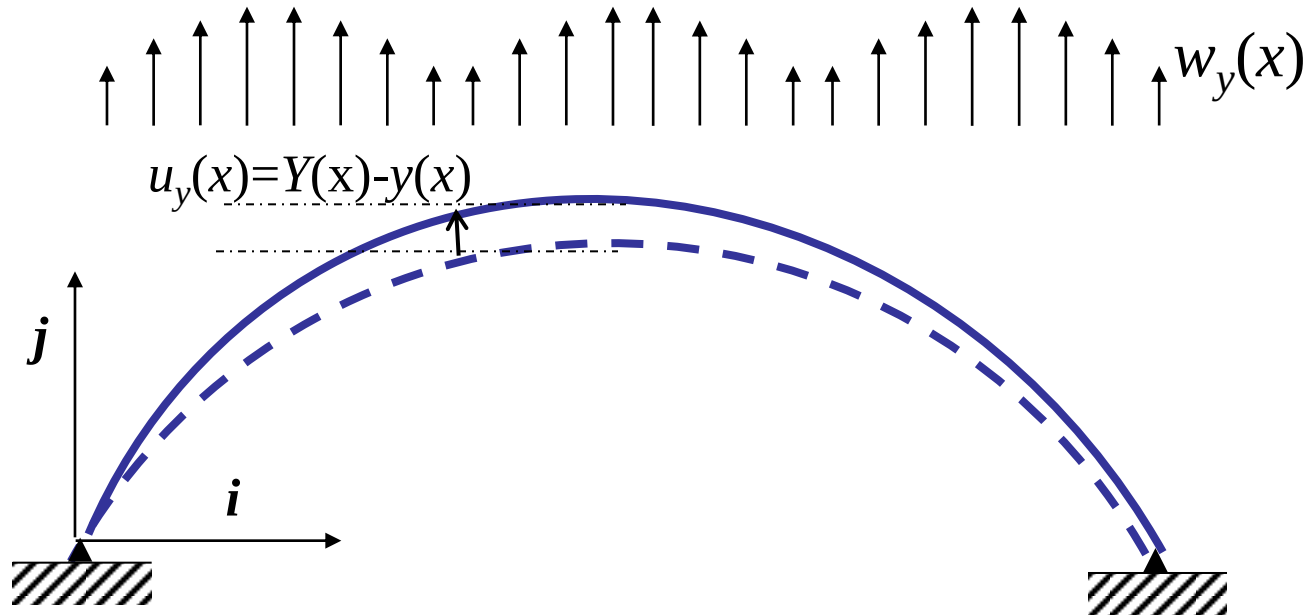


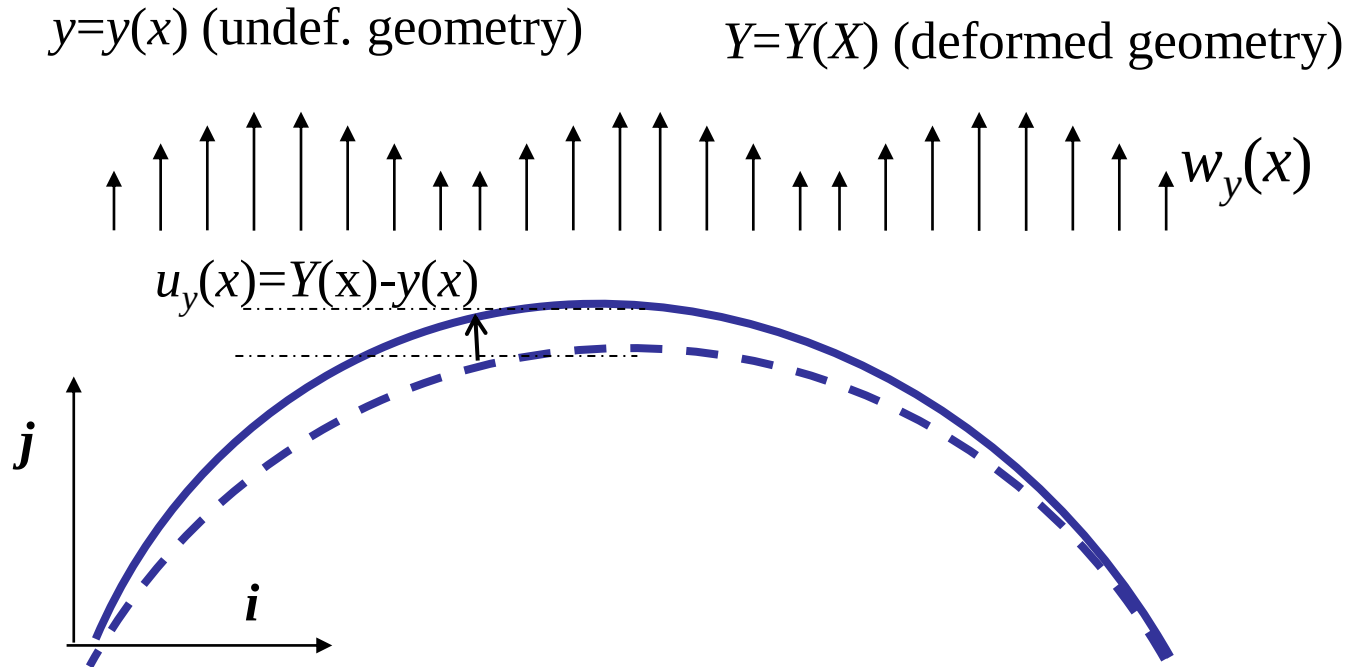
Arch Deflections: Energy Methods

$y=y(x)$ (undef. geometry)

$Y=Y(X)$ (deformed geometry)



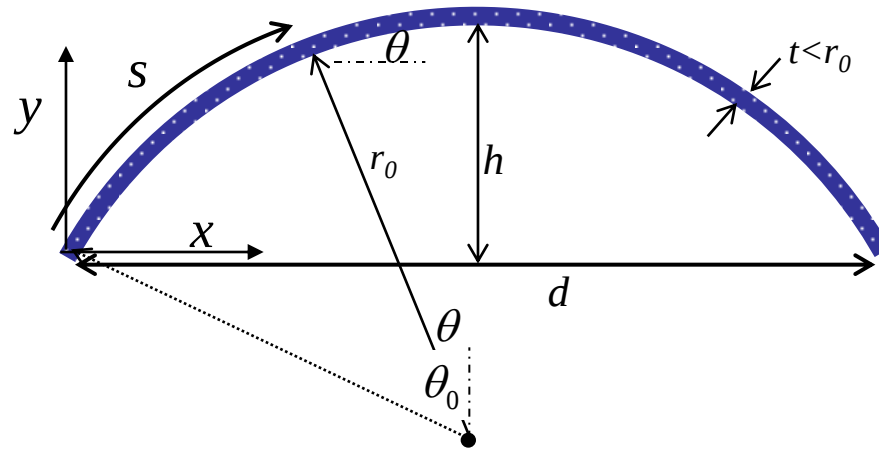
Energy due to the load w_y :



$$V^{\text{load}} = -\int w_y(x) u_y(x) dx$$

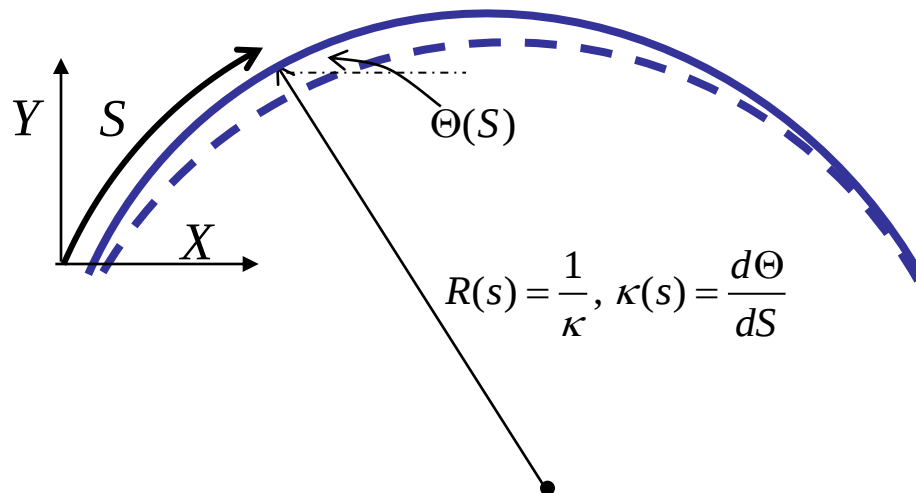
Strain Energy

Undeformed geometry $y=y(x)$: constant curvature κ_0



$$\theta(s) = \theta_0 - \frac{s}{r_0}; \quad \kappa_0 = \frac{d\theta}{ds} = -\frac{1}{r_0}$$

Deformed geometry: $Y=Y(X)$ Curvature $\kappa(s)$



$$R(s) = \frac{1}{\kappa}, \quad \kappa(s) = \frac{d\Theta}{dS}$$

Moment is

$$M = EI (\kappa - \kappa_0)$$

Strain Energy is

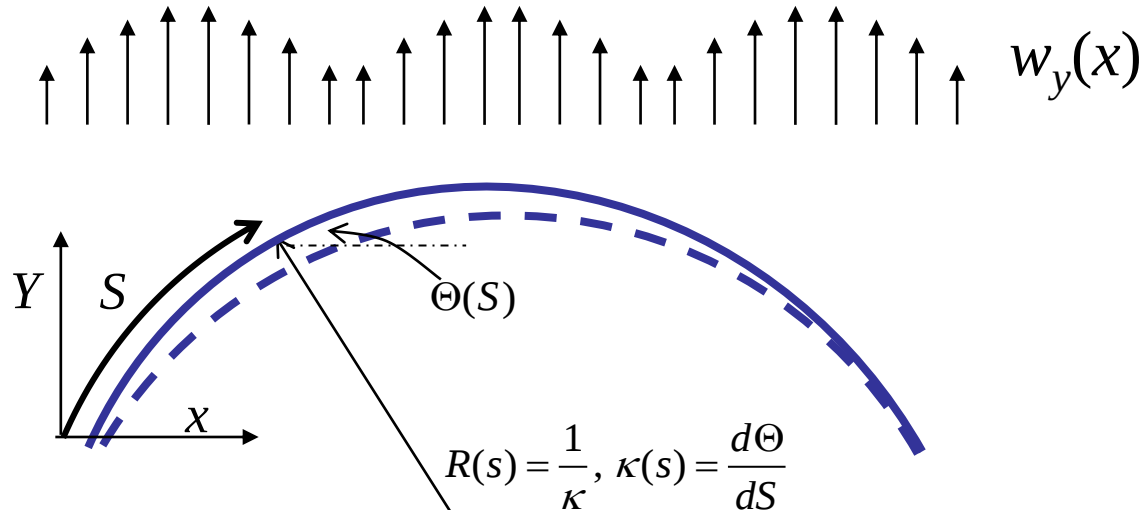
$$V = \int \frac{1}{2} EI (\kappa_0 - \kappa)^2 ds + \int \frac{1}{2} EA \left(\frac{dS}{ds} \right)^2 ds$$

Total Energy to be minimized

$y=y(x)$ (undef. geometry)

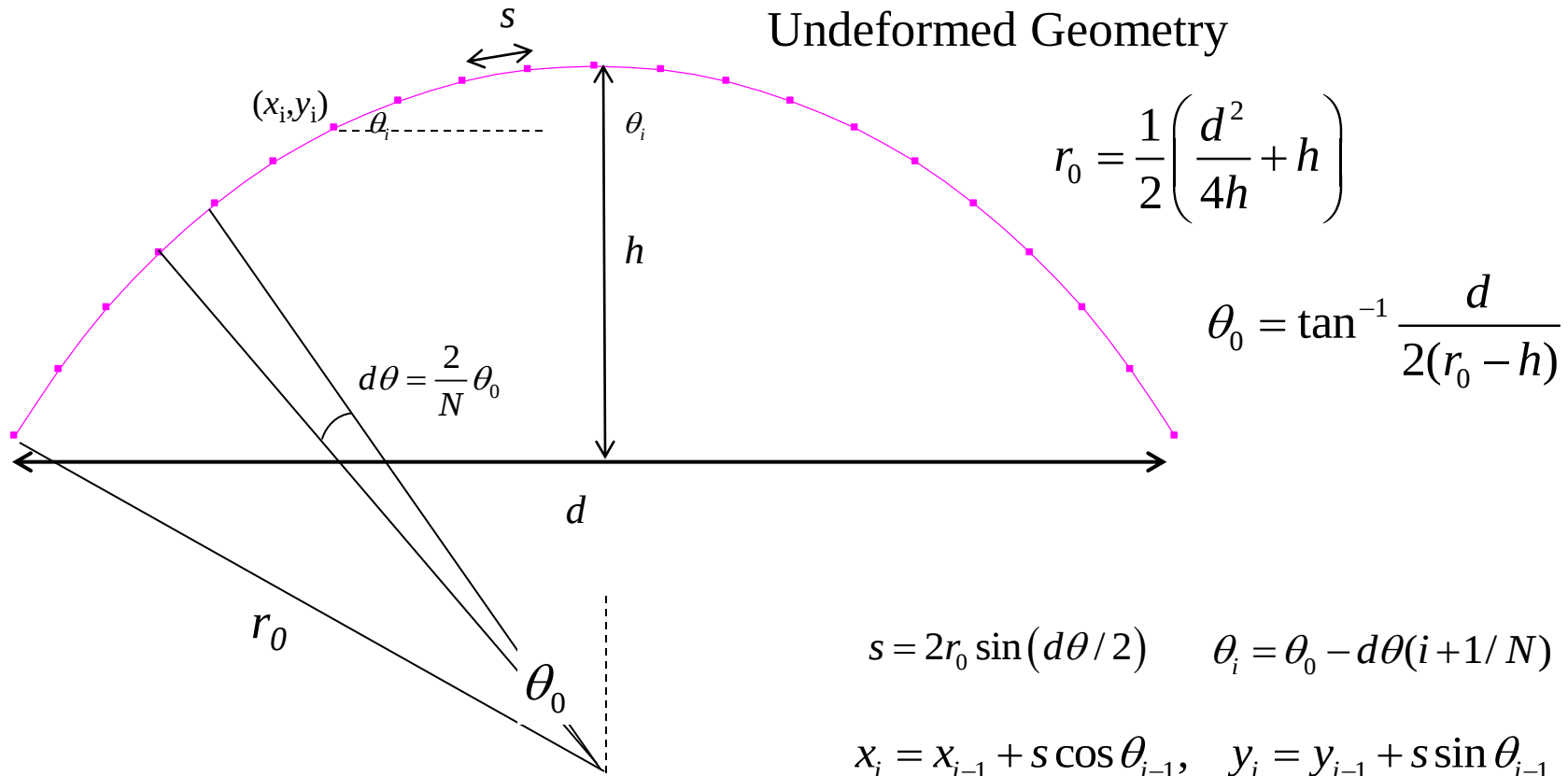
$Y=Y(X)$ (deformed geometry)

$$u_y(x) = Y(x) - y(x)$$

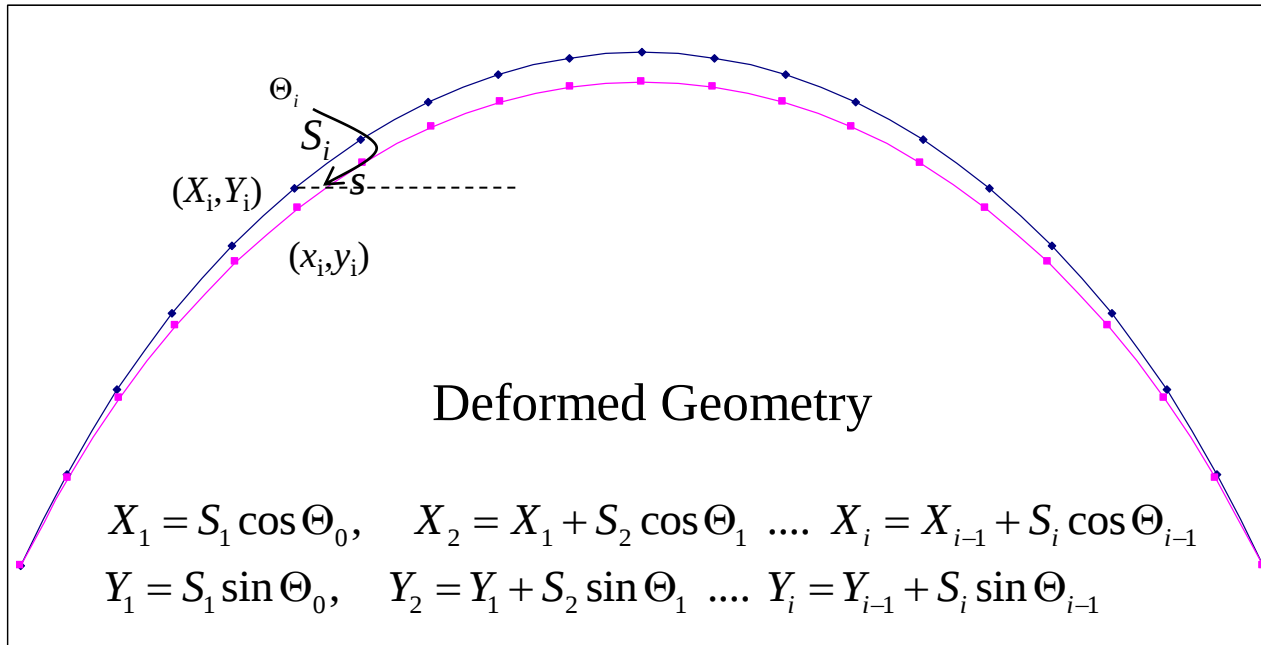


$$V = \int \frac{1}{2} EI (\kappa_0 - \kappa)^2 ds + \int \frac{1}{2} EI \left(\frac{dS}{ds} \right)^2 ds - \int w_y(x) u_y(x) dx$$

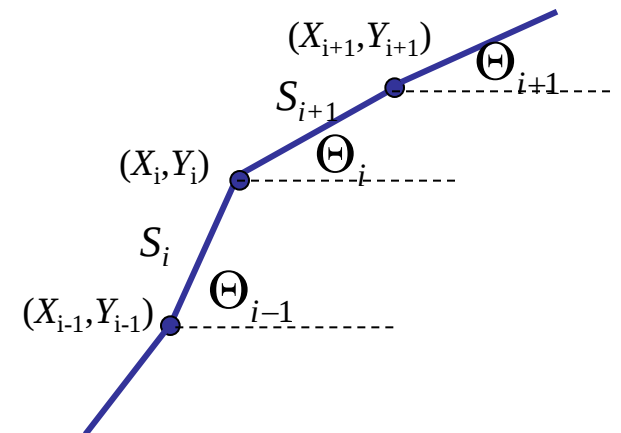
Undeformed Geometry: N segments



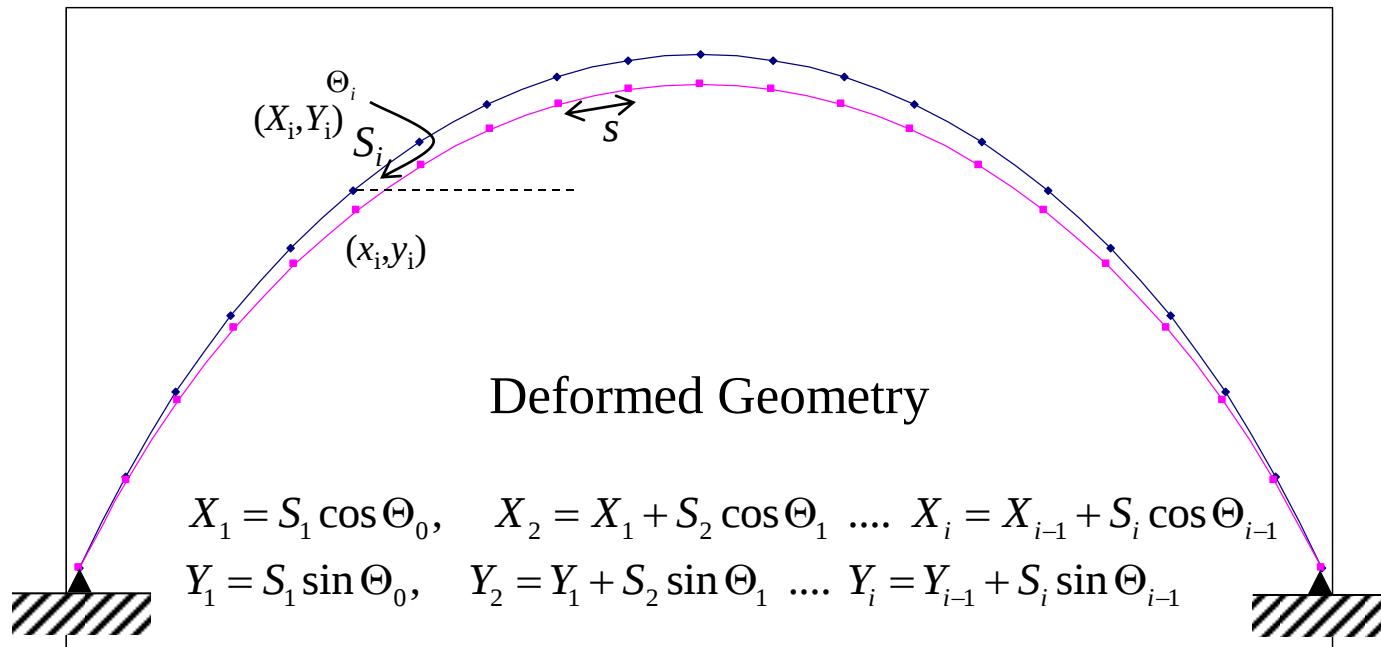
Ugly...



- Curvature in segment i is $\kappa_i = (\Theta_{i-1} - \Theta_i) / S_i$
- Moment at i is $M_i = EI(\kappa_i - \kappa_0)$ $\kappa_0 = 1/r_0$
- Axial force for segment i is $EA(s - S_i)/s$
- Deflection at point i is $u_y = Y_i - y_i$
- Load at i is W_i

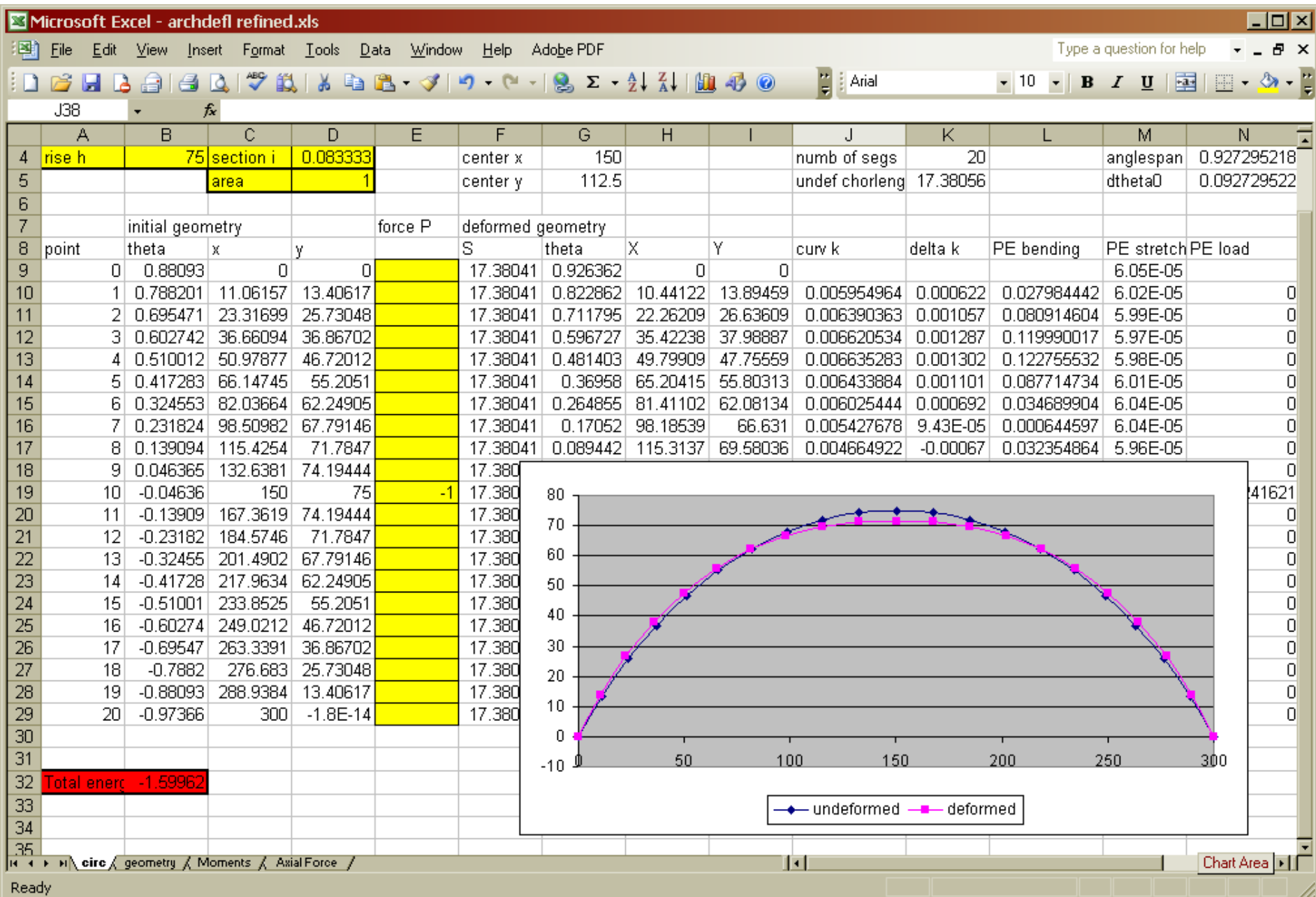


Minimize V , Varying Θ_i, S



$$V = \frac{1}{2} EI \sum_{i=1}^{N-1} s \left(\frac{1}{r_o} - \kappa_i \right)^2 + \frac{1}{2} EA \sum_{i=0}^{N-1} s \left(\frac{S_i - s}{s} \right)^2 - \sum_{i=0}^N W_i u_i$$

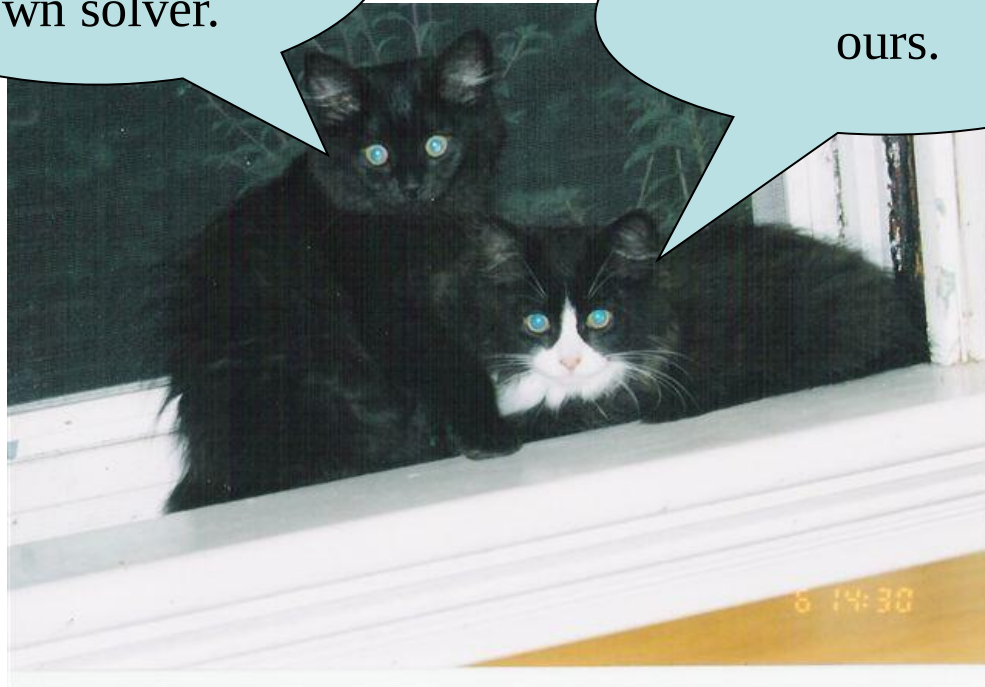
Constraints from BC.



Don't Worry, Be Happy

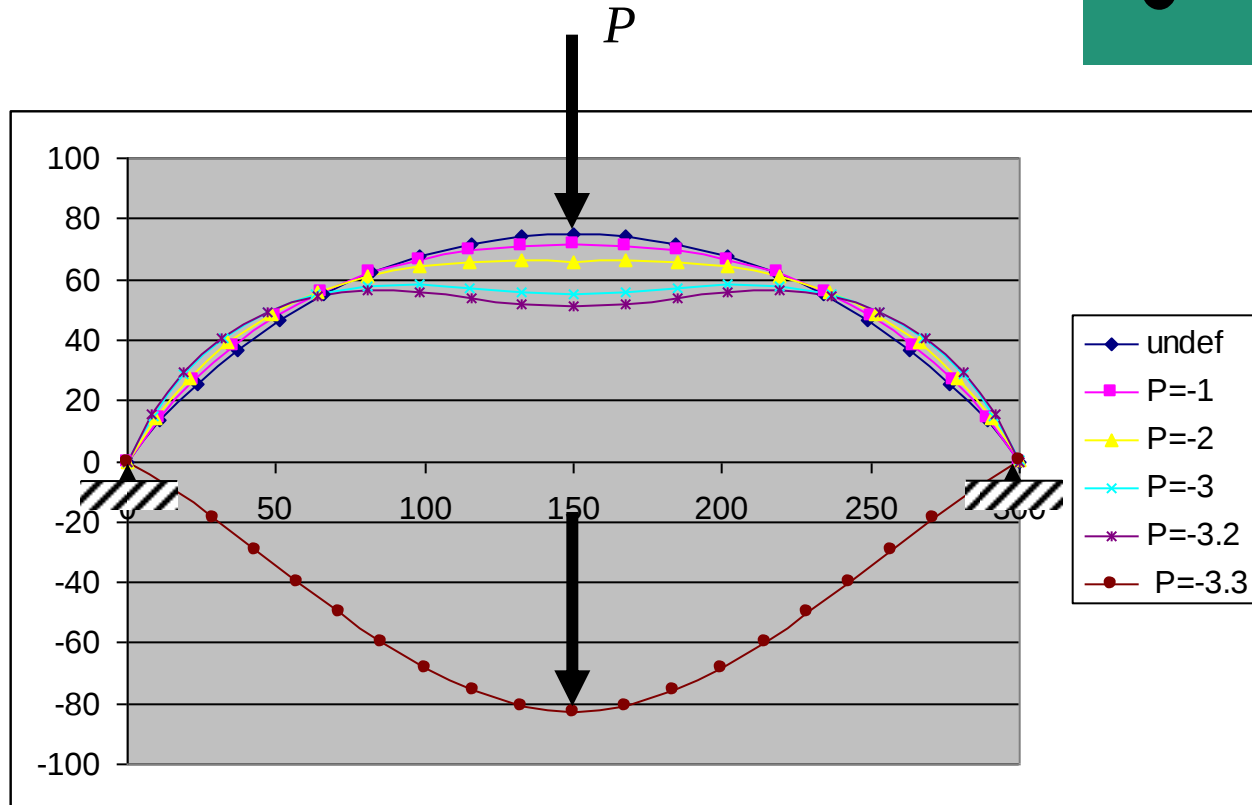
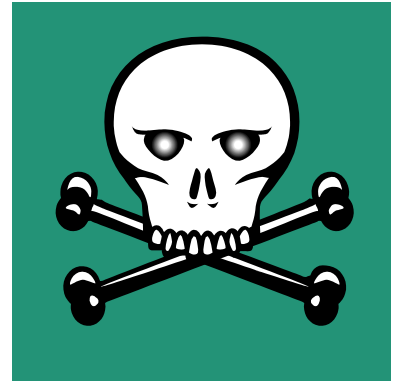
You don't have
to write your
own solver.

You can have
ours.

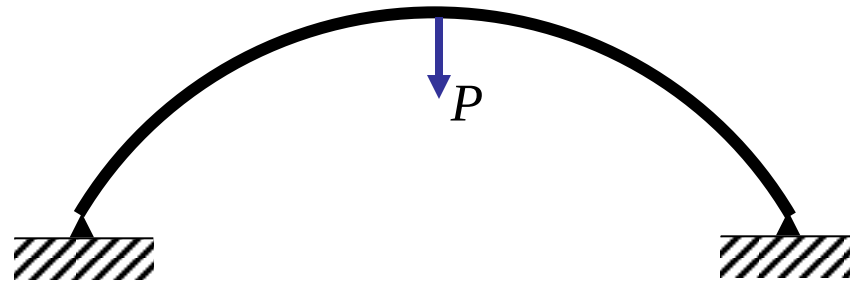


Beware Snap-through Buckling!

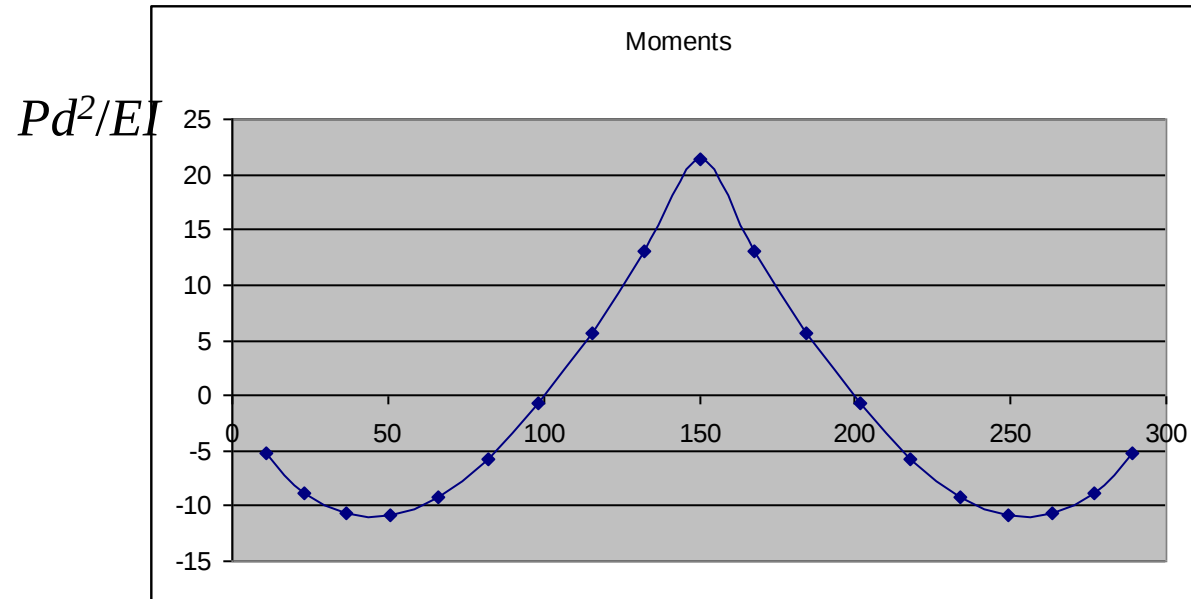
$$E=10^5, I=1/12, A=1, d=300, h=70$$



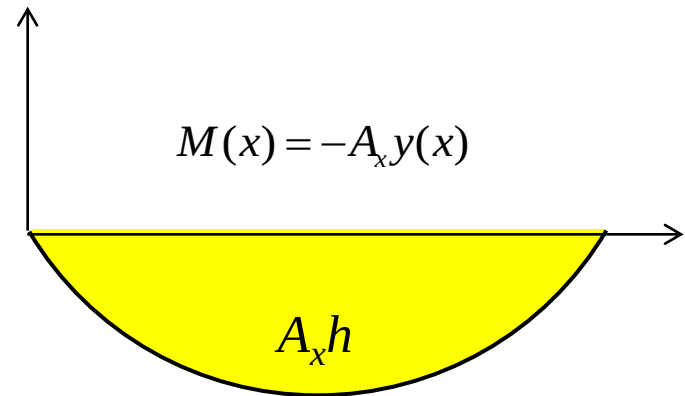
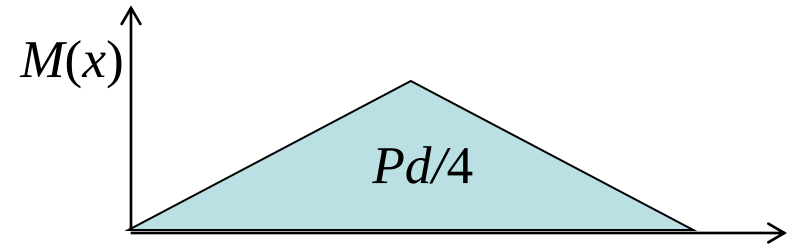
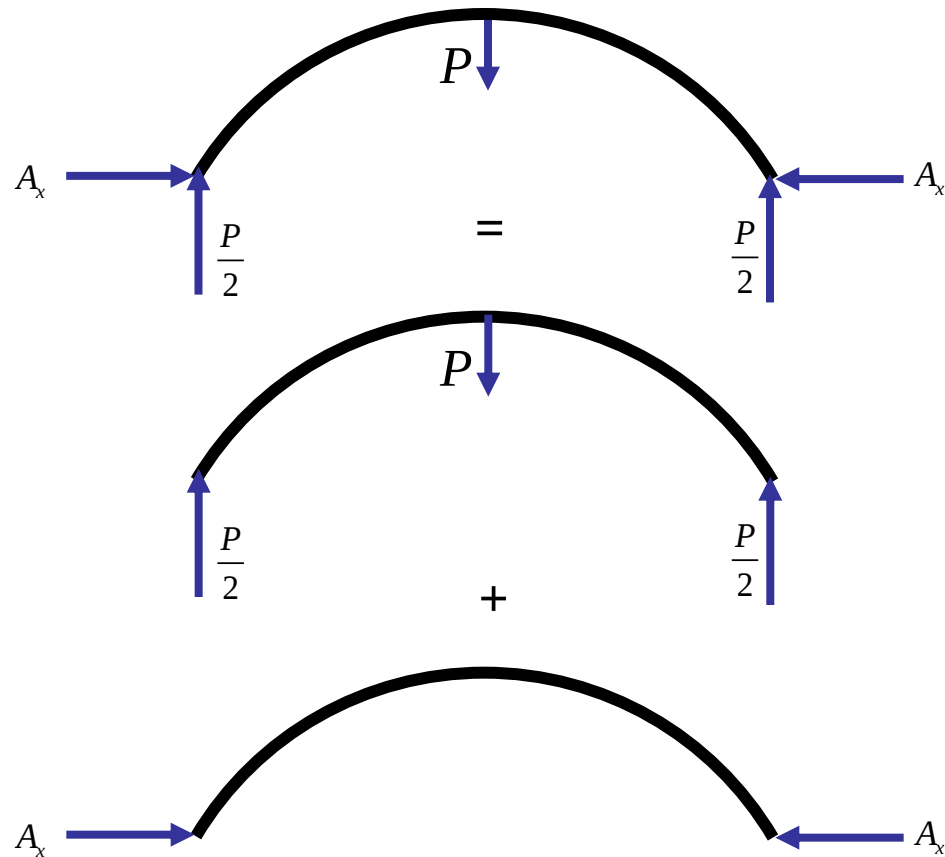
Small deflections not assumed!



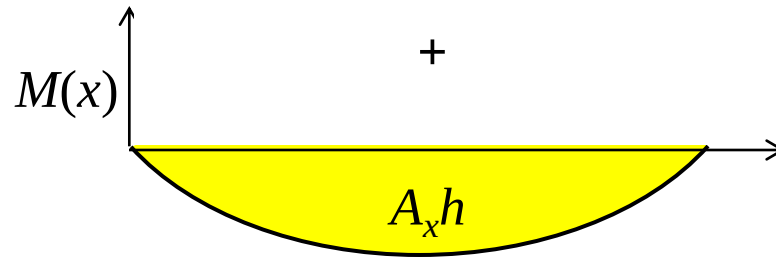
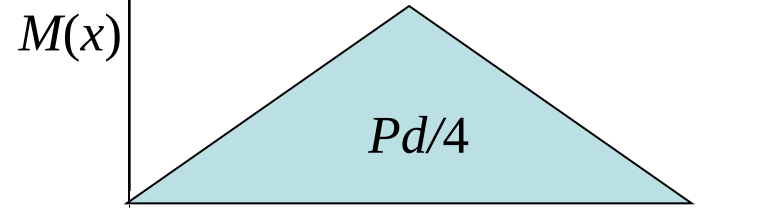
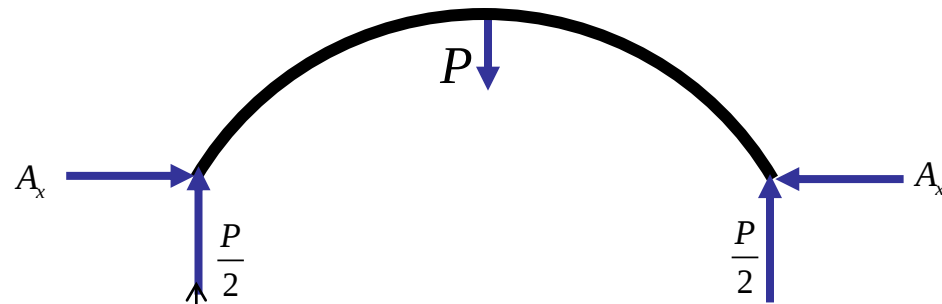
Moments



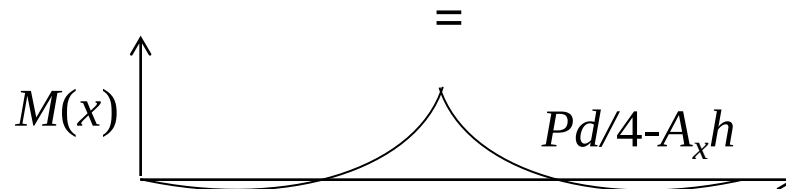
Approximate Moments



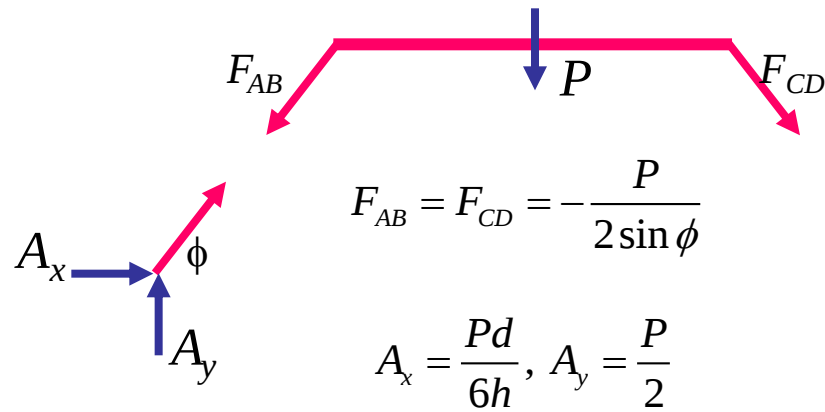
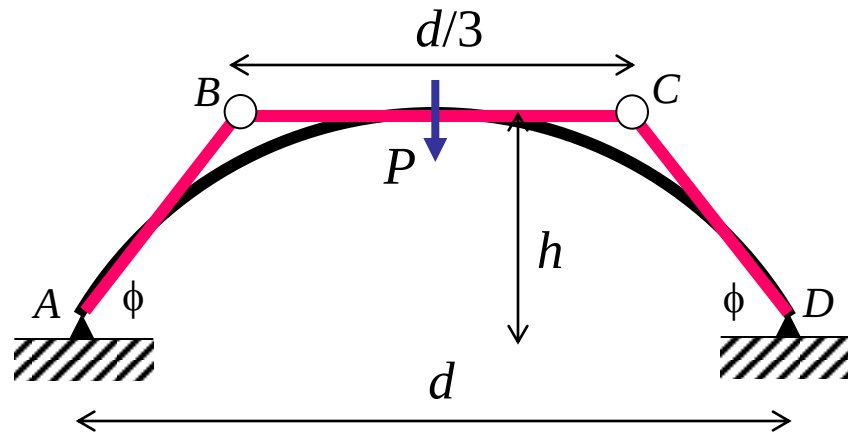
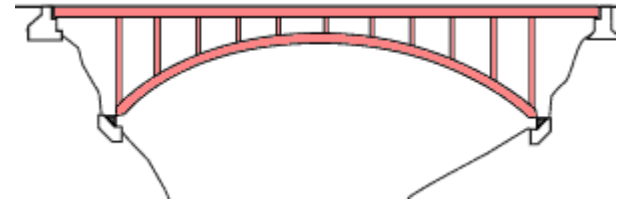
Moments



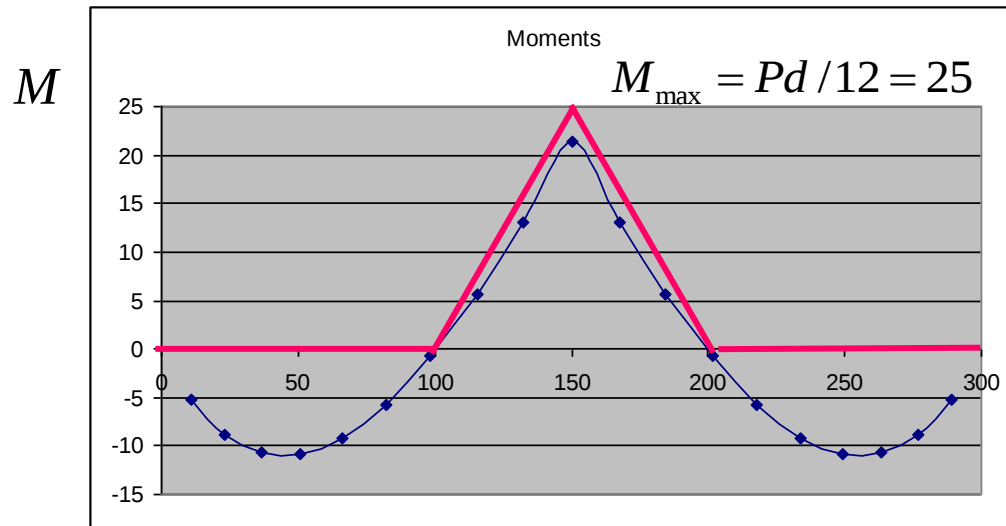
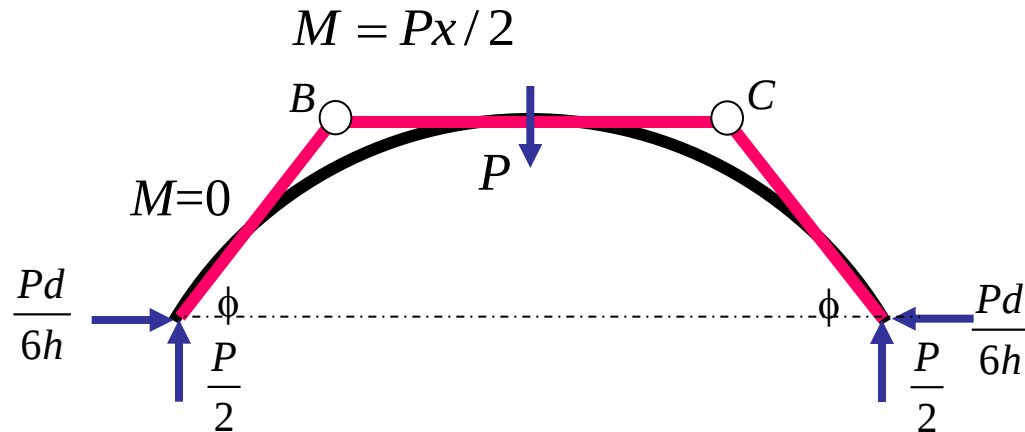
$$M(x) = -A_x y(x)$$



Approximate Analysis

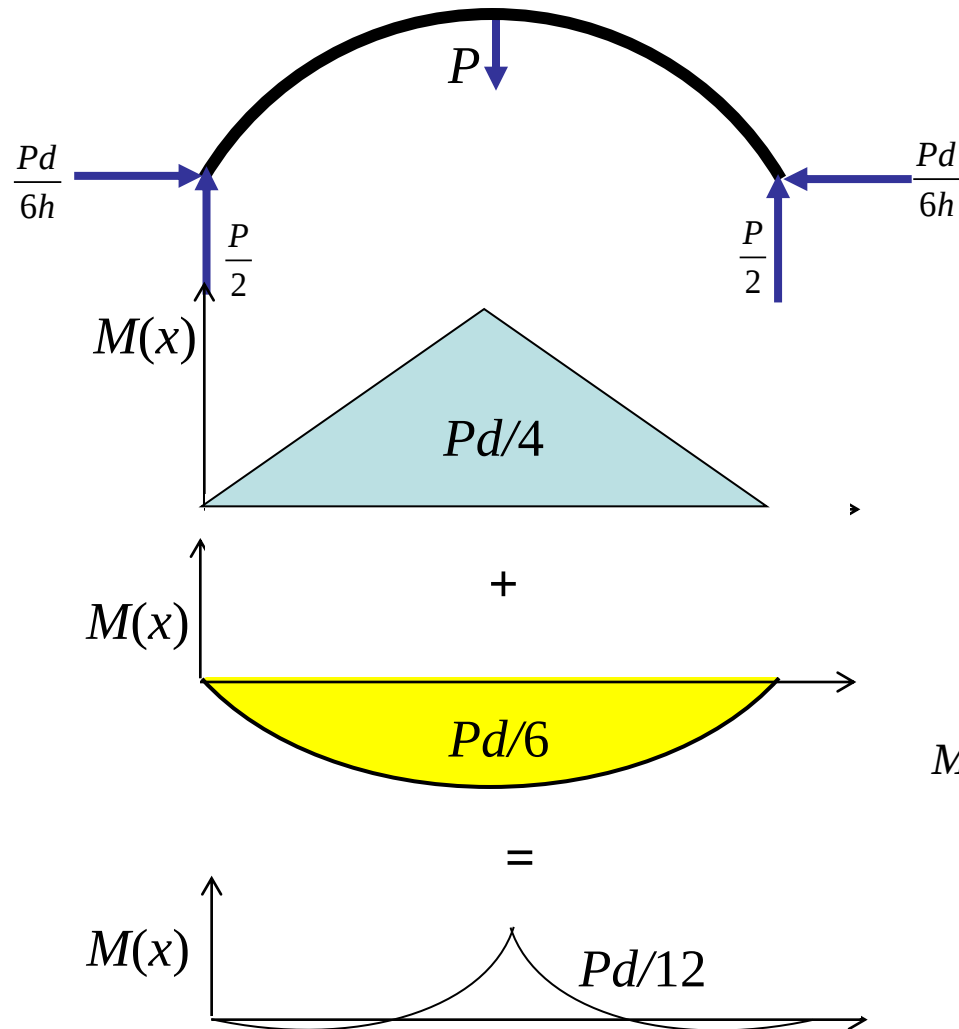


Moments (1)

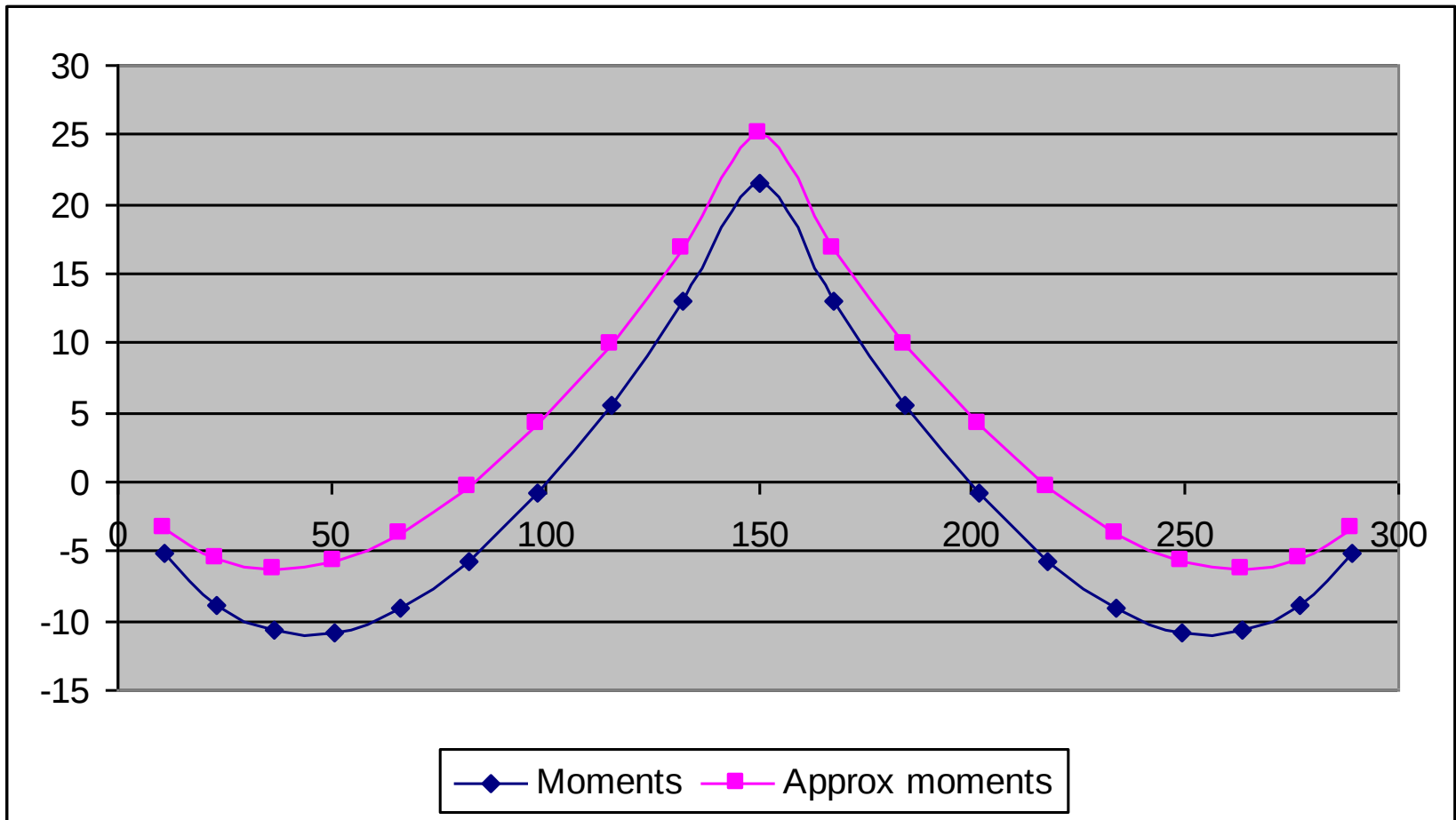


For $E=10^6$, $P=1$, $d=300\text{ft}$, $h=75\text{ft}$, $I=0.0833\text{ft}^4$, $A=1\text{ft}^2$

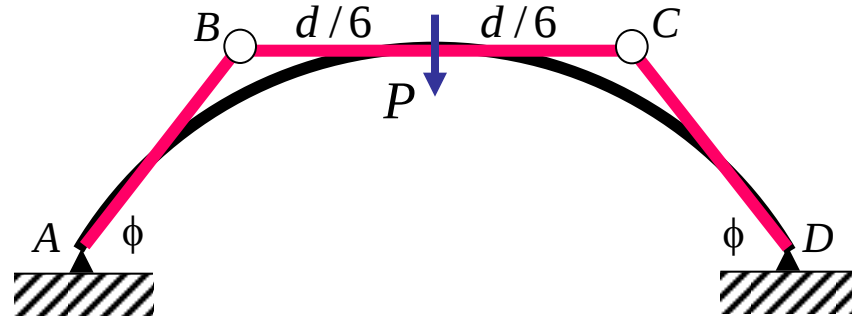
Moments (2)



P=1, d=300ft, h=75ft



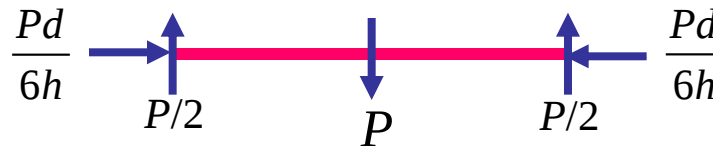
Deflections



$$\tan \phi = \frac{3h}{d},$$

$$L_{AB} = \sqrt{h^2 + d^2/9}$$

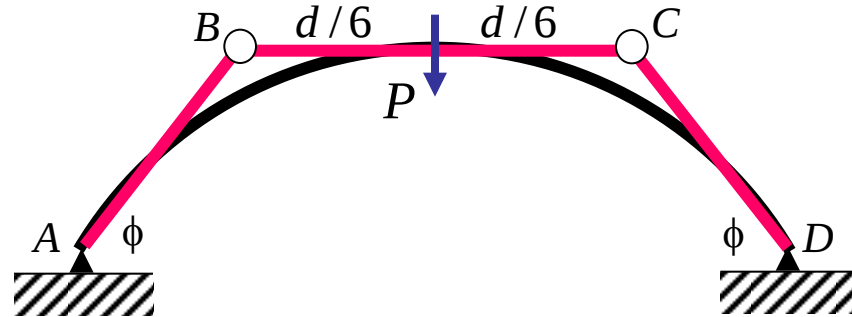
$$\delta_{AB} = \frac{F_{AB} L_{AB}}{EA} = \mathbf{u}_B \cdot \mathbf{n}_{AB} = u_x^B \cos \phi + u_y^B \sin \phi = -\frac{P L_{AB}}{2EA \sin \phi}.$$



$$u_x^B = -\frac{Pd}{6h} \frac{d}{6EA} = \frac{Pd^2}{36hEA}$$

$$u_y^P = u_y^B - \frac{Pd/3}{48EI}$$

Deflections



$$\tan \phi = \frac{3h}{d}$$

$$\sin \phi = \frac{h}{L_{AB}}$$

$$L_{AB} = \sqrt{h^2 + d^2 / 9}$$

$$\frac{Pd^2}{36hEA} \cos \phi + u_y^B \sin \phi = -\frac{PL_{AB}}{2EA \sin \phi}$$

$$\Rightarrow u_y^B = -\frac{PL_{AB}}{2EA \sin^2 \phi} - \frac{Pd^2}{36EAh \tan \phi} = -\frac{P(h^2 + d^2 / 9)^{3/2}}{2EAh^2} - \frac{Pd^3}{108EAh^2}$$

$$u_y^P = u_y^B - \frac{P(d/3)^3}{48EI}$$

For $E=10^6$, $P=1$, $d=300\text{ft}$, $h=75\text{ft}$, $I=0.0833\text{ft}^4$, $A=1\text{ft}^2$

$$u_y^P = -0.25\text{ft}$$

Excel gives 0.28 feet