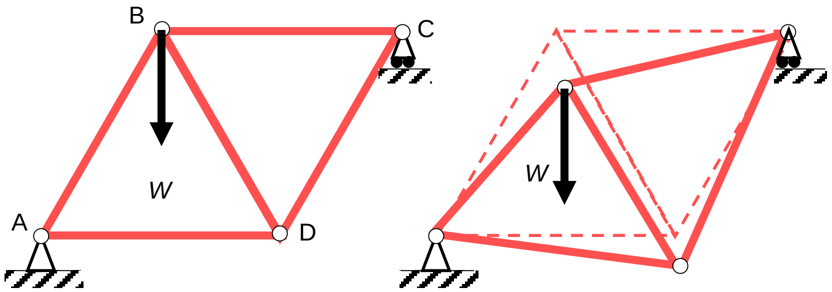
Principle of Stationary Potential Energy

For a system in stable static equilibrium,
the potential energy of the structure is minimized



Find the values of the joint displacements for which the potential energy of the structure and its applied loads is a minimum.

PE consists of elastic energy stored in members + energy due to applied loads.

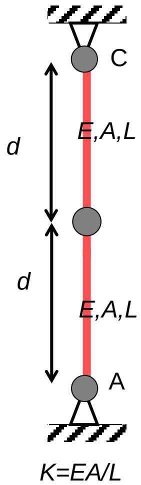
1

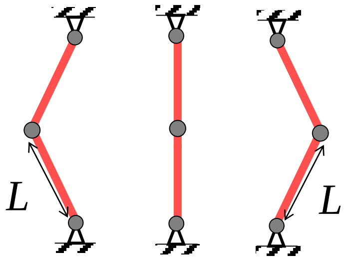
Equlibria by Energy considerations

$$V'(x) = 0 \Rightarrow x = 0 \text{ or } x = \pm\sqrt{L^2 - d^2} \text{ if } (L > d)$$

If *L<d:* equilibrium at *x=0* only

If *L>d: 3* equilibria!



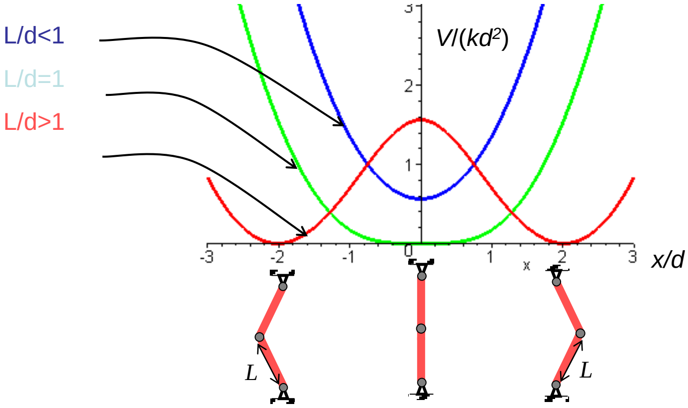


2

Stable Equilibrium: Minimum PE

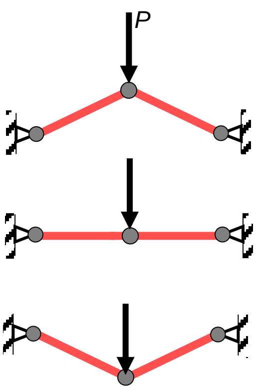
$$V(x) = 2\left(\frac{1}{2}K\left(\sqrt{x^2+d^2}-L\right)^2\right)$$

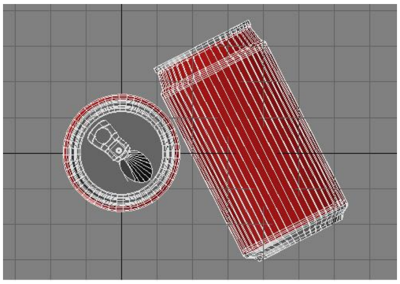
$$V'(x) = 0 \Rightarrow x = 0 \text{ or } \sqrt{x^2+d^2} = L \ (L > d)$$

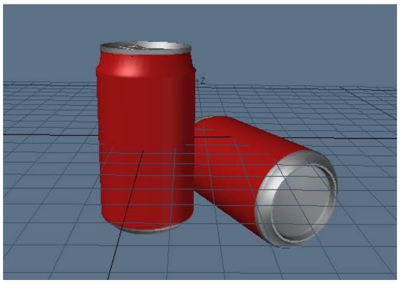


3

Snap Through Buckling: $L<L_0$







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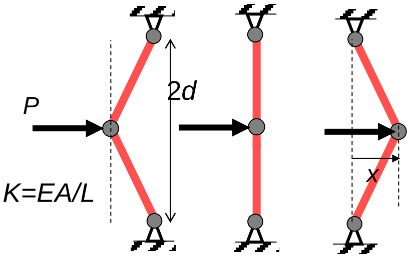
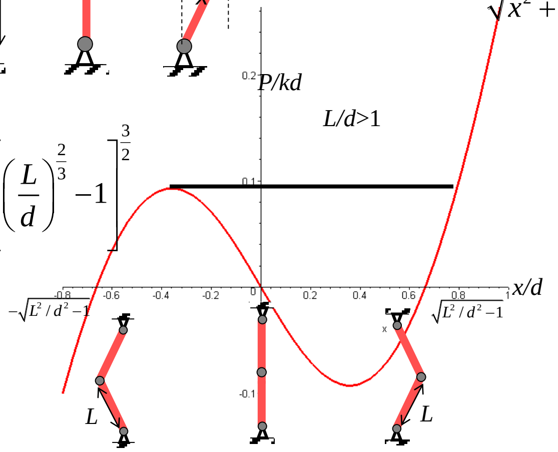
Snap Through Buckling



$$V(x) = 2\left(\frac{1}{2}K\left(\sqrt{x^2+d^2}-L\right)^2\right) - Px$$

$$V'(x) = 2K\frac{x\left(\sqrt{x^2+d^2}-L\right)}{\sqrt{x^2+d^2}} - P = 0$$

$$P_{CR} = 2Kd\left[\left(\frac{L}{d}\right)^{\frac{2}{3}} - 1\right]^{\frac{3}{2}}$$



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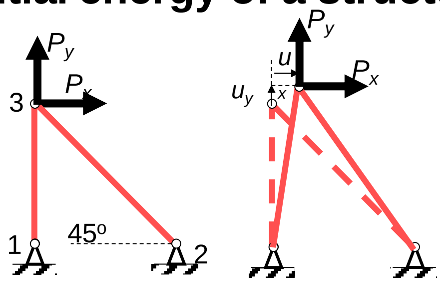
Potential energy of a structure



Elastic Potential energy of member ij $\quad V^{ij} = \frac{1}{2}K^{ij}(\delta^{ij})^2$

Potential energy of applied force at joint i $\quad V^i_{force} = -\mathbf{u}^i \cdot \mathbf{P}^i$

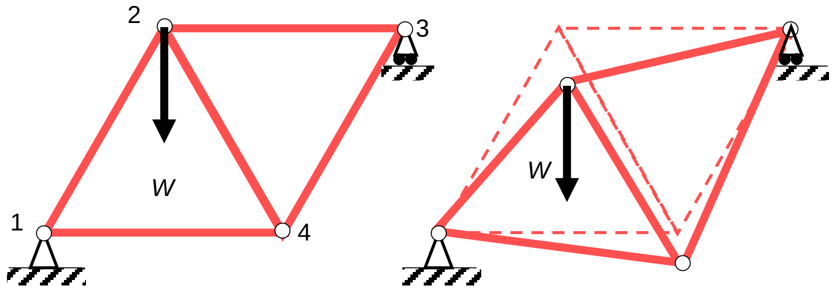
$$\text{Total} \quad V = \sum_{members} \frac{1}{2}K^{ij}(\delta^{ij})^2 - \sum_{\text{joints j}} \mathbf{u}^j \cdot \mathbf{P}^j$$

Find the set of joint displacements that minimizes V.

These minimizing displacements are those attained by the structure in static equilibrium

$$\frac{\partial V}{\partial u_n^j} = 0$$

Note that reaction forces do not contribute to the system PE!

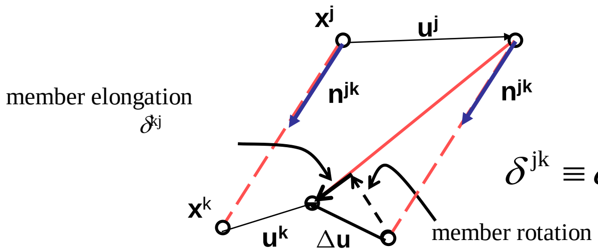


PE due to reaction forces:

$$V_{\text{reactions}} = -R_x^1 u_x^1 - R_y^1 u_y^1 - -R_y^3 u_y^3 = 0$$

Potential Energy of a Single Member (small deformations!)

For a member connecting joints (j) and (k):

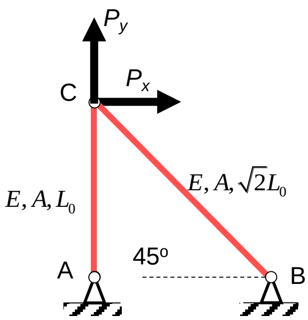


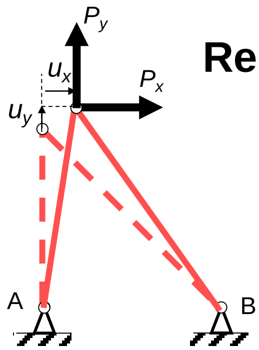
$$\delta^{\text{jk}} \equiv \delta^{\text{kj}} = (\mathbf{u}^{\text{k}} - \mathbf{u}^{\text{j}}) \cdot \mathbf{n}^{\text{jk}}$$

$$V^{\text{jk}} \equiv \frac{1}{2} K^{jk} \left(\delta^{\text{kj}}\right)^2 = \frac{1}{2} K^{jk} \left((\mathbf{u}^{\text{k}} - \mathbf{u}^{\text{j}}) \cdot \mathbf{n}^{\text{jk}}\right)^2$$

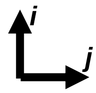
$$V = \sum_{members} \frac{1}{2} K^{ij} (\delta^{ij})^2 - \sum_{\text{joints j}} \mathbf{u}^j \cdot \mathbf{P}^j$$

Quadratic in the displacements.
⇒Minimization leads to linear equations for **u**





ReMember this?

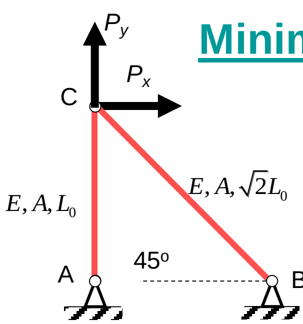


$$\delta_{AC} = (\mathbf{u}^C - \mathbf{u}^A)\bullet\mathbf{n}^{AC} = \mathbf{u}^C\bullet\mathbf{j} = u_y,$$

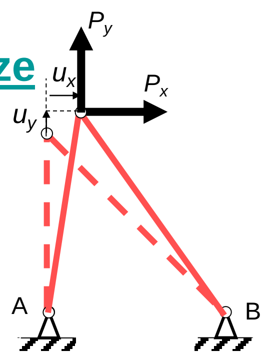
$$\delta_{BC} = (\mathbf{u}^C - \mathbf{u}^B)\bullet\mathbf{n}^{BC} = \mathbf{u}^C\bullet\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j}) = \frac{1}{\sqrt{2}}(u_y - u_x)$$

$$V_{AC} = \frac{1}{2}\frac{EA}{L_0}u_y^{\ 2},\ V_{BC} = \frac{1}{4}\frac{EA}{\sqrt{2}L_0}\left(u_y - u_x\right)^2, V_{load} = -P_x u_x - P_y u_y$$

$$V = V_{AC} + V_{BC} + V_{load}$$



Minimize



$$dux := -\frac{k\sqrt{2}\ uy}{4} + \frac{k\sqrt{2}\ ux}{4} - px$$

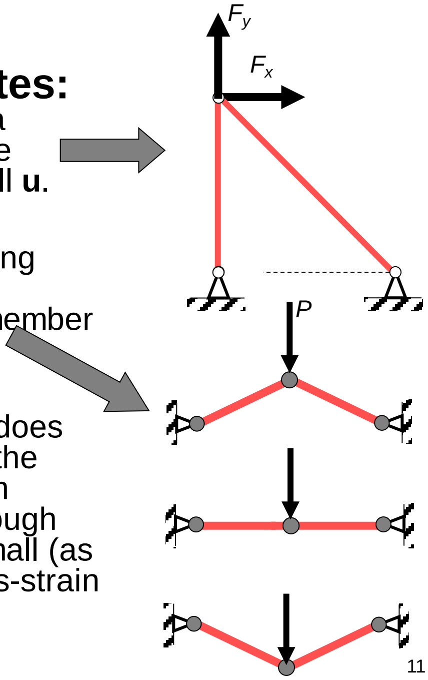
$$duy := k\ uy + \frac{k\sqrt{2}\ uy}{4} - \frac{k\sqrt{2}\ ux}{4} - py$$

$$\frac{EA}{2\sqrt{2}L_0}\begin{bmatrix} 1 & -1 \\ -1 & 1+2\sqrt{2} \end{bmatrix}\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

$$[K] = \begin{bmatrix} \frac{\partial^2 V}{\partial u_x^{\ 2}} & \frac{\partial^2 V}{\partial u_x \partial u_y} \\ \frac{\partial^2 V}{\partial u_x \partial u_y} & \frac{\partial^2 V}{\partial u_x^{\ 2}} \end{bmatrix}$$

Notes:

- For this problem, we used a linearized expression for the elongations, assuming small **u**.
- For the snap through buckling problem, we used the full, nonlinear expressions for member elongations.
- Since energy minimization does **not** enforce equilibrium on the undeformed geometry it can predict things like snap-through buckling. Strains are still small (as required for the linear stress-strain behavior)



11