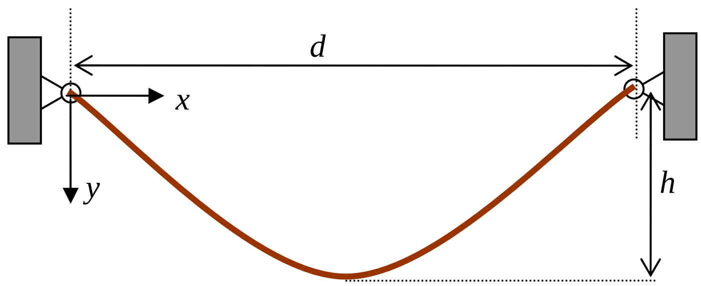
ENGN1300: Structural Analysis

Homework 5
Due Wednesday March 12, 2010

Division of Engineering
Brown University

1. Using the solutions derived in class for a cable hanging under its own weight, find the cable sag *h/d* and length L_0/d that minimizes the tension in the cable at the supports.



From class: $L_0/d = \frac{1}{c}\sinh c, \frac{h}{d} = \frac{1}{2c}\{\cosh(c)-1\}$ and $\frac{T_0}{wd} = \frac{1}{2c}\cosh(c)$. Maple gives c=1.2 for the minimum T_0=0.75wd. Corresponding L_0=1.26d and h=.34d.

```
> restart;
> T:=cosh(c)/c/2;
```

$$T := \frac{1}{2}\frac{\cosh(c)}{c}$$

```
> with(Optimization);
```

[*ImportMPS* , *Interactive* , *LPSolve* , *LSSolve* , *Maximize* , *Minimize* , *NLPSolve* , *QPSolve*]

```
> Minimize(T,c=0..2);
```

[0.754439780769159851 , [c = 1.19967864636161290]]

```
> c:=1.2;
```

$$c := 1.2$$

```
> evalf(T);
```

$$0.7544398195$$

```
> L:=sinh(c)/c;
```

$$L := 1.257884462$$

```
> h:=(cosh(c)-1)/2/c;
```

$$h := 0.3377731529$$

2. For a cable of length L_0 with subjected to a uniformly distributed load $w(x)=w_0$, determine the cable shape $y(x)$ and tension $T(x)$. Neglect the cable self-weight, and assume that $y(0)=y(d)=0$. Plot the length L_0/d versus the sag h/d. Also, plot $T/(2wd)$ as a function of x/d, for h/d=0.1,0.3,0.5,0.7,0.9.

$$\frac{d^2y}{dx^2}=-\frac{w_0}{R_x}\Rightarrow y(x)=-\frac{w_0}{2R_x}x^2+Cx.\quad y(d)=0\Rightarrow\frac{w_0}{2R_x}d=C$$

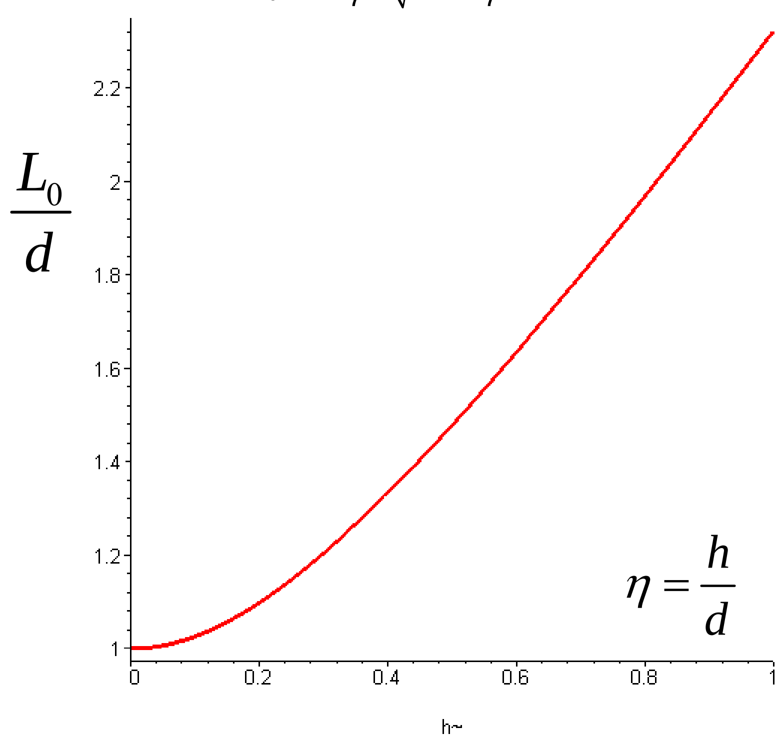
$$y(x)=\frac{w_0}{2R_x}x(d-x)=d\frac{w_0 d}{2R_x}\frac{x}{d}\left(1-\frac{x}{d}\right)$$

$$y(d/2)=h=d\frac{w_0 d}{8R_x}\Rightarrow y(x)=4h\frac{x}{d}\left(1-\frac{x}{d}\right)$$

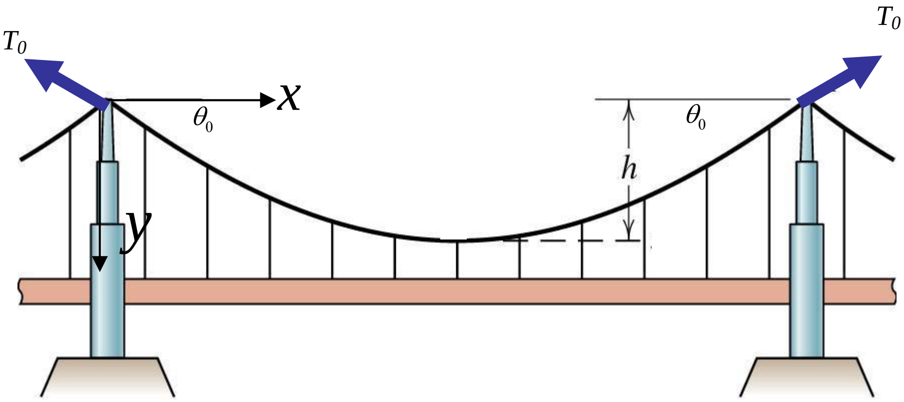
Relate h to L_0:

$$L_0=\int_0^d\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=\int_0^d\sqrt{1+\left(4\frac{h}{d}\right)^2\left(2\frac{x}{d}-1\right)^2}dx=d\int_0^1\sqrt{1+(4\eta)^2(2\xi-1)^2}d\xi$$

$$\frac{L_0}{d}=\frac{1}{2}\sqrt{1+16\eta^2}-\frac{1}{16\eta}\ln\frac{-4\eta+\sqrt{1+16\eta^2}}{4\eta+\sqrt{1+16\eta^2}}\quad\eta=\frac{h}{d}$$



Tension:



$$\sum F_y = 0 \Rightarrow 2T_0 \sin\theta_0 = wd . \tan\theta_0 = y'(0) = 4h/d.$$

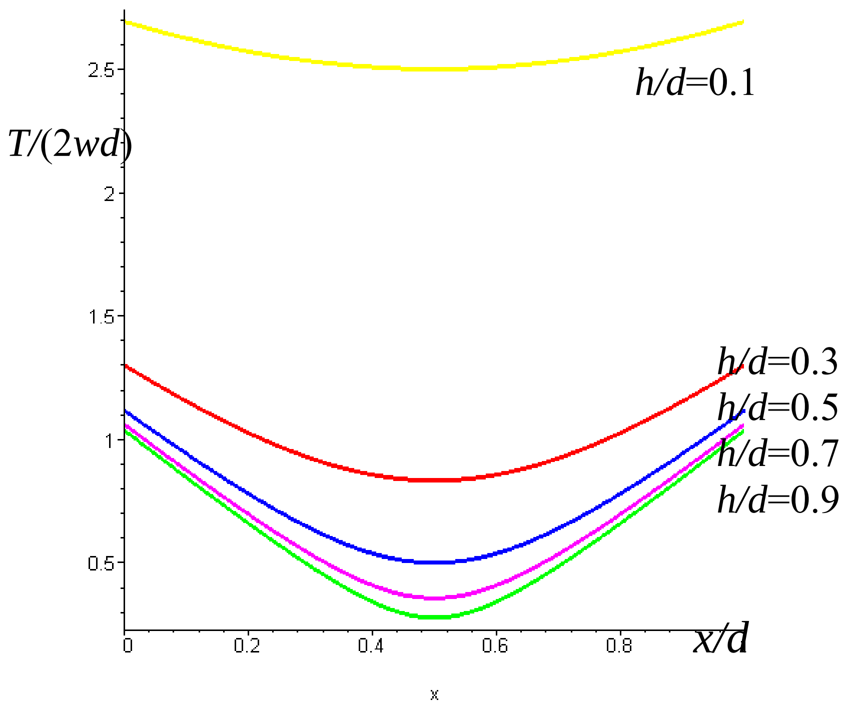
$$\Rightarrow T_0 = \frac{wd}{2\sin\theta_0} = \frac{wd}{2}\sqrt{1+\left(\frac{d}{4h}\right)^2}.$$

$$T(x) = T_0 \frac{\cos\theta_0}{\cos\theta(x)}$$

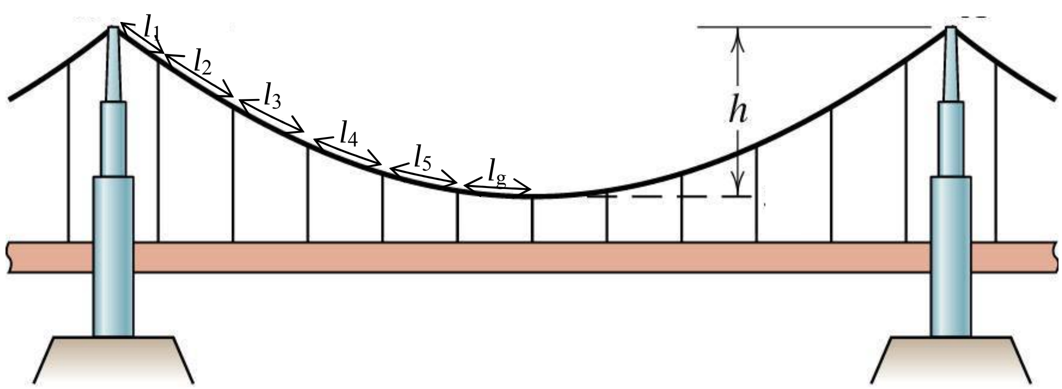
$$\cos\theta(x) = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{1}{\sqrt{1+16\frac{h^2}{d^2}\left(2\frac{x}{d}-1\right)^2}}$$

$$T(x) = T_0 \frac{\cos\theta_0}{\cos\theta(x)} = \frac{wd\cot\theta_0}{2\cos\theta(x)} = \frac{wd^2}{8h}\sqrt{1+16\frac{h^2}{d^2}\left(2\frac{x}{d}-1\right)^2}$$

$$T(x) = \frac{wd}{2}\sqrt{\frac{d^2}{16h^2}+\left(2\frac{x}{d}-1\right)^2}$$



3. The picture below shows a suspension bridge with 11 vertical hangers. The tower spacing d and sag h are given with h/d=0.075 (Approximate dimensions for the Golden Gate Bridge). Assuming that the tension in each hanger is the same, find the locations l_i along the cable at which each hanger should be attached. Neglect the weight of the cable and hangers, and assume that the hangers are so closely spaced that the main cable shape is well approximated by the solution found in problem 2. Calculate the total length L_0 of the cable too. Give the answers l_i/d and also as l_i/L_0.



The length along the cable is $l_i = \int_0^{d_i} \sqrt{1+(y'(x))^2}\,dx$. where d_i is the horizontal position (x-coordinate) of the hanger. $d_1=(d/11)/2$. $d_i=d_{i-1}+d/11$ (i=2,3,…11)

```
> restart;#problem 3.
> y:=h*4*(x/d)*(1-x/d);h:=0.075*d;
```

$$y := \frac{4\,h\,x\left(1-\frac{x}{d}\right)}{d}$$

$$h := 0.075\ d$$

```
> dy:=simplify(diff(y,x));x:=xi*d;
```

$$dy := -\frac{0.3000000000\ \ (-1.\ d + 2.\ x)}{d}$$

$$x := \xi\ d$$

```
> ds:=simplify(sqrt(1+dy^2));
```

$$ds := 0.1000000000\ \sqrt{109. - 36.\ \xi + 36.\ \xi^2}$$

```
> delta:=1/11;#hanger spacing normalized by d
```

$$\delta := \frac{1}{11}$$

```
> l1:=d*int(ds,xi=0..delta/2); #first hanger
```

$$l1 := 0.04728280963\ \ d$$

```
>
l2:=d*int(ds,xi=0..delta/2+delta);l3:=d*int(ds,xi=0..delta/
2+2*delta);l4:=d*int(ds,xi=0..delta/2+3*delta);
>
l5:=d*int(ds,xi=0..delta/2+4*delta);l6:=d*int(ds,xi=0..delt
a/2+5*delta);
```

$$l2 := 0.1409007149\ \ d$$

$$l3 := 0.2334623088\ \ d$$

$$l4 := 0.3252236972\ \ d$$

$$l5 := 0.4164477193\ \ d$$

$$l6 := 0.5074018690\ \ d$$

```
> lnth:=sqrt(1+16*eta^2)/2-ln((-
4*eta+sqrt(1+16*eta^2))/(4*eta+sqrt(1+16*eta^2)))/16/eta;
```

$$lnth := \frac{\sqrt{1+16\,\eta^2}}{2} - \frac{1}{16}\,\frac{\ln\left(\frac{-4\,\eta+\sqrt{1+16\,\eta^2}}{4\,\eta+\sqrt{1+16\,\eta^2}}\right)}{\eta}$$

```
> L0:=d*evalf(subs(eta=0.075,lnth)); #overall cable length:
```

$$L0 := 1.014803738\ \ d$$

```
> L1:=l1/L0;L2:=l2/L0;L3:=l3/L0;L4:=l4/L0;L5:=l5/L0;L6:=l6/L0
; #hanger locations as fractions of L0:
```

$$L1 := 0.04659305820$$

$$L2 := 0.1388452857$$

$$L3 := 0.2300566110$$

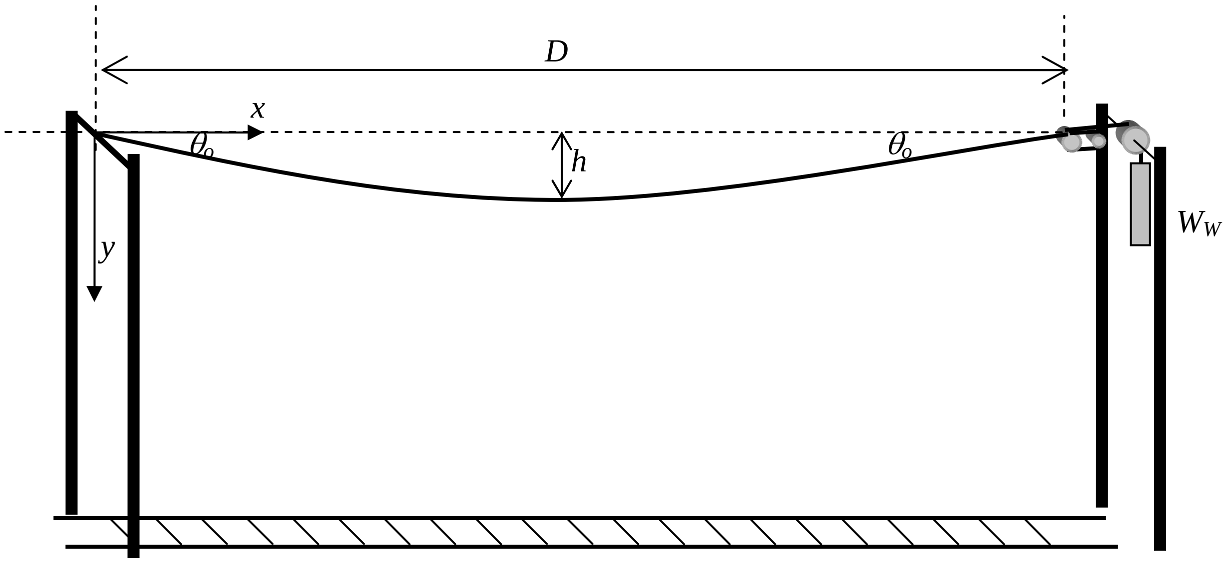
$$L4 := 0.3204794041$$

$$L5 := 0.4103726698$$

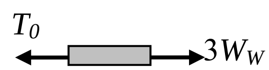
$$L6 := 0.5000000000$$

4. The tension and sag in the overhead cables for electric railways may be controlled by a pulley-counter weight system. Without the counterweight in place, the tension and sag would vary as the cable underwent thermal expansion or contraction. The sag must be carefully controlled in order that the train pantographs do not loose contact with the wire. The tension must be controlled for another reason: the wavespeed of the cable depends on the cable tension, and if a train travels at or near the cable wavespeed, a sort of resonance can occur, again causing the cable fluctuate wildly. The photos of the Amtrak rail line below were taken from the Home Depot parking lot in Providence.

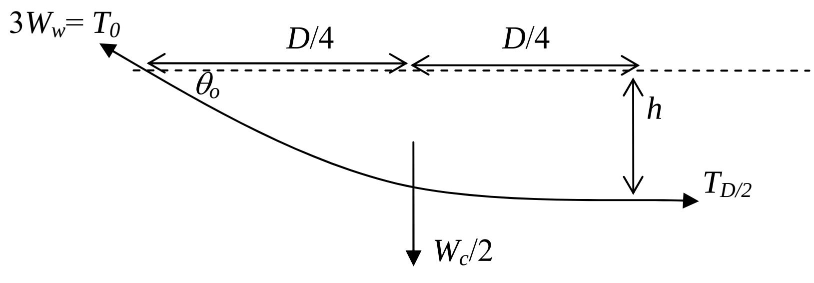
The system is shown below. The total weight of the cable is W_C and the counter weight is W_W. The pulley is frictionless. The cable hangs symmetrically about its midpoint.



a. Determine the tension T_o in the electric cable at the endpoints in terms of W_w and/or W_c. Neglect the weight of the pulleys and the small segment of cable hanging over the pulley.



b. Draw a free body diagram showing all forces acting on the left half of the cable ($0 < x \leq D/2$). Note that the cable is horizontal at $x=D/2$. Determine the angle θ_o shown in the picture in terms of W_W and W_C..



$$\sum F_y = 0 \Rightarrow T_0 \sin\theta_0 = W_c/2 \Rightarrow \sin\theta_o = \frac{W_c}{2T_0} = \frac{W_c}{6W_W}$$

c. Find expressions for the tension $T_{D/2}$ sag h in the cable at its midpoint as a function of W_W, W_C , and D. You may assume that the center of mass of this half of the cable is at a distance $x_c=D/4$ from the left support. (This is a good assumption if the sag h is small compared with the distance D).

$$\sum F_x = 0 \Rightarrow T_0 \cos\theta_0 = T_{D/2} = 3W_w \cos\theta_0 = 3W_w\sqrt{1-\sin^2\theta_0} = 3W_w\sqrt{1-\left(\frac{W_c}{6W_W}\right)^2}$$

$$\sum \mathbf{M}_o = \mathbf{0} \Rightarrow (T_{D/2}h - W_c D/8)\mathbf{k} = \mathbf{0} \Rightarrow h = \frac{D}{24}\frac{W_c/W_w}{\sqrt{1-\left(\frac{W_c}{6W_W}\right)^2}}$$

d. The sag is generally desired to be less than 5% of the span length. Find the necessary counterweight W_w for a given cable weighing W_c.

$$h = \frac{D}{20} = \frac{D}{24}\frac{W_c/W_w}{\sqrt{1-\left(\frac{W_c}{6W_W}\right)^2}} \Rightarrow W_w = 0.85W_c$$

e. Assume that the span distance is 100 meters, and the cable is made from copper, with a cross sectional area of $200mm^2$. For such small sags, you may assume that the cable length $L \approx D$. Find the minimum required counterweight *Ww*.

Density of copper is 8230 kg/m^3. For a cross sectional area of 200mm^2=0.0002m^2, the mass/unit length is 1.65 kg/meter. For a 100-meter span, W_c=164.6 kg. The counter weight must be W_w=0.85(164.6 kg)=140 kg.

f. The cable tension must be high enough to ensure that the speed of traveling waves along the cable exceeds the maximum speed of the train. The traveling wave speed is calculated from the formula $c = \sqrt{T_{min}/m}$ where T_{min} is the lowest value of the tension in the cable and *m* is the mass per unit length of the cable. For the copper cable with the counterweight calculated from part (e), find the traveling wave speed *c* (in meters per second and miles per hour) for the sag-controlled cable.

$$T_{min} = T_{D/2} = 3W_w\sqrt{1-\left(\frac{W_c}{6W_W}\right)^2} = 3(140\text{kg})(9.8\text{m/s}^2)\sqrt{1-\left(\frac{164.6}{6(140)}\right)^2} = 4036\text{N}$$

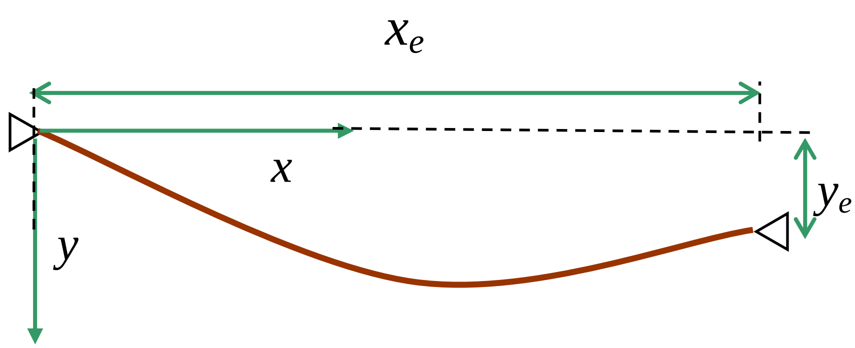
$$c = \sqrt{T_{min}/m} = \sqrt{\frac{4036\text{N}}{1.65\text{kg/m}}} = 49.5\text{m/sec}=110\text{ mph}$$

g. The Acela travels at a top speed 150 mph. If the wave speed calculated in part f falls below the speed of the Acela, then the Acela will induce resonance in the cable when it travels at the wave speed *c*. If this is the case, then the weight W_c which keeps sag to a minimum of 5% of the span is not sufficient to control the wavespeed. Calculate the minimum weight W_w which will ensure that the wave speed c is greater than the top speed of the Acela, 150 mph.

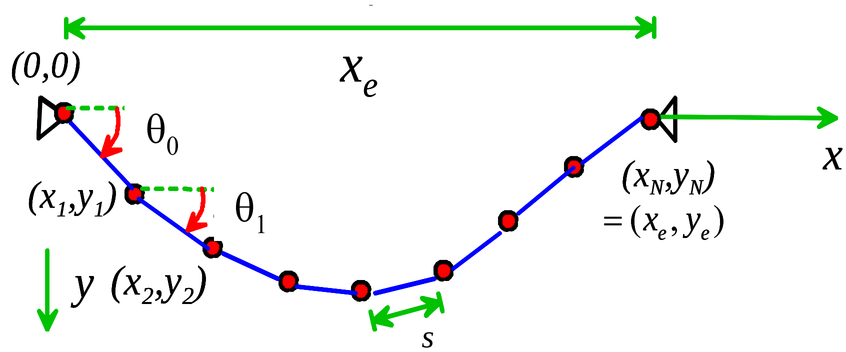
$$c = 110 \text{ mph} = 67.05\text{m/sec} = \sqrt{T_{\min}/\mu} \Rightarrow T_{\min} = \mu c^2 = 1.65\text{kg/m}^2 \times (67.05\text{m/s})^2 = 7420\text{N}$$

$$T_{\min} = 7420\text{N} = T_{D/2} = 3W_w \sqrt{1 - \left(\frac{W_c}{6W_W}\right)^2} \Rightarrow W_w = 2488\text{N} = 254\text{kg}$$

5. In this problem you will use the principle of minimum potential energy to calculate the shape of a flexible, inextensible cable of length L_0, with weight per unit length ω, which is suspended between two points (0,0) and (x_e, y_e) .



As described in class, the cable will be approximated as a series of N short segments, as shown in the picture below. Each segment has length $s=L/N$. The shape of the curve will be determined by computing the angle of each segment $\theta_1, \theta_2 .... \theta_N$ using an EXCEL spreadsheet.



To this end, note that the coordinates of successive points on the curve follow as

$$x_1 = s\cos\theta_0, \quad x_2 = x_1 + s\cos\theta_1 \;\; .... \;\; x_i = x_{i-1} + s\cos\theta_{i-1}$$
$$y_1 = s\sin\theta_0, \quad y_2 = y_1 + s\sin\theta_1 \;\; .... \;\; y_i = y_{i-1} + s\sin\theta_{i-1}$$

The i^{th} segment, connecting points (x_i,y_i) and (x_{i+1},y_{i+1}) has potential energy $V_i = -\omega s \frac{1}{2}(y_i + y_{i+1})$ and so the total potential energy of the cable is $V = \sum_{i=0}^{N-1} V_i$

You can use the solver on Excel to find the values of the angles θ_0, θ_1, θ_2, … θ_{N-1} that minimize the potential energy V for the cable, subject to the constraint that the end point $(x_N,y_N) =(x_e,y_e)$.

a. Set up an EXCEL spreadsheet, which, given a column of values of θ_i, will compute x_i, y_i, V_i and the total potential energy V. Hand in a copy of your spreadsheet with brief annotations indicating the formulas in the various cells.

   Have excel calculate the tension in each segment. To do this, use the expression for the tension in the cable near the left endpoint:

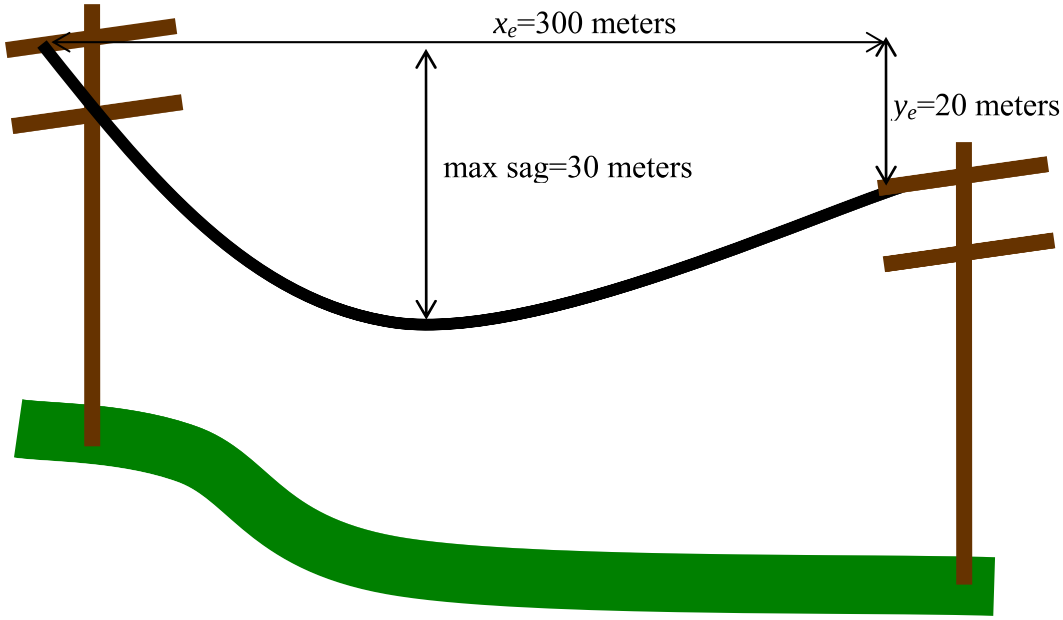
$$T_0 = \frac{\omega L}{\sin\theta_0 - \cos\theta_0 \tan\theta_e}$$

   The tension in successive segments is found from the fact that the horizontal component of force in the cable is constant: $T_i=T_0\cos\theta_0/\cos\theta_i$. Show a plot which gives the cable tension as a function of position in the cable. Note that you are actually find the tension at the midpoint of each segment.

   Show your results for a cable of length 2 with x_e=1, and each of y_e=0 and y_e=1.

b. The electric transmission cable shown has a mass density of 1.96kg per meter at 50°f (10°C). It is to be strung over a 300 meter span (x_e=300) and connected to a pole 20 meters below the start point. Determine the minimum cable length (to the nearest decimeter) required and the maximum tension in the cable. The ultimate tensile strength is 153 kN. Is this exceeded?

See excel. Minimum cable length is 303.7 meters, max tension is 12861 newtons (2890 lbs) No where near the Ultimate tensile strength.



c. The coefficient of linear thermal expansion of the cable in part (b) is 20×10^{-6} per degrees C. Suppose a cable with length 304 meters at 10^{o}C is strung. Find the tension and sag of the cable at 120^{o} f (50^{o}C) and at -4^{o} f (-20^{o} C).

The cable length and density as a function of temperature t are

$$L(T) = L_0 + (t - t_0)a,\ \ a = 20\times10^{-6}\,/^{0}\,\mathrm{C},\ \ t_0 = 10^{0}\,\mathrm{C},\ L_0\text{=301m}$$
$$\mu(t) = \mu_0 L_0 / L(t)$$

At 50 degrees C, the new length is $L(50)$=304×(1+20×10^{-6}×(50-10))=304.24m
The density is $\mu(50)$=1.96kg/m(304/304.24)=1.96km/m (no appreciable change).

From excel: the max sag is 31.4 m and the max tension is 11925 N

At -20 degrees C, the new length is $L(50)$=304×(1+20×10^{-6}×(-20-10))=303.82m
The density is $\mu(-20)$=1.96kg/m(304/303.182)=1.96km/m (no appreciable change)

From excel: the max sag is 30.27 m and the max tension is 12633 N