we have seen so far, the product variety model with just horizontal innovations and the Schumpeterian model with just vertical innovations, predict that increased population leads to increased growth. This is because increased population raises the size of the market that can be captured by a successful entrepreneur and also because it raises the supply of potential researchers.

This prediction has been challenged however on empirical grounds. In particular, Jones (1995) has pointed out that the number of scientists and engineers engaged in R&D has grown almost ninefold since 1953 with no significant trend increase in productivity growth. The present section shows how the counterfactual scale effect can be eliminated from the theory by allowing for both horizontal and vertical innovations.

The way to deal with this problem in Schumpeterian theory is to incorporate Young's (1998) insight that as population grows, *proliferation of product varieties* reduces the effectiveness of research aimed at quality improvement, by causing it to be spread more thinly over a larger number of different sectors, thus dissipating the effect on the overall rate of productivity growth.

So the first thing we need to do is assume a final-good production function that allows for a variable number of intermediate products:

$$Y_t = (L/M)^{1-\alpha} \int_0^M A_{it}^{1-\alpha} y_{it}^{\alpha} di, \tag{A.4.1}$$

which is the same as the production function (5.18) above except that now the intermediate products are indexed over the interval $[0, M]$ instead of $[0, 1]$. Thus M is our measure of product variety.

This production function is the same as the function assumed in the product-variety model, except that (i) each product has its own unique productivity parameter A_{it} instead of having $A_{it} = 1$ for all products, and (ii) we assume that what matters is not the absolute input L of labor but the input per product L/M. Thus the contribution each intermediate product to final output is now

$$Y_{it} = {A_{it}}^{1-\alpha} y_{it}^{\alpha} \left(L/M\right)^{1-\alpha},$$

which indicates that as the number of intermediate products goes up, there will be less labor to work with each one, so each will contribute less to final output unless the quality A_{it} or the quantity y_{it} is increased.[17]

[17] The production function is a special case of the one that Benassy (1998) showed does not necessarily yield a positive productivity effect of product variety.

The next thing we have to do is model the process by which product variety increases. The simplest scheme is to suppose that each person has a probability λ of inventing a new intermediate product, with no expenditure at all on research. Suppose also that the exogenous fraction ϕ of products disappears each year. If population is constant, then each year the length M_t of the list of intermediate products will change by the amount

$$\lambda L - \phi M_t$$

and will eventually stabilize[18] at a steady-state value:

$$M = \frac{\lambda L}{\phi}$$

which is proportional to population. So if population were to increase permanently, the number of products would eventually grow in proportion.

Thus in the long run the final-good production function will be:

$$Y_t = \left(\frac{\phi}{\lambda}\right)^{1-\alpha} \int_0^M A_{it}^{1-\alpha} y_{it}^{\alpha} di,$$

and the contribution of each intermediate product to final output will be

$$Y_{it} = \left(\frac{\phi}{\lambda}\right)^{1-\alpha} {A_{it}}^{1-\alpha} x_{it}^{\alpha},$$

which does not depend at all on the size of the economy.

Proceeding as above we see that the price of each intermediate product will be its marginal product in the final sector:

$$p_{it} = \frac{\partial Y_{it}}{\partial y_{it}} = \alpha \left(\frac{\phi}{\lambda}\right)^{1-\alpha} A_{it}^{1-\alpha} y_{it}^{\alpha-1}. \tag{A.4.2}$$

[18] That is, the difference equation

$$M_{t+1} = M_t + \lambda L - \phi M_t$$

starting at M_0 has the unique solution

$$M_t = (\lambda/\phi) L + (1-\phi)^t \left[M_0 - (\lambda/\phi) L\right]$$

which converges to $(\lambda/\phi) L$ as $t \to \infty$ because $(1-\phi)$ is between zero and one.

Therefore the monopolist in sector i chooses the quantity y_{it} that maximizes her profit: $\pi_{it} = \max_{p_{it},y_{it}} \{p_{it}y_{it} - y_{it}\}$ subject to (A.4.2), which is equivalent to

$$\pi_{it} = \max_{y_{it}} \left\{ \alpha \left(\frac{\phi}{\lambda} \right)^{1-\alpha} A_{it}^{1-\alpha} y_{it}^{\alpha} - y_{it} \right\},$$

which implies an equilibrium quantity:

$$y_{it} = \alpha^{\frac{2}{1-\alpha}} \frac{\phi}{\lambda} A_{it}$$

and an equilibrium profit:

$$\pi_{it} = \pi \frac{\phi}{\lambda} A_{it}.$$

where the parameter π is the same as in the analogous equation (5.9) of the one-sector model.

According to these equations, both the monopolists' equilibrium quantity and equilibrium profit are both independent of the scale of the economy as measured by population L, because her demand function (A.4.2) is independent of L. Because of this, the net benefit to research will be independent of scale and so will the equilibrium intensity of research, the frequency of innovation and the economy's growth rate. More specifically, the net benefit will be

$$\max_{z_{it}} \left\{ z_{it} \pi \frac{\phi}{\lambda} \gamma A_{it-1} - c(z_{it}) A_{it-1} \right\}$$

which is maximized by z_{it} satisfying the research arbitrage equation:

$$z_t = z = \pi \frac{\phi}{\lambda} \gamma.$$

Therefore the frequency of innovation z and the growth rate $g = z \cdot (\gamma - 1)$ are also independent of scale L.

A.4.3 Nondrastic Innovations

Until this point the analysis has assumed that innovations are drastic: that the intermediate monopolist is not constrained by potential competition from owners of previous patents. The present section shows that the analysis of stationary equilibria in Section 5.3 can be generalized to the case where innovations are *nondrastic*.

Innovations are nondrastic if and only if the previous incumbent could make a positive profit when the current one is charging the monopolistic price $p_t = \alpha A_t y_t^{\alpha-1} = \frac{1}{\alpha} w_t$ which yields an unconstrained maximum to the current incumbent's profit. If innovations are nondrastic, then the current incumbent sets the maximum price that gives the previous incumbent nonpositive profits and satisfies all the demand at that price, leaving none to the previous incumbent.

The previous incumbent could make a positive profit if and only if a competitive producer of consumption goods could produce at a smaller cost using the previous incumbent's intermediate good, buying the latter at a price equal to its average cost of production w_t. The cost of producing Y units of consumption would be

$$C_{t-1}(w_t, Y) = w_t y \quad \text{where} \quad Y = A_{t-1} y^{\alpha},$$

that is,

$$C_{t-1}(w_t, Y) = w_t \left(\frac{Y}{A_{t-1}} \right)^{1/\alpha}.$$

The cost of producing Y units of consumption good using the new intermediate input priced at the unconstrained monopolistic maximum $p_t = \frac{1}{\alpha} w_t$ is

$$C_t(p_t, Y) = p_t y \quad \text{where} \quad Y = A_t y^{\alpha},$$

that is,

$$C_t(p_t, Y) = \frac{1}{\alpha} w_t \left(\frac{Y}{A_t} \right)^{1/\alpha}.$$

It follows that innovations are *drastic* if and only if $C_t(p_t, Y) \leq C_{t-1}(w_t, Y)$ for all Y, which, given that $A_t = \gamma A_{t-1}$, is equivalent to

$$\gamma \geq \alpha^{-\alpha}.$$

However, innovations are *nondrastic* whenever

$$\gamma < \alpha^{-\alpha}.$$

In that case, the maximum price that can be charged by the current incumbent to the consumption good sector is $\widehat{p}_t$ such that

$$C_t(\widehat{p}, y) = C_{t-1}(w_t, Y),$$

that is,

$$\widehat{p} = \gamma^{1/\alpha} w_t.$$

The corresponding profit flow $\widehat{\pi}_t$ and labor demand $\widehat{y}_t$ are respectively given by

$$\widehat{\pi}_t = (\gamma^{1/\alpha} - 1) w_t \widehat{y}_t$$

and

$$\widehat{y}_t = (\gamma^{1/\alpha} \omega_t / \alpha)^{\frac{1}{\alpha - 1}},$$

where $\omega_t = \frac{w_t}{A_t}$ is the productivity-adjusted wage rate.

These expressions are almost identical to those in the drastic case, except that the markup $\gamma^{1/\alpha}$ replaces the markup $1/\alpha$ in the drastic case. The equation defining the stationary (or steady-state) equilibrium level of research in the nondrastic case will thus be

$$1 = \frac{\lambda \gamma (\gamma^{1/\alpha} - 1)(L - \widehat{z})}{r + \lambda \widehat{z}}. \tag{A.4.3}$$

It is straightforward to check that all the comparative statics results derived for the case of drastic innovations are valid also when innovations are nondrastic. Furthermore, the comparison between (A.4.3) and (5.33) shows that the same welfare effects analyzed in Section 5.3.3 operate in the case where innovations are nondrastic, again with the result that research and growth under laissez-faire may be more or less than optimal.

As is customary in the patent-race literature, this analysis has ruled out the possibility that the current and previous incumbent might contract to share the higher monopoly profits that could be earned if the previous incumbent agreed never to compete. For example, the previous incumbent might sell its patent to the current one; in the extreme case where the previous incumbent always had no bargaining power in negotiation with the current one, competition from previous vintages of the intermediate good would never constrain the monopolist, and the earlier analysis of drastic innovations would apply no matter how small the innovations were.