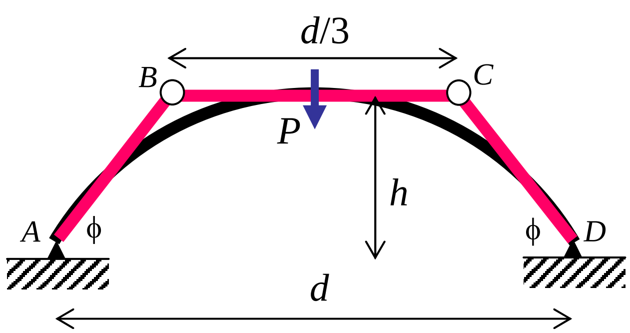
ENGN1300: Structural Analysis

Homework 9
Due Wednesday April 23, 2010

Division of Engineering
Brown University|

1. In class, we talked about a way to approximate the moments and deflection of a two-hinged arch subjected to a point load at its center. For the purposes of the calculation, the arch is represented as the frame shown, with hinges at points B and C.

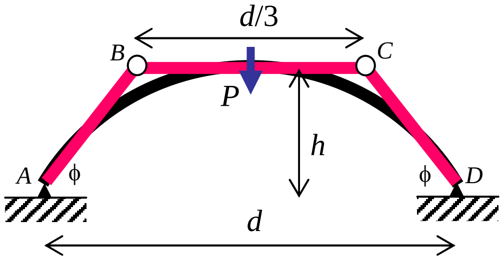


Assume all members of the approximating truss have identical properties *E*, *I*, and *A*, and calculate the deflection under the load.

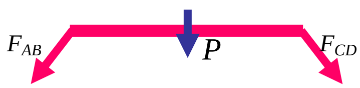
To do this, note that members AB and CD are two-force members, and the axial forces in those members can be calculated from static equilibrium. This in turn allows you to calculate the axial force in member BC. This member is not a two force member---it deflects due to bending as well as due to the axial load that it carries. Bending does not induce any axial elongation, but the axial load does. If the axial load in that member is F_{BC}, then $F_{BC} = EA(u_x^C - u_x^B)/L_{BC}$. Note that symmetry requires that $u_x^C = -u_x^B$, so $F_{BC} = -2EAu_x^B/L_{BC}$. This means that you can solve for u_x^B in terms of the given parameters of the problem.

Knowing u_x^B and the forces in the two-force members AB and CD, you can now calculate $u_y^B = u_y^C$ in terms of the parameters of the problem. And once you know these, you can find the vertical deflection under the load. This comes from the deflection of the cross beam (due to bending) and the deflections $u_y^B = u_y^C$.

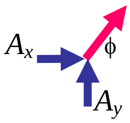
Evaluate your answer for the case For E=4x10^6 ksf, P=1kip, d=300ft, h=75ft, I=0.0833ft$_4$, A=1ft^2. Run the Excel solver for the two hinged arch and compare.



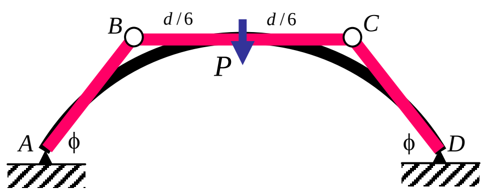
$$\tan\phi = \frac{3h}{d},\quad \sin\phi = \frac{h}{L_{AB}},\quad \cos\phi = \frac{d}{3L_{AB}},\quad L_{AB} = \sqrt{h^2 + d^2/9}$$



$$F_{AB} = F_{CD} = -\frac{P}{2\sin\phi}$$

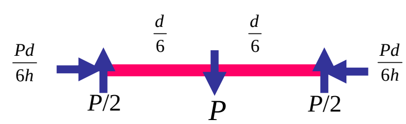


$$A_x = \frac{Pd}{6h},\ A_y = \frac{P}{2}$$



$$\tan\phi = \frac{3h}{d},\quad L_{AB} = \sqrt{h^2 + d^2/9}$$

(1) Axial extension in AB $\delta_{AB} = \frac{F_{AB}L_{AB}}{EA} = \mathbf{u}_B \bullet \mathbf{n}_{AB} = u_x^B\cos\phi + u_y^B\sin\phi = -\frac{PL_{AB}}{2EA\sin\phi}.$

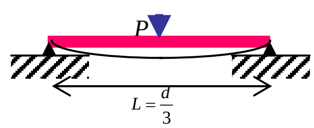


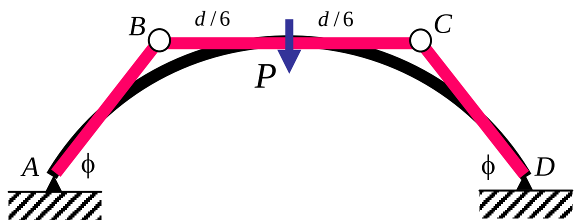
(2) Axial extension in BC $-2u_x^B = -\frac{Pd}{6h}\frac{d}{6EA} = -\frac{Pd^2}{36hEA} \Rightarrow u_x^B = -\frac{Pd^2}{72hEA}$

Solve for u_y^B from (1) and (2)

Then, from the simply supported beam center deflection $\frac{PL^3}{48EI}$

$$u_y^P = u_y^B - \frac{P(d/3)^3}{48EI}$$





$$\tan\phi = \frac{3h}{d}$$

$$\sin\phi = \frac{h}{L_{AB}}$$

$$L_{AB} = \sqrt{h^2 + d^2/9}$$

$$\frac{Pd^2}{72hEA}\cos\phi + u_y^B \sin\phi = -\frac{PL_{AB}}{2EA\sin\phi}$$

$$\Rightarrow u_y^B = -\frac{PL_{AB}}{2EA\sin^2\phi} - \frac{Pd^2}{72EAh\tan\phi} = -\frac{P(h^2 + d^2/9)^{3/2}}{2EAh^2} - \frac{Pd^3}{216EAh^2}$$

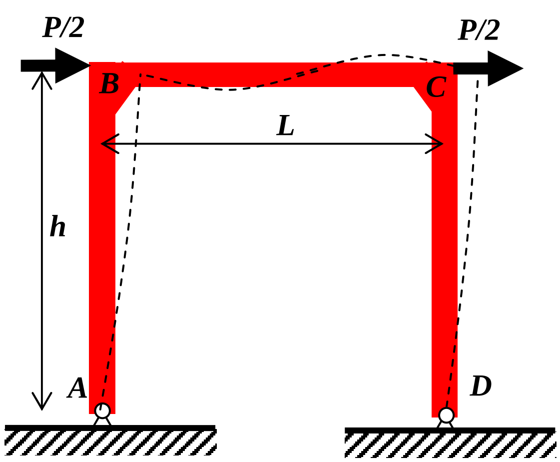
$$u_y^P = u_y^B - \frac{P(d/3)^3}{48EI}$$

For $E=10^6$, $P=1$, $d=300$ft, $h=75$ft, $I=0.0833\text{ft}^4$, $A=1\text{ft}^2$

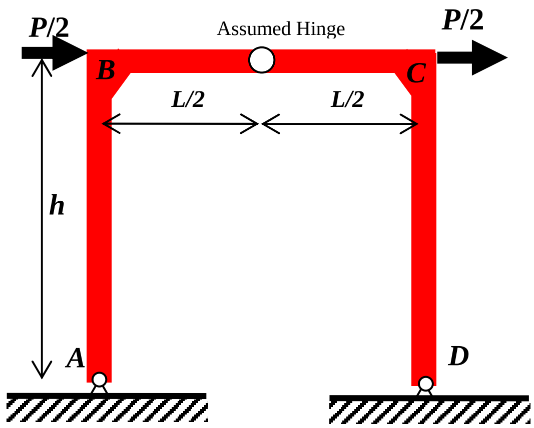
$$u_y^P = -0.063\text{ft}$$

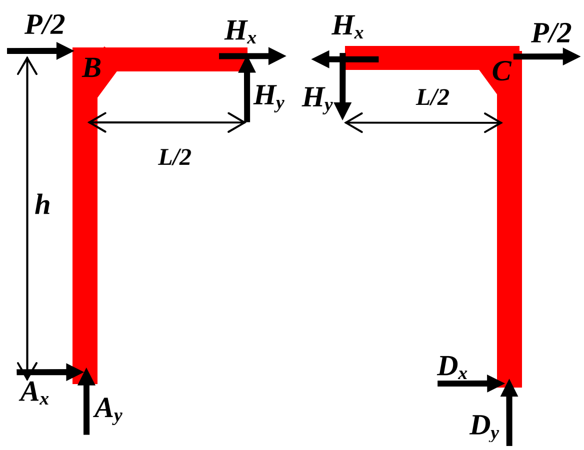
Excel gives -0.068 feet.

2. The pinned portal frame shown below is subjected to a lateral load P due to wind. An estimate of the deformed shape is shown in dashed lines. The deformed shape appears to have an inflection point near the midpoint of the cross beam. At such a point, the internal moment must be zero. (why?)



   a. To approximately analyze a pinned frame subject to a horizontal load, a hinge is often assumed at the crossbeam midpoint. Find the base reactions as well as the forces and moments transmitted through the joints B, and C, and the forces transmitted through the central hinge. Draw the moment diagram for the members.



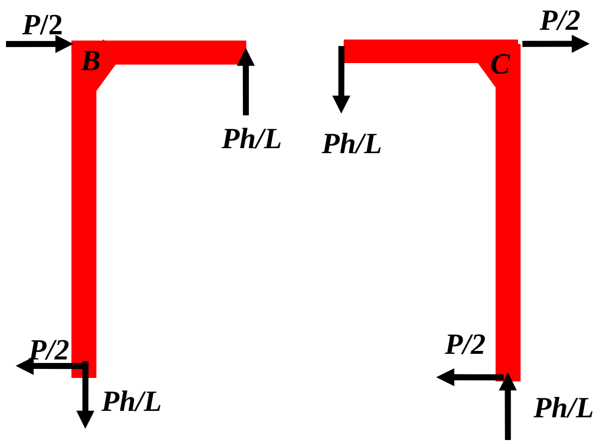


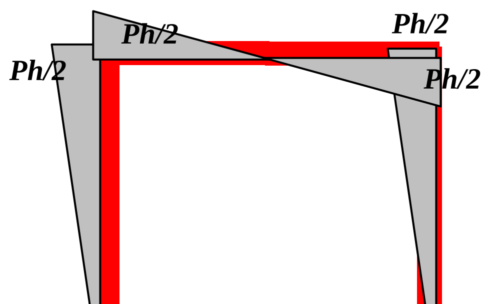
Sum forces and moments for each half of the frame:

A_y=-Ph/L=-D_y=H_y

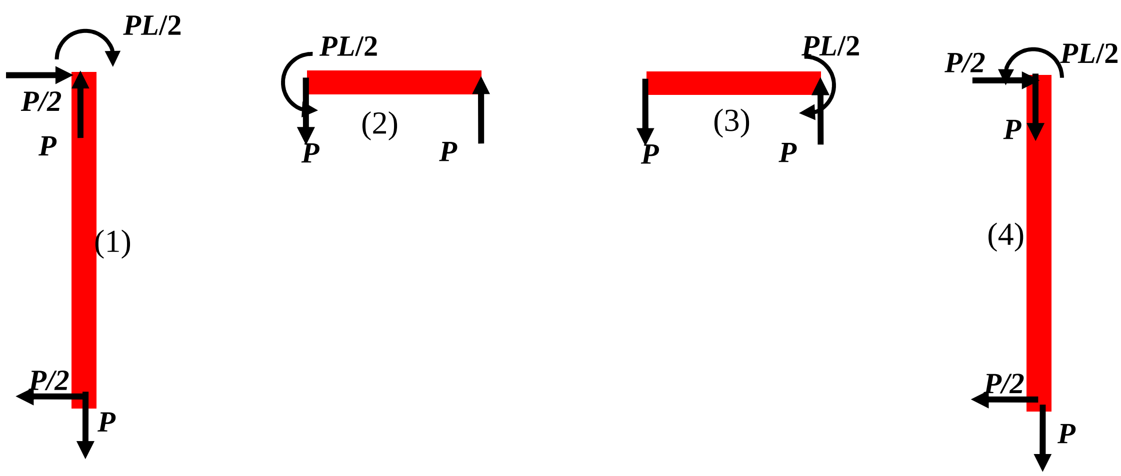
A_x=-P/2,= D_x

H_x=0

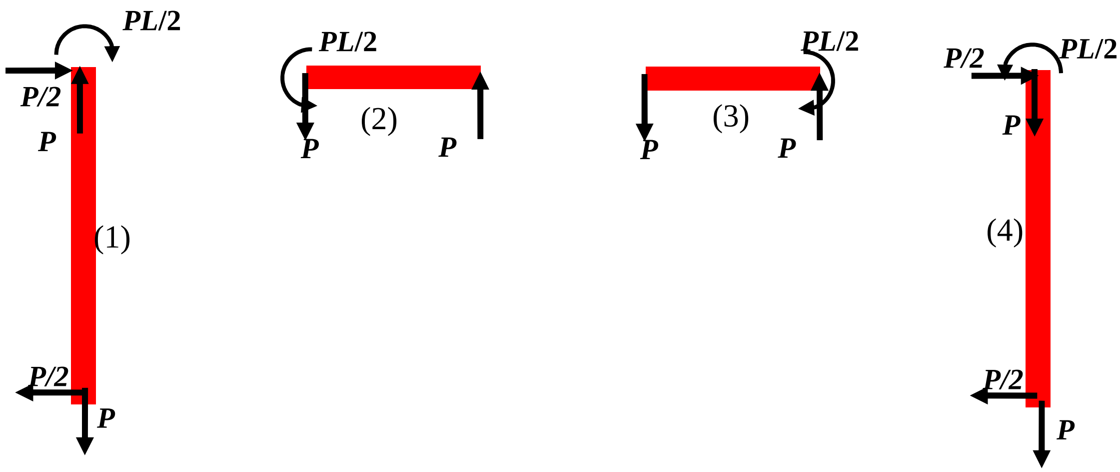




Moments



b. Calculate the deflections and rotations in the structure at points B, C, and at the central hinge. You can neglect deflections due to axial loads—include only deflections due to bending. The members have identical values of *EI*, and let *h*=*L*.



Beam 1: Rotation at A is $\theta_A = \theta_1^A + \frac{PL^2}{6EI}$, where θ_1^A is the rigid rotation of the member at the base. The second term comes from the beam theory result for a simply supported member with a concentrated moment PL/2 acting on one end. There are no displacements at joint A.

At joint B, the lateral deflection is $u_{Bx} = -\theta_1^A L$ and the rotation is $\theta_B = \theta_1^A - \frac{PL^2}{12EI}$. Axial displacement is negligible.

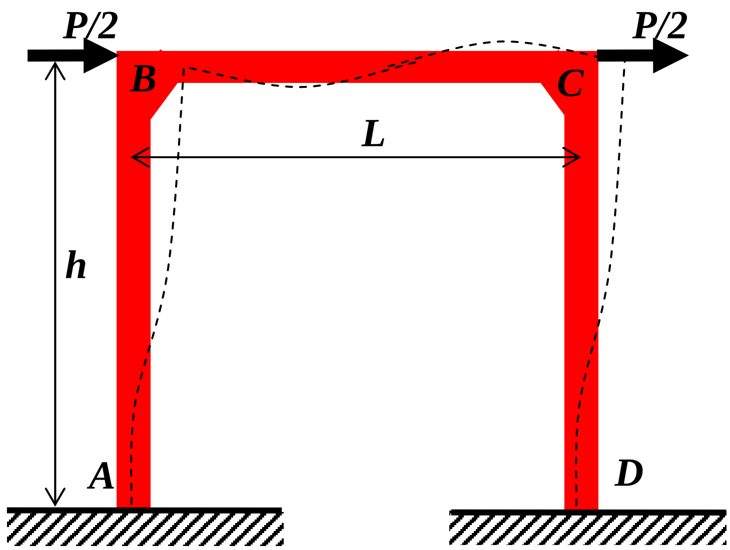
Beam 2: This beam can be thought of as cantilever with an end load. The right-hand support, however, has a rotation θ_B. Thus, $\theta_C = \theta_B + \frac{PL^2}{4EI} = \theta_1^A + \frac{PL^2}{6EI}$. The deflections are

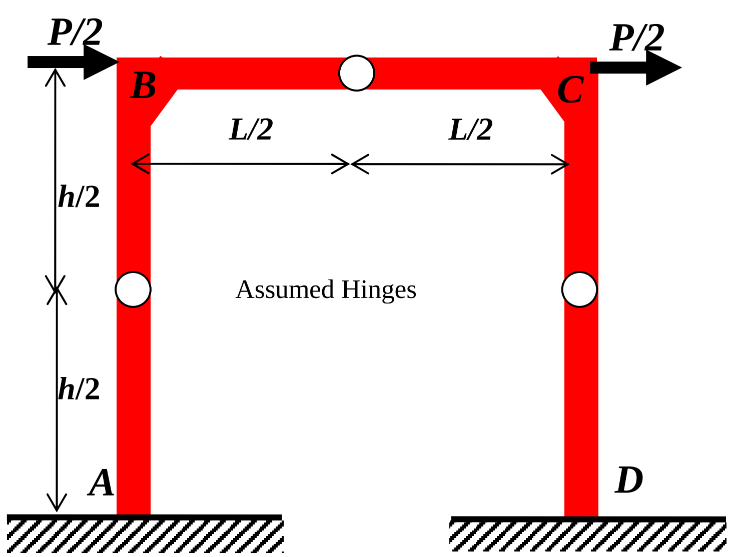
$$u_{Bx} = u_{Cx} = -\theta_1 L,\ \ u_{Cy} = \frac{PL^3}{3EI} + \theta_B L = \theta_1^A L + \frac{PL^3}{3EI}.$$

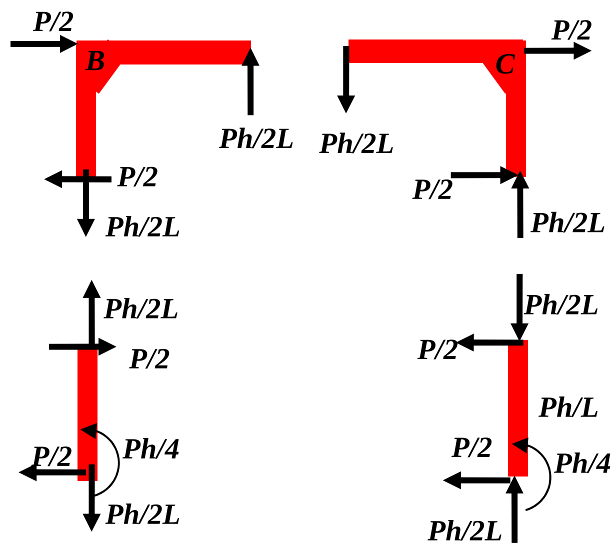
If we use symmetry, then we expect $u_{Cy} = 0$. Thus $\theta_1^A = -\frac{PL^2}{3EI}$. And finally:

$$\theta_A = \theta_1^A + \frac{PL^2}{6EI} = \frac{PL^2}{6EI},\ u_{Bx} = \frac{PL^3}{3EI},\ u_{Cy} = \theta_1^A L + \frac{PL^3}{3EI} = 0.$$

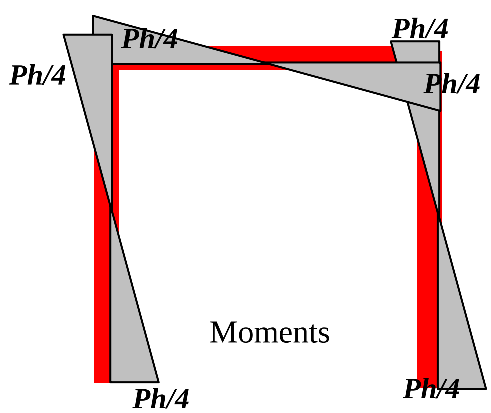
c. If the frame is fixed at the base, rather than hinged, the deflection might looks something like the dotted line shown below. In addition to an infection point at the crossbeam midpoint, there are also inflection points in the columns. Assuming these to lie at the midpoint of each member, fund the reactions and internal moment distribution in the members of the fixed-base frame. Which case gives higher moments?


P/2
P/2
B
C
L
h
A
D


P/2
P/2
B
C
L/2
L/2
h/2
Assumed Hinges
h/2
A
D

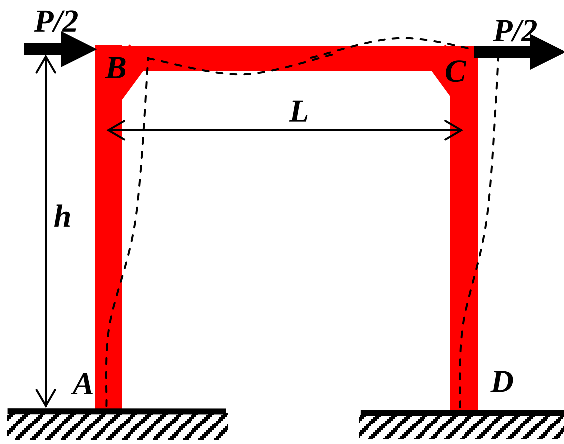


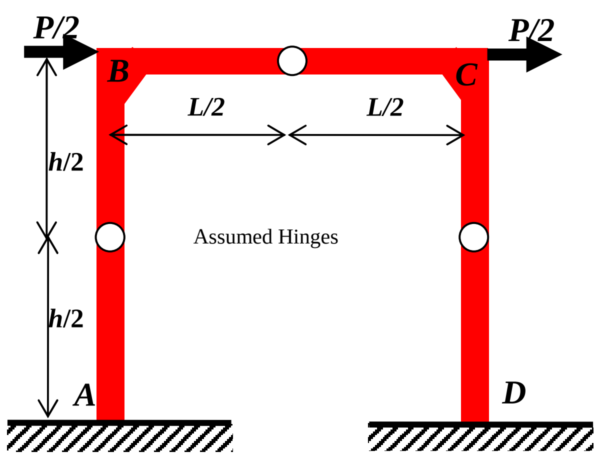
P/2
B
Ph/2L
Ph/2L
C
P/2
P/2
Ph/2L
P/2
Ph/2L
Ph/2L
P/2
Ph/4
P/2
Ph/2L
Ph/2L
P/2
Ph/L
P/2
Ph/4
Ph/2L



Ph/4
Ph/4
Ph/4
Ph/4
Moments
Ph/4
Ph/4

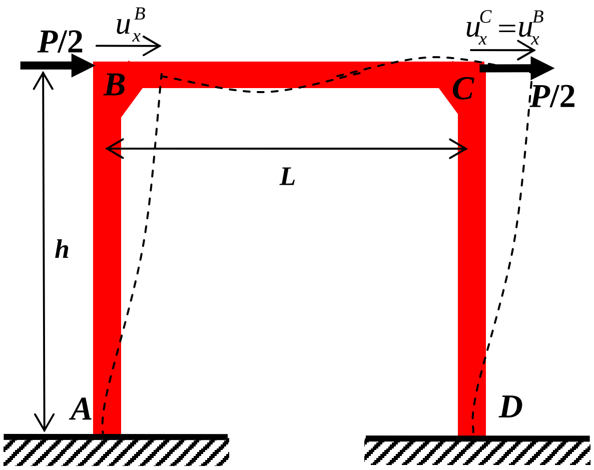
d.



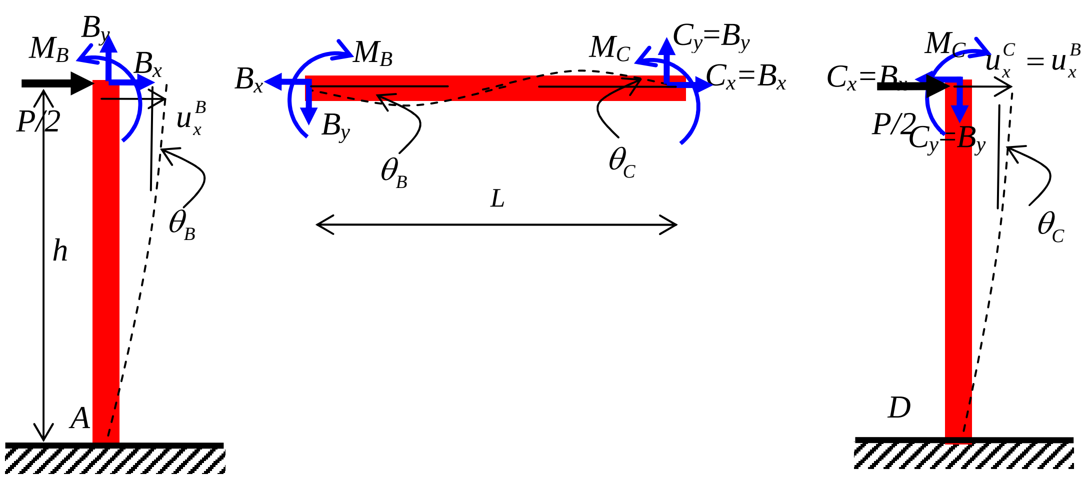


.

3. In this problem you will find the actual deflections and moments in the frame that is approximated in the previous problem. Assume that the properties *EI* are the same in the columns and the crossbeam. Neglect axial deformation throughout, so that $u_x^B = u_x^C$ and $u_y^B = u_y^C = 0$ and assume full moment connections at B and C.



In the figure below, the forces and moments transmitted through joints B and C are shown, along with rotations and displacements of the joints. Note that $\sum \mathbf{F} = \mathbf{0}$ for the cross beam implies $B_x = C_x$ and $B_y = C_y$.



a. Look at the left hand column and relate the forces P and B_x and moment M_B to u_x^B and θ_B. You can do this by looking up or deriving the displacement and slope of a cantilever beam subjected to an end load (P+B_x) and an end moment M_B. **Be very careful about signs!**

b. Derive similar relations between the force C_x=B_x and moment M_C and $u_x^B = u_x^C$ and θ_C by looking at the right hand column.

c. Now consider the cross beam. Relate the moments M_B and M_C to the rotations θ_B and θ_C. You can do this by looking up or deriving the slope of a simply supported beam subjected to end moments M_B and M_C.

d. Finally, use these equations to solve for M_B, B_x, M_c, the rotations θ_B and θ_C, and the lateral displacement $u_x^B = u_x^C$.

e. Look at the crossbeam. Use the equations of static equilibrium to solve for B_y=C_y in terms of M_B.

f. Compare the corner moments M_B obtained here with the approximate result from the previous problem set. Plot $M_B/(Ph/4)$ as a function of L/h. For what range of L/h is the approximate solution valid?

Solutions:

(a) Column AB: end deflection due to load P+B_x:

$$u_B^{(1)} = \frac{\left(P/2 + B_x\right)h^3}{3EI},\ \theta_B^{(1)} = \frac{\left(P/2 + B_x\right)h^2}{2EI}$$

End deflection and rotation due to moment M_B:

$$u_B^{(2)} = -\frac{M_B h^2}{2EI},\ \theta_B^{(2)} = -\frac{M_B h}{EI}.$$

Total $u_x^B \equiv u = u_B^{(1)} + u_B^{(2)},\ \theta_B = \theta_B^{(1)} + \theta_B^{(2)}$

(b) Column CD: end deflection due to load B_x:

$$u_C^{(1)} = \frac{(P/2 - B_x)h^3}{3EI},\ \theta_C^{(1)} = \frac{(P/2 - B_x)h^2}{2EI}$$

End deflection and rotation due to moment M_C:

$$u_C^{(2)} = \frac{M_C h^2}{2EI},\ \theta_C^{(2)} = \frac{M_C h}{EI}.$$

Total $u_x^C \equiv u = u_C^{(1)} + u_C^{(2)},\ \theta_C = \theta_C^{(1)} + \theta_C^{(2)}$

(c) Beam BC: end rotations due to moment M_B:

$$\theta_B^{M_B} = \frac{M_B L}{3EI}, \theta_C^{M_B} = -\frac{M_B L}{6EI}.$$

end rotations due to moment M_B:

$$\theta_B^{M_C} = \frac{M_C L}{6EI}, \theta_C^{M_C} = -\frac{M_C L}{3EI}.$$

Total $\theta_B = \theta_B^{M_B} + \theta_B^{M_C} \quad \theta_C = \theta_C^{M_B} + \theta_C^{M_C}$

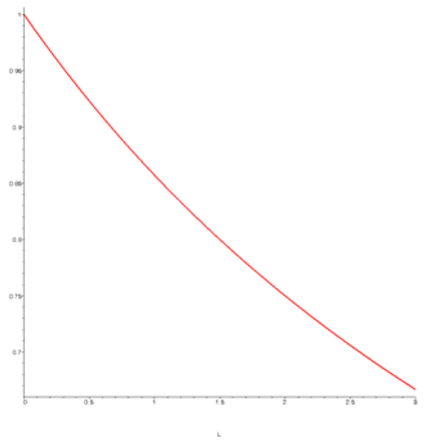
(d) Put all of this into Maple to solve. We have 6 equations for 6 unknowns: M_B, B_x, M_c, $u_x^B = u_x^C$, θ_B and θ_C. In maple, t_B=θ_B, etc.

$$\left\{ tc = \frac{L\ h^2\ P}{4\ ei\ (6\ h + L)},\ tb = \frac{L\ h^2\ P}{4\ ei\ (6\ h + L)},\ ubx = \frac{h^3\ P\ (3\ h + 2\ L)}{12\ ei\ (6\ h + L)},\right.$$
$$\left. Mc = -\frac{3\ h^2\ P}{2\ (6\ h + L)},\ Bx = -\frac{P}{2},\ Mb = \frac{3\ h^2\ P}{2\ (6\ h + L)} \right\}$$

(e) B_y=C_y, $M_C - M_B + C_y L = 0 \Rightarrow C_y = \frac{M_B - M_C}{L} = B_y$:

$$By := \frac{3\ h^2\ P}{(6\ h + L)\ L}$$

(f) $\frac{M_B}{Ph/4} = \frac{1}{1 + L/(6h)}$ is close to one when L<<6h. See plot of $M_B/(Ph/4)$ vs L/h


1
0.95
0.9
0.85
0.8
0.75
0.7
0
0.5
1
1.5
2
2.5
3
L