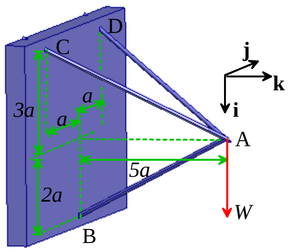
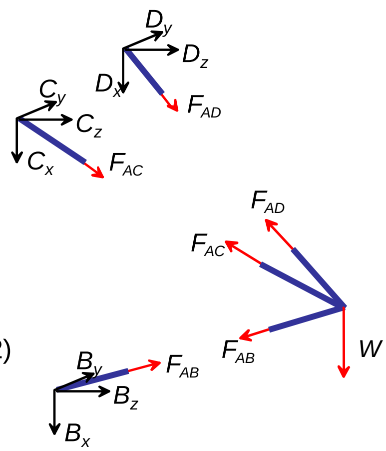
3D problem: $\sum F_x = \sum F_y = \sum F_z = 0$

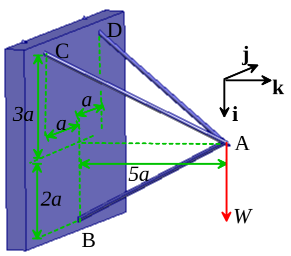
At each joint

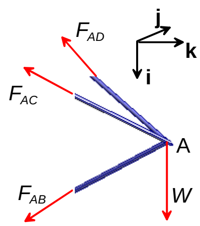




Equations: $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$ at each joint (12)
Unknowns: Total of 12:
Member forces F_{AC}, F_{AD}, F_{AB}, (3)
Reactions : B_x, B_y, B_z, C_x, C_y, C_z, D_x, D_y, D_z (9)

Forces act parallel to the members

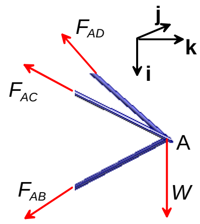




$$F_{AB}\mathbf{n}_{AB} + F_{AC}\mathbf{n}_{AC} + F_{AD}\mathbf{n}_{AD} + W\mathbf{i} = 0$$

Member	Unit Vector pointing away from Joint A.
Member AB	$\mathbf{n}_{AB} = (2a\mathbf{i} - 5a\mathbf{k}) / \sqrt{29a^2} = (2\mathbf{i} - 5\mathbf{k}) / \sqrt{29}$
Member AC	$\mathbf{n}_{AC} = (-3\mathbf{i} - \mathbf{j} - 5\mathbf{k})\sqrt{35}$
Member AD	$\mathbf{n}_{AD} = (-3\mathbf{i} + \mathbf{j} - 5\mathbf{k})\sqrt{35}$

Joint A



$$F_{AB}\mathbf{n}_{AB} + F_{AC}\mathbf{n}_{AC} + F_{AD}\mathbf{n}_{AD} + W\mathbf{i} = 0$$

$$F_{AB}(2\mathbf{i}-5\mathbf{k})/\sqrt{29} + F_{AC}(-3\mathbf{i}-\mathbf{j}-5\mathbf{k})/\sqrt{35} + F_{AD}(-3\mathbf{i}+\mathbf{j}-5\mathbf{k})/\sqrt{35} + W\mathbf{i} = \mathbf{0}$$

$$\Sigma F_{Ax} = 2F_{AB}/\sqrt{29} - 3F_{AC}/\sqrt{35} - 3F_{AD}/\sqrt{35} + W = 0$$

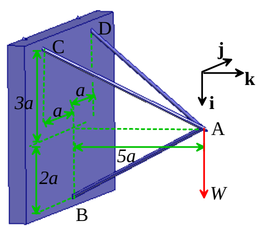
$$\Sigma F_{Ay} = -F_{AC}/\sqrt{35} + F_{AD}/\sqrt{35} = 0$$

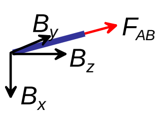
$$\Sigma F_{Az} = -5F_{AB}/\sqrt{29} - 5F_{AC}/\sqrt{35} - 5F_{AD}/\sqrt{35} = 0$$

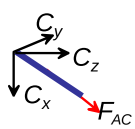
$$F_{AD} = F_{AC} = (\sqrt{35}/10)W \qquad F_{AB} = -(\sqrt{29}/5)W$$

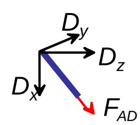
3

Joints B,C,D









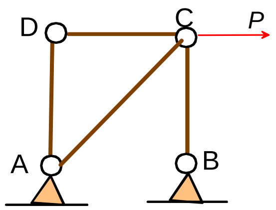
$$F_{AB}\mathbf{n}_{BA} + B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k} = \mathbf{0}$$
$$-F_{AB}\mathbf{n}_{BA} + B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k} = \mathbf{0}$$
$$\sum F_{Bx} = -F_{AB}2/\sqrt{29} + B_x = 0$$
$$\sum F_{By} = B_y = 0$$
$$\sum F_{Bz} = 5F_{AB}/\sqrt{29} + B_z = 0$$

$$F_{AC}\mathbf{n}_{CA} + C_x\mathbf{i} + C_y\mathbf{j} + C_z\mathbf{k} = \mathbf{0}$$
$$-F_{AC}\mathbf{n}_{AC} + C_x\mathbf{i} + C_y\mathbf{j} + C_z\mathbf{k} = \mathbf{0}$$
$$\sum F_{Cx} = -F_{AC}3/\sqrt{35} + C_x = 0$$
$$\sum F_{Cy} = F_{AC}/\sqrt{35} + C_y = 0$$
$$\sum F_{Cz} = 5F_{AC}/\sqrt{35} + C_z = 0$$

$$F_{AD}\mathbf{n}_{DA} + D_x\mathbf{i} + D_y\mathbf{j} + D_z\mathbf{k} = \mathbf{0}$$
$$-F_{AD}\mathbf{n}_{AD} + D_x\mathbf{i} + D_y\mathbf{j} + D_z\mathbf{k} = \mathbf{0}$$
$$\sum F_{Dx} = F_{AD}3/\sqrt{35} + D_x = 0$$
$$\sum F_{Dy} = F_{AD}/\sqrt{35} + D_y = 0$$
$$\sum F_{Dz} = 5F_{AD}/\sqrt{35} + D_z = 0$$

4

Does this procedure always work?



Equations (**8**):
$\Sigma F_x = \Sigma F_y = 0$ at each of the 4 joints

$$A_x + F_{AC}/\sqrt{2} = 0 \qquad (Ax)$$
$$A_y + F_{AD} + F_{AC}/\sqrt{2} = 0 \qquad (Ay)$$
$$B_x = 0 \qquad (Bx)$$
$$B_y + F_{BC} = 0 \qquad (By)$$
$$P - F_{CD} - F_{AC}/\sqrt{2} = 0 \qquad (Cx)$$
$$-F_{CB} - F_{CA}/\sqrt{2} = 0 \qquad (Cy)$$
$$F_{CD} = 0 \qquad (Dx)$$
$$-F_{AD} = 0 \qquad (Dy)$$

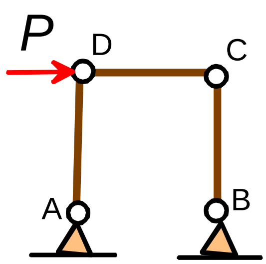
Unknowns (8):
4 Member forces: $F_{AD}, F_{AC}, F_{BC}, F_{CD}$
4 Reaction forces: A_x, A_y, B_x, B_y

$$\{Bx = 0,\ fcd = 0,\ fad = 0,\ fac = P\sqrt{2},\ fbc = -P,\ By = P,\ Ay = -P,\ Ax = -P\}$$

A structure like this is *Statically Determinate*

5

Does this procedure always work?



Equations (**8**):
$\Sigma F_x = \Sigma F_y = 0$ at each of the 4 joints

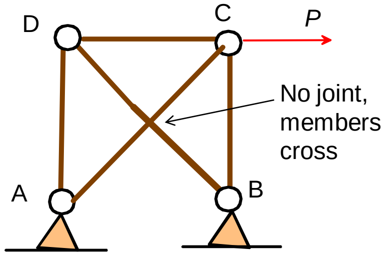
$$A_x = 0 \qquad (Ax)$$
$$A_y + F_{AD} = 0 \qquad (Ay)$$
$$B_x = 0 \qquad (Bx)$$
$$B_y + F_{BC} = 0 \qquad (By)$$
$$-F_{CD} = 0 \qquad (Cx)$$
$$-F_{CB} = 0 \qquad (Cy)$$
$$P + F_{CD} = 0 \qquad (Dx)$$
$$-F_{AD} = 0 \qquad (Dy)$$

Unknowns (7):
3 Member forces: F_{AD}, F_{BC}, F_{CD}
4 Reaction forces: A_x, A_y, B_x, B_y

No solution that satisfies all the equations. This is a *mechanism*

6

Does this procedure always work?



Equations (**8**):
$\Sigma F_x = \Sigma F_y = 0$ at each of the 4 joints

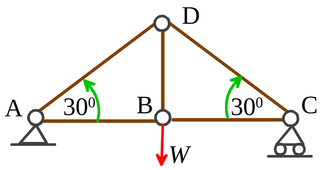
$$A_x + F_{AC}/\sqrt{2} = 0 \qquad (Ax)$$
$$A_y + F_{AD} + F_{AC}/\sqrt{2} = 0 \qquad (Ay)$$
$$B_x - F_{BD}/\sqrt{2} = 0 \qquad (Bx)$$
$$B_y + F_{BC} + F_{BD}/\sqrt{2} = 0 \qquad (By)$$
$$P - F_{CD} - F_{AC}/\sqrt{2} = 0 \qquad (Cx)$$
$$-F_{CB} - F_{AC}/\sqrt{2} = 0 \qquad (Cy)$$
$$F_{CD} + F_{BD}/\sqrt{2} = 0 \qquad (Dx)$$
$$-F_{AD} - F_{BD}/\sqrt{2} = 0 \qquad (Dy)$$

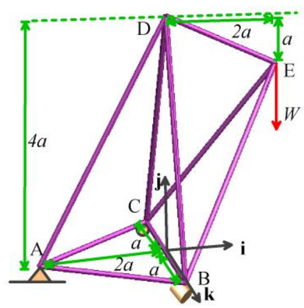
Unknowns (9):
5 Member forces: $F_{AD}, F_{AB}, F_{AC}, F_{BC}, F_{BD}, F_{CD}$
4 Reaction forces: A_x, A_y, B_x B_y

A structure like this is *Statically Indeterminate*

7

Structures for which internal forces can be calculated using equilibrium equations for joints are *statically determinate*





2D structure	3D structure
M = 5 members	M = 9 members
R = 3 reaction force components	R = 6 reaction force components
J = 4 joints	J = 5 joints
No. unknowns = M+R = 8	No. unknowns = M + R = 15
No. equations = 2J=8	No. equations = 3J = 15

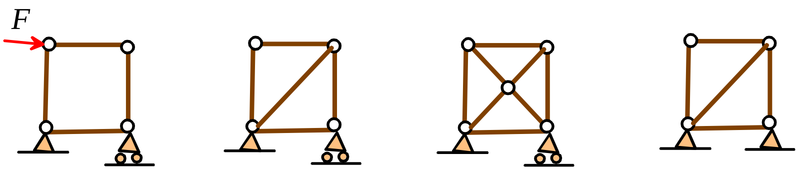
8

How can you tell if a structure is statically determinate, indeterminate or a mechanism?

- *Maxwell's conditions* are a guide (not perfect but easy).
- Procedure – count no. members *M;*
  count no. support force components *R*;
  count no. joints *J.*
- For 2D problem:
  M+R>2J (indeterminate)
  M+R=2J (determinate)
  M+R<2J (mechanism)
- *3D* problem:
  M+R>3J (indeterminate)
  M+R=3J (determinate)
  M+R<3J (mechanism)

9

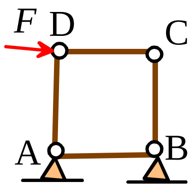
Examples where Maxwell works



M=4 R=3 J=4	*M=5 R=3 J=4*	*M=8 R=3 J=5*	*M=5 R=4 J=4*
M+R=7 2J=8	*M+R=8 2J=8*	*M+R=11 2J=10*	*M+R=9 2J=8*
M+R<2J	*M+R=2J*	*M+R>2J*	*M+R>2J*
Mechanism	Determinate	Indeterminate	Indeterminate

10

An example where Maxwell fails



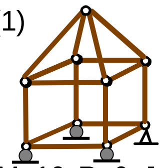
M=4 R=4 J=4
M+R=8 2J=8
M+R=2J
????????

$$R_{Ax} + T_{AB} = 0 \quad (Ax)$$
$$R_{Ay} + T_{AD} = 0 \quad (Ay)$$
$$R_{Bx} - T_{AB} = 0 \quad (Bx)$$
$$R_{By} + T_{BC} = 0 \quad (By)$$
$$-T_{CD} = 0 \quad (Cx)$$
$$-T_{CB} = 0 \quad (Cy)$$
$$F + T_{CD} = 0 \quad (Dx)$$
$$-T_{AD} = 0 \quad (Dy)$$

11

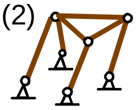
3D

(1)



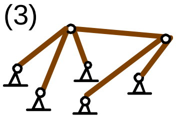
M=16, R=6, J=9
M+R<3J
Mechanism

(2)



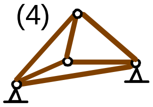
M=7, R=12, J=7
M+R<3J
Mechanism

(3)



M=6, R=15, J=7
M+R=3J
Determinate

(4)

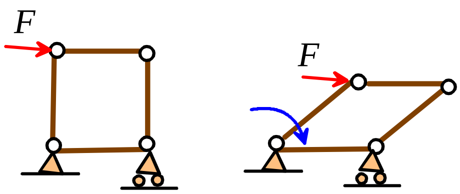


M=6, R=6, J=4
M+R=3J
Determinate?

12

Why do we care if a structure is determinate?

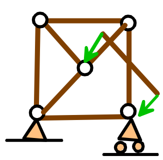
A Mechanism will move if subjected to forces. Sometimes OK.
Bad news if your house is a mechanism, though…



$M=4 \quad R=3 \quad J=4$
$M+R=7 \quad 2J=8$
$M+R<2J$
Mechanism

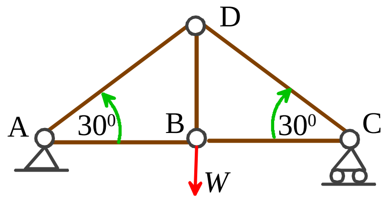
Why do we care?

An indeterminate structure:
1. Is hard to assemble
2. Is hard to analyze
3. Weighs more than it has to
4. May have forces in members even without loading
5. Small motions at joints induce big forces
6. Thermal expansion of some members induce big forces



$M=8 \quad R=3 \quad J=5$
$M+R=11 \quad 2J=10$
$M+R>2J$
Indeterminate

Why do we care…

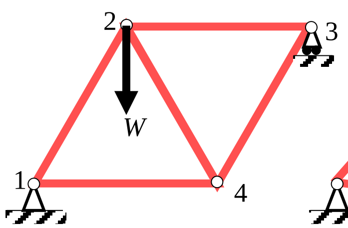


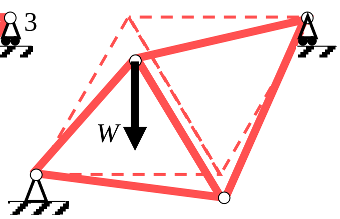
An determinate structure:

1. Won't fall down when it's loaded
2. Is easy to assemble
3. Is easy to analyze (yeah right)
4. Won't have forces in members even without loading
5. Small motions at joints won't induce forces
6. Thermal expansion of some members won't induce big forces

BUT it will become a mechanism if a member fails!

Stiffness Formulation of Truss analysis: forces and deflections

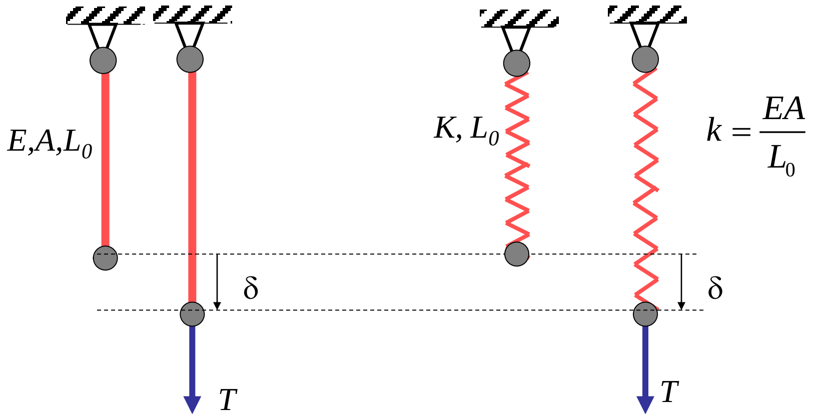




$$\underline{u} = \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ \vdots \end{bmatrix}$$

Fundamental Unknowns
are Nodal (joint) Displacements

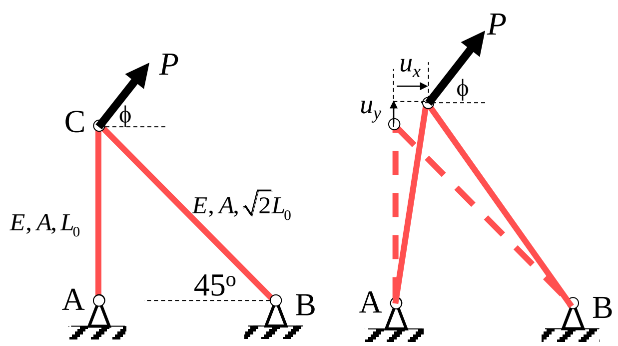
Elastic rods are linear springs



$$k\delta = T$$

$$\sigma = \frac{T}{A} = E\frac{\delta}{L_o} \Rightarrow \frac{EA}{L_0}\delta = T$$

Example



Method of Joints:

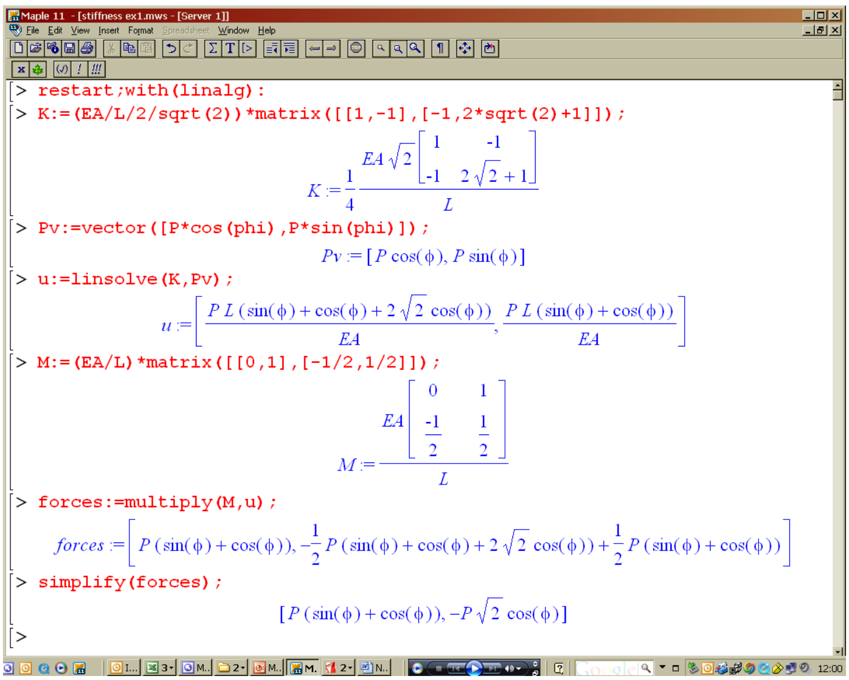
$$\sum F_{Cx} = F_{BC}/\sqrt{2} + P\cos\phi = 0$$

$$\sum F_{Cy} = -F_{AC} - F_{BC}/\sqrt{2} + P\sin\phi = 0$$

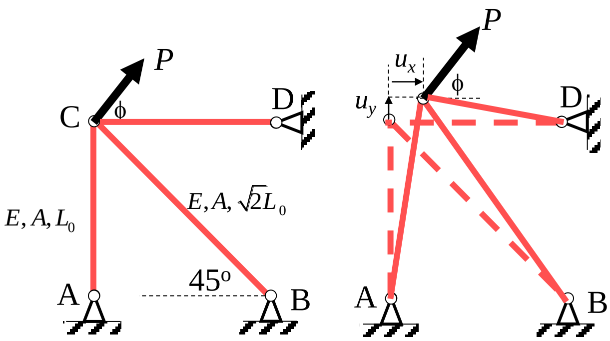
$$F_{AC} = P(\sin\phi + \cos\phi),\ F_{BC} = -P\sqrt{2}\cos\phi$$

Stiffness Matrix

- Deflection in AC $\delta_{AC} = u_y$
- Force in AC $F_{AC} = \dfrac{EA}{L_0} u_y$
- Deflection in BC $\delta_{BC} = \dfrac{1}{\sqrt{2}}\left(u_y - u_x\right)$
- Force in BC: $F_{BC} = \dfrac{EA}{2L_0}\left(u_y - u_x\right)$
- Matrix Equation: $\dfrac{EA}{2\sqrt{2}L_0}\begin{bmatrix} 1 & -1 \\ -1 & 1+2\sqrt{2} \end{bmatrix}\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} P\cos\phi \\ P\sin\phi \end{bmatrix}$



Works for Statically indeterminate too.



In General

1. Express M member elongations δ_i in terms of $\underline{u}$ (M eqns)
2. Enforce the boundary constraints on the displacements (R eqns)
3. Express M member forces F^{ij} in terms of δ_i (M eqns)
4. Enforce static equilibrium at each node (2J or 3J eqns)

Total number of equations: 2M+R+DJ

Unknowns

Nodal displacements $\underline{u}$: DJ

Member elongations δ_i: M

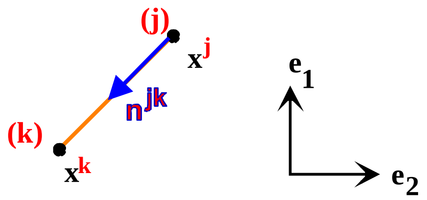
Member forces *Fij* : **M**

Reaction forces R^k: *R*

Total number of unknowns: 2M+R+DJ

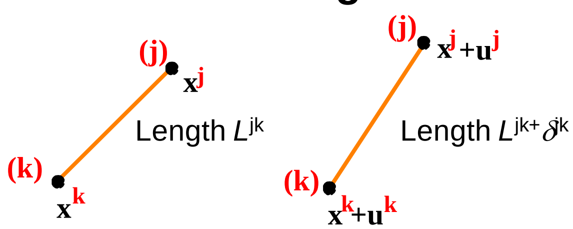
Member Forces and elongations in terms of ***u***:

For a member connecting joints (j) and (k):



$$L^{jk} = \left|(\mathbf{x}^k - \mathbf{x}^j)\right|, \quad \mathbf{n}^{jk} = (\mathbf{x}^k - \mathbf{x}^j) / L^{jk}$$

Member Elongations



Before Displacement **After Displacement**

$$L^{jk} + \delta^{jk} \equiv \left|\mathbf{x}^k + \mathbf{u}^k - \mathbf{x}^j - \mathbf{u}^j\right| =$$

$$\sqrt{(\mathbf{x}^k + \mathbf{u}^k - \mathbf{x}^j - \mathbf{u}^j)\bullet(\mathbf{x}^k + \mathbf{u}^k - \mathbf{x}^j - \mathbf{u}^j)}$$

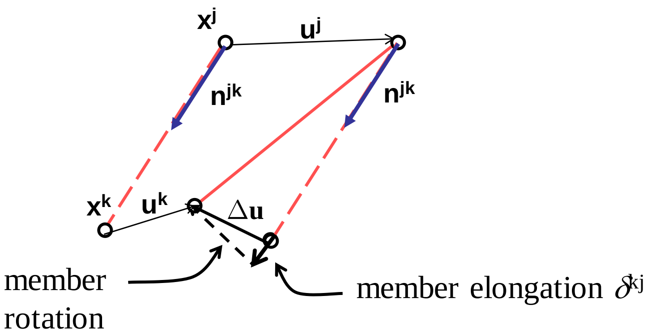
$$= \sqrt{\left|\mathbf{x}^k - \mathbf{x}^j\right|^2 + 2(\mathbf{x}^k - \mathbf{x}^j)\bullet(\mathbf{u}^k - \mathbf{u}^j) + \left|\mathbf{u}^k - \mathbf{u}^j\right|^2}$$

$$\approx L^{jk}\left(1 + (\mathbf{u}^k - \mathbf{u}^j)\bullet(\mathbf{x}^k - \mathbf{x}^j) / L^{jk}\right)$$

$$= L^{jk} + (\mathbf{u}^k - \mathbf{u}^j)\bullet\mathbf{n}^{jk}$$

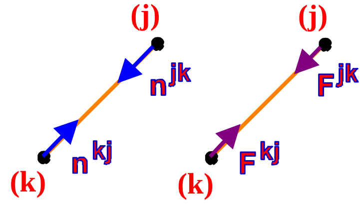
Finally

$$\delta^{jk} \equiv \delta^{kj} = (\mathbf{u}^{k} - \mathbf{u}^{j}) \cdot \mathbf{n}^{jk}$$



$$\mathbf{n}^{jk} = (\mathbf{x}^{k} - \mathbf{x}^{j}) / L^{jk}, \quad L^{jk} = \left| \mathbf{x}^{k} - \mathbf{x}^{j} \right|$$

Axial Force in the member:

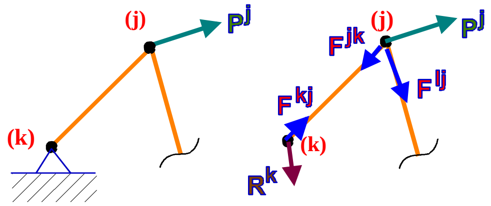


$$F^{jk} = \frac{E^{jk} A^{jk}}{L^{jk}} \delta^{jk} \equiv F^{kj} = \left(\frac{EA}{L} \right)^{jk} (\mathbf{u}^{k} - \mathbf{u}^{j}) \cdot \mathbf{n}^{jk}$$

Vectorial force on joint j through the member jk:

$$\mathbf{F}^{jk} = F^{jk} \mathbf{n}^{jk}$$

Static Equilibrium: The net force on every joint is zero.



P^j is the externally applied force (known) on joint *j*
R^k is the reaction force (unknown) on joint *k*

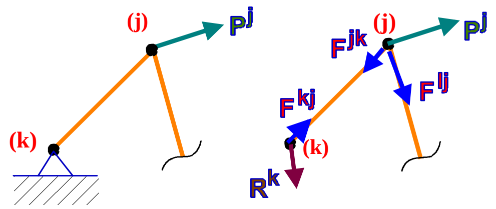
$$\sum_{k=1}^{N} \mathbf{F}^{jk} + \mathbf{P}^{j} + \mathbf{R}^{j} = \mathbf{0} \quad j = 1, 2, 3, \ldots N$$

$$\sum_{k=1}^{N} \left(\frac{EA}{L}\right)^{jk} \mathbf{n}^{jk} \cdot (\mathbf{u}^{k} - \mathbf{u}^{j})\mathbf{n}^{jk} + \mathbf{P}^{j} + \mathbf{R}^{j} = \mathbf{0} \quad j = 1, 2, 3, \ldots N$$

Constraints

For each reaction force, there is a corresponding constraint:
Using prescribed displacement information,
the unknown reactions R^j, (j=1,2,…*N*) can be written in terms of the nodal displacements.

$$\mathbf{u}^{k} = \mathbf{0}$$



$$\sum_{j=1}^{N} \left(\frac{EA}{L}\right)^{jk} \mathbf{n}^{kj} \cdot (\mathbf{u}^{j} - \cancel{\mathbf{u}^{k}})\mathbf{n}^{kj} + \cancel{\mathbf{P}^{k}} + \mathbf{R}^{k} = \mathbf{0} \Rightarrow$$

$$\mathbf{R}^{k} = -\sum_{j=1}^{N} \left(\frac{EA}{L}\right)^{jk} \left(\mathbf{n}^{kj} \cdot \mathbf{u}^{j}\right)\mathbf{n}^{kj}$$

The end results is a system of linear equations for the nodal displacements:

$$[\mathbf{K}]\underline{\mathbf{u}} = \underline{\mathbf{r}}$$

The coefficient matrix [K] and right-hand side vector r are fully determined by:

- **Truss geometry (x^k)**
- **Material properties $(EA)^k$**
- **Applied loads P^k**
- **Prescribed Displacements**

And If You Try Some Time…

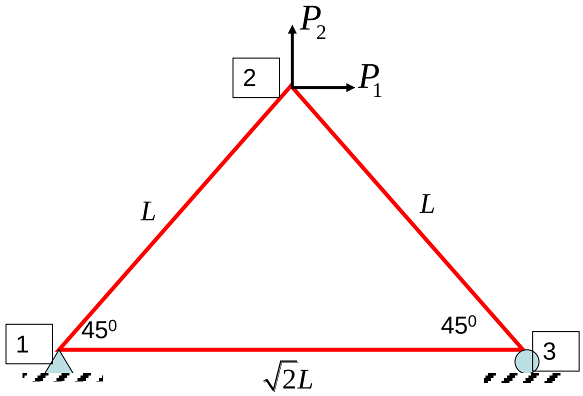
$$\underline{\mathbf{u}} = [\mathbf{K}]^{-1}\underline{\mathbf{r}}$$

You Might Find…

You Get What You Need:

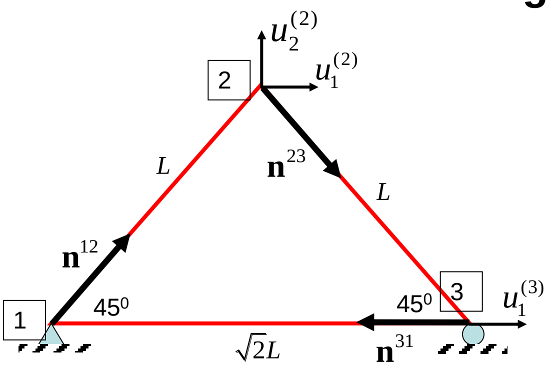
- Element Forces
- Element strains
- Reaction Forces

EXAMPLE



All members have equal cross section and material *EA*

Member Elongations



$$\mathbf{n}^{12} = \frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})$$

$$\mathbf{n}^{23} = \frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j})$$

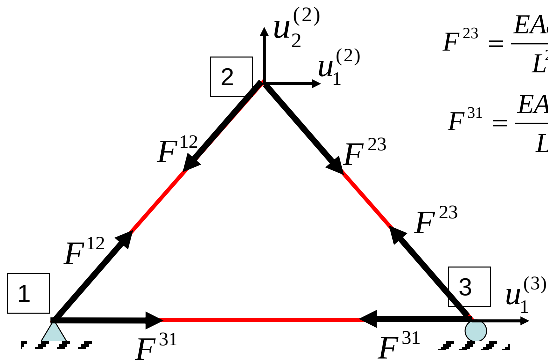
$$\mathbf{n}^{31} = -\mathbf{i}$$

$$\delta^{12} = (\mathbf{u}^{(2)} - \cancel{\mathbf{u}^{(1)}})\cdot\mathbf{n}^{12} = \frac{\sqrt{2}}{2}\left(u_1^{(2)} + u_2^{(2)}\right)$$

$$\delta^{23} = (\mathbf{u}^{(3)} - \mathbf{u}^{(2)})\cdot\mathbf{n}^{23} = \frac{\sqrt{2}}{2}\left(u_1^{(3)} - u_1^{(2)} - (\cancel{u_2^{(3)}} - u_2^{(2)})\right)$$

$$\delta^{31} = (\cancel{\mathbf{u}^{(1)}} - \mathbf{u}^{(3)})\cdot\mathbf{n}^{31} = u_1^{(3)}$$

Member Forces

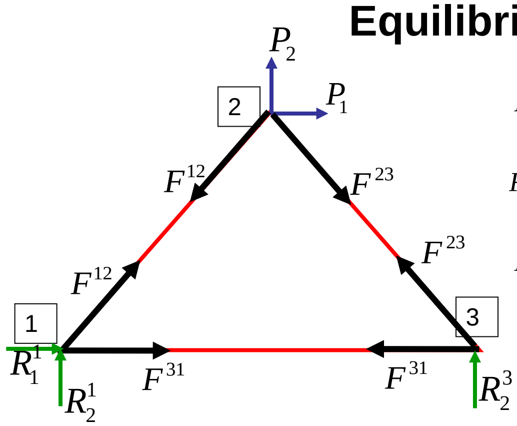


$$F^{12} = \frac{EA\delta^{12}}{L^{12}} = \frac{\sqrt{2}EA}{2L}\left(u_1^{(2)} + u_2^{(2)}\right)$$

$$F^{23} = \frac{EA\delta^{23}}{L^{23}} = \frac{\sqrt{2}EA}{2L}\left(u_1^{(3)} - u_1^{(2)} + u_2^{(2)}\right)$$

$$F^{31} = \frac{EA\delta^{31}}{L^{31}} = \frac{\sqrt{2}EA}{2L}u_1^{(3)}$$

Equilibrium



$$F^{12} = \frac{EA\delta^{12}}{L^{12}} = \frac{\sqrt{2}EA}{2L}\left(u_1^{(2)} + u_2^{(2)}\right)$$

$$F^{23} = \frac{EA\delta^{23}}{L^{23}} = \frac{\sqrt{2}EA}{2L}\left(u_1^{(3)} - u_1^{(2)} + u_2^{(2)}\right)$$

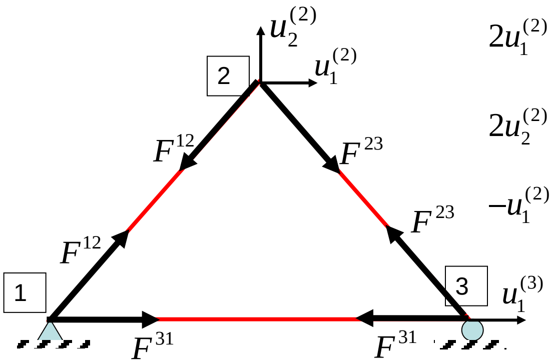
$$F^{31} = \frac{EA\delta^{31}}{L^{31}} = \frac{\sqrt{2}EA}{2L}u_1^{(3)}$$

$$P_1 + \left(F^{23} - F^{12}\right)\Big/\sqrt{2} = 0,\ P_2 + \left(F^{23} + F^{12}\right)\Big/\sqrt{2} = 0$$

$$R_1^1 + F^{31} + F^{12}/\sqrt{2} = 0,\ R_2^1 + F^{12}/\sqrt{2} = 0$$

$$-F^{31} - F^{23}/\sqrt{2} = 0,\ R_2^3 + F^{23}/\sqrt{2} = 0$$

Equations for <u>u</u>



$$2u_1^{(2)} - u_2^{(3)} = \frac{2P_1L}{EA}$$

$$2u_2^{(2)} + u_2^{(3)} = \frac{2P_2L}{EA}$$

$$-u_1^{(2)} + u_2^{(2)} + \left(1+\sqrt{2}\right)u_2^{(3)} = 0$$

Matrix Form

$$\frac{EA}{2L}\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1+\sqrt{2} \end{bmatrix}\begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ 0 \end{bmatrix}$$

$$[\mathbf{K}]\underline{\mathbf{u}} = \underline{\mathbf{r}}$$

$$U := \left[\frac{(\sqrt{2}+4)\,(-2\sqrt{2}\,P2 + 7\,P1 + P2)}{56}, -\frac{\sqrt{2}\,(P1 - P2 - 2\sqrt{2}\,P2)}{8}, \frac{\sqrt{2}\,(P1 - P2)}{4} \right]$$

$$\underline{u} = \frac{2L}{EA}\underline{U}$$

Member forces

$$F^{12} = \frac{EA\delta^{12}}{L^{12}} = \frac{\sqrt{2}EA}{2L}\left(u_1^{(2)} + u_2^{(2)}\right)$$

$$F^{23} = \frac{EA\delta^{23}}{L^{23}} = \frac{\sqrt{2}EA}{2L}\left(u_1^{(3)} - u_1^{(2)} + u_2^{(2)}\right)$$

$$F^{31} = \frac{EA\delta^{31}}{L^{31}} = \frac{\sqrt{2}EA}{2L}u_1^{(3)}$$

$$\begin{bmatrix} F^{12} \\ F^{23} \\ F^{31} \end{bmatrix} = \frac{\sqrt{2}EA}{2L}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \end{bmatrix}$$

$$F := \left[\frac{3}{4}P1 - \frac{3}{4}P2 + \frac{1}{2}\sqrt{2}\,P1, \frac{\sqrt{2}\,P2}{2} - \frac{\sqrt{2}\,P1}{2}, \frac{P1}{2} - \frac{P2}{2}\right]$$

What if the structure is a mechanism?

You can follow the procedure and arrive at the system of equations:

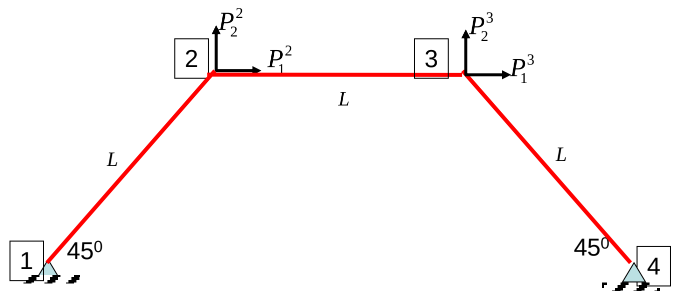
$$[\mathbf{K}]\underline{\mathbf{u}} = \underline{\mathbf{r}}$$

Since nonzero displacements can occur without inducing member forces, matrix **K** will be singular.

•The number of zero eigenvalues corresponds to the degree of indeterminacy: (number of missing members or reactions.

•Null vectors of **K** correspond to the motion allowed by the mechanism

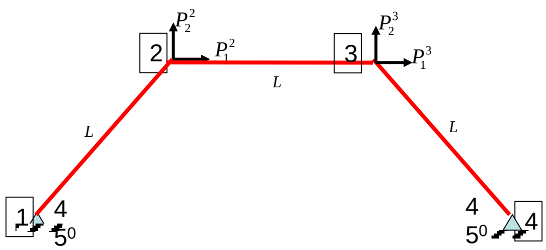
EXAMPLE



All members have equal cross section and material *EA*

$$\mathbf{n}^{12} = \frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j}),\ \mathbf{n}^{23} = \mathbf{i},\ \mathbf{n}^{34} = \frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j})$$

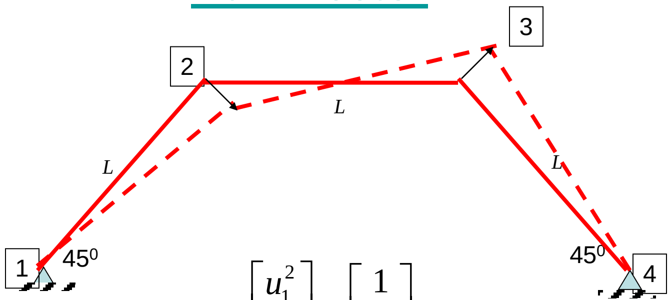
Stiffness



$$K\begin{bmatrix} 3/2 & 1/2 & -1 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ -1 & 0 & 3/2 & 1/2 \\ 0 & 0 & -1/2 & 1 \end{bmatrix}\begin{bmatrix} u_1^2 \\ u_2^2 \\ u_1^3 \\ u_2^3 \end{bmatrix} = \begin{bmatrix} P_1^2 \\ P_2^2 \\ P_1^3 \\ P_2^3 \end{bmatrix}$$

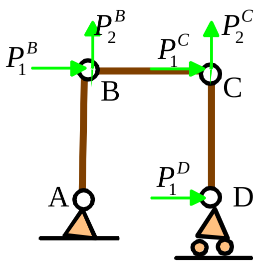
Eigenvalues: $0,\ 1,\ \frac{3}{2}+\frac{\sqrt{5}}{2},\ \frac{3}{2}-\frac{\sqrt{5}}{2}$

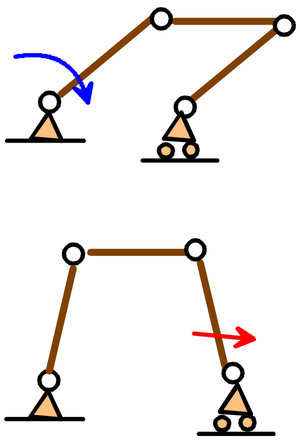
Null Vector



$$\begin{bmatrix} u_1^2 \\ u_2^2 \\ u_1^3 \\ u_2^3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

Example

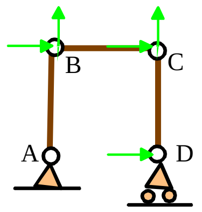




M=3 *R*=3 *J*=4
M+*R*=6 2*J*=8
M+*R*<2*J*
Mechanism

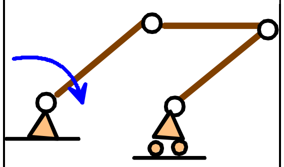
K has two zero eigenvalues;

$$K\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1^B \\ u_2^B \\ u_1^C \\ u_2^C \\ u_1^D \end{bmatrix} = \begin{bmatrix} P_1^B \\ P_2^B \\ P_1^C \\ P_2^C \\ P_1^D \end{bmatrix}$$



Null vectors are:

$$\begin{bmatrix} u_1^B \\ u_2^B \\ u_1^C \\ u_2^C \\ u_1^D \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} u_1^B \\ u_2^B \\ u_1^C \\ u_2^C \\ u_1^D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$