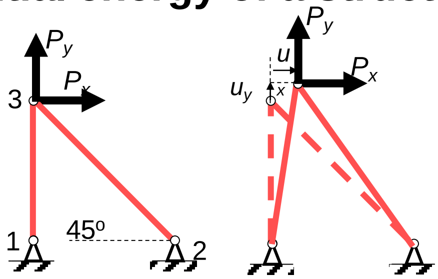
Potential energy of a structure



Elastic Potential energy of member *ij* $V^{ij} = \frac{1}{2} K^{ij} (\delta^{ij})^2$

Potential energy of applied force at joint *i* $V^{i}_{force} = -\mathbf{u}^i \cdot \mathbf{P}^i$

Total $V = \sum_{members} \frac{1}{2} K^{ij} (\delta^{ij})^2 - \sum_{\text{joints j}} \mathbf{u}^j \cdot \mathbf{P}^j$

Find the set of joint displacements that minimizes *V*.

These minimizing displacements are those attained by the structure in static equilibrium

$$\frac{\partial V}{\partial u_n^j} = 0$$

Problem is, in general, nonlinear.

- To linearize, assume that the displacements are small compared with structure dimensions.
- Elongations are linear in displacement.
- Energy is quadratic in displacement

Minimizing energy leads to a general linear matrix method which can handle very large problems

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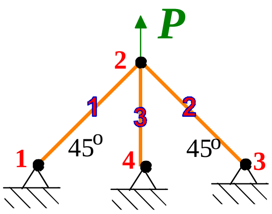
Matrix Finite Element Methods By Energy Minimization: Small Deflections



$$\underline{u} = \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ \vdots \end{bmatrix}$$

- **Write Potential Energy as a function of nodal displacements:** $V(\underline{u}) = \frac{1}{2}\underline{u}\bullet[K]\underline{u} - \underline{r}\bullet\underline{u}$
- **Minimize *V(u)*:** $\frac{\partial V}{\partial \underline{u}}(\underline{u}) = 0 \Rightarrow [K]\underline{u} = \underline{r}$
- **Solve for *u*** $\underline{u} = [K]^{-1}\underline{r}$

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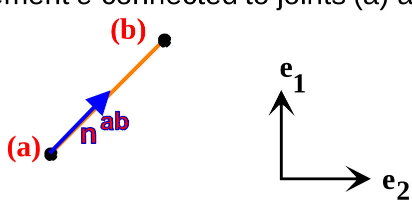
To calculate *V(u)*:

Write the strain and axial force of each member as a function of nodal displacements.

For Element *e* connected to joints (a) and (b):



$$\delta^{ab} = (\mathbf{u}^{\mathrm{b}} - \mathbf{u}^{\mathrm{a}}) \cdot \mathbf{n}^{\mathrm{ab}},$$

$$F^e \equiv F^{ab} = \left(\frac{EA}{L}\right)^e (\mathbf{u}^{\mathrm{b}} - \mathbf{u}^{\mathrm{a}}) \cdot \mathbf{n}^{\mathrm{ab}}$$

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Write the Strain Energy for each member:

$$V^e = \frac{1}{2}\left(\frac{EA}{L}\right)^e \left(\delta^e\right)^2 = \frac{1}{2}\left(\frac{EA}{L}\right)^e \left(\left(\mathbf{u}^{\mathrm{b}} - \mathbf{u}^{\mathrm{a}}\right)\cdot \mathbf{n}^{\mathrm{ab}}\right)^2$$

$$V^e = \frac{1}{2}\left(\frac{EA}{L}\right)^e \begin{bmatrix} u_1^{\mathrm{a}} & u_2^{\mathrm{a}} & u_1^{\mathrm{b}} & u_2^{\mathrm{b}} \end{bmatrix} \cdot \begin{bmatrix} \left(n_1^{\mathrm{ab}}\right)^2 & n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} & -\left(n_1^{\mathrm{ab}}\right)^2 & -n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} \\ n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} & \left(n_2^{\mathrm{ab}}\right)^2 & -n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} & -\left(n_2^{\mathrm{ab}}\right)^2 \\ -\left(n_1^{\mathrm{ab}}\right)^2 & -n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} & \left(n_1^{\mathrm{ab}}\right)^2 & n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} \\ -n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} & -\left(n_2^{\mathrm{ab}}\right)^2 & n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} & \left(n_2^{\mathrm{ab}}\right)^2 \end{bmatrix} \begin{bmatrix} u_1^{\mathrm{a}} \\ u_2^{\mathrm{a}} \\ u_1^{\mathrm{b}} \\ u_2^{\mathrm{b}} \end{bmatrix}$$

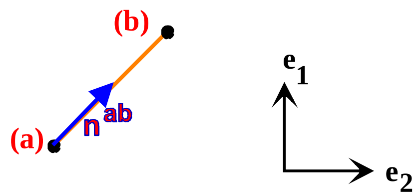
$$V^e = \frac{1}{2}\underline{u}^e \cdot \left[K^e\right] \underline{u}^e$$

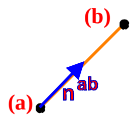
K^e and $\underline{u}^e$ are the element stiffness matrix and element displacement vector.

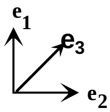
K^e is Symmetric

$$V^e = \frac{1}{2}\underline{u}^e \cdot \left[K^e\right] \underline{u}^e$$

$$K^e = \left(\frac{EA}{L}\right)^e \begin{bmatrix} \left(n_1^{\mathrm{ab}}\right)^2 & n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} & -\left(n_1^{\mathrm{ab}}\right)^2 & -n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} \\ n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} & \left(n_2^{\mathrm{ab}}\right)^2 & -n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} & -\left(n_2^{\mathrm{ab}}\right)^2 \\ -\left(n_1^{\mathrm{ab}}\right)^2 & -n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} & \left(n_1^{\mathrm{ab}}\right)^2 & n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} \\ -n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} & -\left(n_2^{\mathrm{ab}}\right)^2 & n_1^{\mathrm{ab}} n_2^{\mathrm{ab}} & \left(n_2^{\mathrm{ab}}\right)^2 \end{bmatrix}, \quad \underline{u}^e = \begin{bmatrix} u_1^{\mathrm{a}} \\ u_2^{\mathrm{a}} \\ u_1^{\mathrm{b}} \\ u_2^{\mathrm{b}} \end{bmatrix}$$







In 3D

(Same as it ever was…)

$$V^e = \frac{1}{2}\left(\frac{EA}{L}\right)^e \left(\delta^e\right)^2 = \frac{1}{2}\left(\frac{EA}{L}\right)^e \left(\left(\mathbf{u}^{\mathrm{b}} - \mathbf{u}^{\mathrm{a}}\right)\cdot \mathbf{n}^{\mathrm{ab}}\right)^2$$

$$V^e = \frac{1}{2}\underline{u}^e \cdot \left[K^e\right]\underline{u}^e$$

$$K^e = \left(\frac{EA}{L}\right)^e \begin{bmatrix} \left(n_1^{ab}\right)^2 & n_1^{ab}n_2^{ab} & n_1^{ab}n_3^{ab} & -\left(n_1^{ab}\right)^2 & -n_1^{ab}n_2^{ab} & -n_1^{ab}n_3^{ab} \\ & \left(n_2^{ab}\right)^2 & n_2^{ab}n_3^{ab} & -n_1^{ab}n_2^{ab} & -\left(n_2^{ab}\right)^2 & -n_2^{ab}n_3^{ab} \\ & & \left(n_3^{ab}\right)^2 & -n_1^{ab}n_3^{ab} & -n_2^{ab}n_3^{ab} & -\left(n_3^{ab}\right)^2 \\ & & & \left(n_1^{ab}\right)^2 & n_1^{ab}n_2^{ab} & n_1^{ab}n_3^{ab} \\ & & & & \left(n_2^{ab}\right)^2 & n_2^{ab}n_3^{ab} \\ & & & & & \left(n_3^{ab}\right)^2 \end{bmatrix}, \quad \underline{u}^e = \begin{bmatrix} u_1^a \\ u_2^a \\ u_3^a \\ u_1^b \\ u_2^b \\ u_3^b \end{bmatrix}$$

The total strain energy

The total strain energy of the truss may be computed by adding together the strain energy of each element

$$V_{\text{elastic}} = \sum_{\text{elements}} V^{\text{element}} = \sum_{\text{elements}} \frac{1}{2}\underline{u}^{\text{element}} \cdot \left[K^{\text{element}}\right]\underline{u}^{\text{element}}$$

Want to write this in the form:

$$V_{\text{elastic}} = \frac{1}{2}\underline{u}\cdot\left[K\right]\underline{u}$$

Using the global displacement vector $\underline{u}$.
[K] is the global stiffness matrix

Using the global displacement vector, *u*

This vector contains all of the joint displacement components.

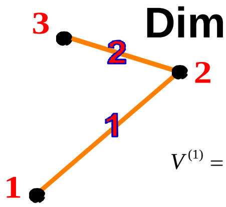
2-D: vector u has length 2J and $u_i^{(b)} = u_{2(b-1)+i}$

$$\underline{u} = \begin{bmatrix} u_1^{(1)} & u_2^{(1)} & \cdots & u_1^{(b)} & u_2^{(b)} & \cdots & u_1^{(N)} & u_2^{(N)} \end{bmatrix}$$

3-D: vector u has length 3J and $u_i^{(b)} = u_{3(b-1)+i}$

$$\underline{u} = \begin{bmatrix} u_1^{(1)} & u_2^{(1)} & u_3^{(1)} & \cdots & u_1^{(b)} & u_2^{(b)} & u_3^{(b)} & \cdots & u_1^{(N)} & u_2^{(N)} & u_3^{(N)} \end{bmatrix}$$

Consider a Two-Member (Two-Dimensional) Truss:



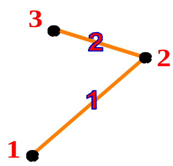
$$V^{(1)} = \frac{1}{2}\begin{bmatrix} u_1^{(1)} & u_2^{(1)} & u_1^{(2)} & u_2^{(2)} \end{bmatrix} \cdot \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{12}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{13}^{(1)} & K_{23}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ K_{14}^{(1)} & K_{24}^{(1)} & K_{34}^{(1)} & K_{44}^{(1)} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \end{bmatrix}$$

$$V^{(2)} = \frac{1}{2}\begin{bmatrix} u_1^{(2)} & u_2^{(2)} & u_1^{(3)} & u_2^{(3)} \end{bmatrix} \cdot \begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ K_{12}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ K_{13}^{(2)} & K_{23}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ K_{14}^{(2)} & K_{24}^{(2)} & K_{34}^{(2)} & K_{44}^{(2)} \end{bmatrix} \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \end{bmatrix}$$

$$V = V^{(1)} + V^{(2)}$$

$V^{(1)}$ and $V^{(2)}$ in terms of the global displacement vector $\underline{u}$:

$$V^{(1)}=\frac{1}{2}\begin{bmatrix}u_1^{(1)}\\u_2^{(1)}\\u_1^{(2)}\\u_2^{(2)}\\u_1^{(3)}\\u_2^{(3)}\end{bmatrix}\cdot\begin{bmatrix}K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} & 0 & 0\\ & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} & 0 & 0\\ & & K_{33}^{(1)} & K_{34}^{(1)} & 0 & 0\\ & & & K_{44}^{(1)} & 0 & 0\\ & & & & 0 & 0\\ & & & & & 0\end{bmatrix}\begin{bmatrix}u_1^{(1)}\\u_2^{(1)}\\u_1^{(2)}\\u_2^{(2)}\\u_1^{(3)}\\u_2^{(3)}\end{bmatrix}$$



$$V^{(2)}=\frac{1}{2}\begin{bmatrix}u_1^{(1)}\\u_2^{(1)}\\u_1^{(2)}\\u_2^{(2)}\\u_1^{(3)}\\u_2^{(3)}\end{bmatrix}\cdot\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ & 0 & 0 & 0 & 0 & 0\\ & & K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)}\\ & & & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)}\\ & & & & K_{33}^{(2)} & K_{34}^{(2)}\\ & & & & & K_{44}^{(2)}\end{bmatrix}\begin{bmatrix}u_1^{(1)}\\u_2^{(1)}\\u_1^{(2)}\\u_2^{(2)}\\u_1^{(3)}\\u_2^{(3)}\end{bmatrix}$$

Total Elastic Energy

$$V=V^{(1)}+V^{(2)}$$

$$=\frac{1}{2}\begin{bmatrix}u_1^{(1)}\\u_2^{(1)}\\u_1^{(2)}\\u_2^{(2)}\\u_1^{(3)}\\u_2^{(3)}\end{bmatrix}\cdot\begin{bmatrix}K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} & 0 & 0\\ & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} & 0 & 0\\ & & K_{33}^{(1)}+K_{11}^{(2)} & K_{34}^{(1)}+K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)}\\ & & & K_{44}^{(1)}+K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)}\\ & & & & K_{33}^{(2)} & K_{34}^{(2)}\\ & & & & & K_{44}^{(2)}\end{bmatrix}\begin{bmatrix}u_1^{(1)}\\u_2^{(1)}\\u_1^{(2)}\\u_2^{(2)}\\u_1^{(3)}\\u_2^{(3)}\end{bmatrix}$$

$$V=\frac{1}{2}\underline{u}\cdot[K]\underline{u}$$

- **[*K*]** is the *Global Stiffness Matrix*. It is the sum of all the element stiffness matrices.
- **Because the element stiffness matrix is symmetric, the global stiffness matrix must also be symmetric.**
- **CAUTION:** SOME ASSEMBLY REQUIRED!!!

Bookkeeping:

- The displacement of node #*b* in the e_i-direction, $u_i^{(b)}$ (i=1,2) is the (2(*b*-1)+*i*)th element of the global displacement vector, $\underline{u}$:

$$\underline{u} = \begin{bmatrix} u_1^{(1)} & u_2^{(1)} & \cdots & u_1^{(b)} & u_2^{(b)} & \cdots & u_1^{(N)} & u_2^{(N)} \end{bmatrix}$$

$$u_i^{(\mathrm{b})} = u_{2(b-1)+i}$$

The **degree of freedom number** **assigned to the displacement of node *b* in the *i*-direction is (2(*b*-1)+*i*).**

In 3D:

$$\underline{u} = \begin{bmatrix} u_1^{(1)} & u_2^{(1)} & u_3^{(1)} & \cdots & u_1^{(b)} & u_2^{(b)} & u_3^{(b)} & \cdots & u_1^{(N)} & u_2^{(N)} & u_3^{(N)} \end{bmatrix}$$

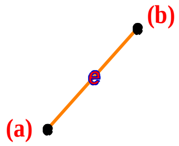
$$u_i^{(\mathrm{b})} = u_{3(b-1)+i}$$

and the **degree of freedom number** **assigned to the displacement of node *b* in the *i*-direction is (3(*b*-1)+*i*).)**

The global stiffness positioning (2D):

$$\begin{array}{c} \quad u_1^{(1)} \quad u_2^{(1)} \qquad\quad u_1^{(b)} \qquad u_2^{(b)} \qquad\qquad u_2^{(N)} \\ \begin{bmatrix} K_{1,1} & K_{1,2} & \cdots & K_{1,2b-1} & K_{1,2b} & \cdots & \cdots & K_{1,2N} \\ K_{1,2} & K_{11} & & & & & & \vdots \\ \vdots & & \ddots & & & & & \\ \vdots & & & & & & & \vdots \\ K_{1,2c-1} & & & K_{2b-1,2c-1} & K_{2b,2c-1} & & & \\ K_{1,2c} & & & K_{2b-1,2c} & K_{2b,2c} & & & \vdots \\ \vdots & & & & & & \ddots & \\ K_{1,2N} & \cdots & & \cdots & & \cdots & & K_{2N,2N} \end{bmatrix} \end{array} \begin{array}{c} u_1^{(1)} \\ u_2^{(1)} \\ \\ \\ u_1^{(c)} \\ u_2^{(c)} \\ \\ u_2^{(N)} \end{array}$$

Where do the element stiffness components land?



$$\begin{array}{cccc} u_1^{(a)} & u_2^{(a)} & u_1^{(b)} & u_2^{(b)} \end{array}$$

$$\begin{bmatrix} K_{11}^{(e)} & K_{12}^{(e)} & K_{13}^{(e)} & K_{14}^{(e)} \\ & K_{22}^{(e)} & K_{23}^{(e)} & K_{24}^{(e)} \\ & & K_{33}^{(e)} & K_{34}^{(e)} \\ & & & K_{44}^{(e)} \end{bmatrix} \begin{matrix} u_1^{(a)} \\ u_2^{(a)} \\ u_1^{(b)} \\ u_2^{(b)} \end{matrix}$$

$$\begin{array}{llll} K_{11}^{(e)} \to K_{2a-1,2a-1}, & K_{12}^{(e)} \to K_{2a-1,2a}, & K_{13}^{(e)} \to K_{2a-1,2b-1} & K_{14}^{(e)} \to K_{2a-1,2b} \\ & K_{22}^{(e)} \to K_{2a,2a}, & K_{23}^{(e)} \to K_{2a,2b-1} & K_{24}^{(e)} \to K_{2a,2b} \\ & & K_{33}^{(e)} \to K_{2b-1,2b-1} & K_{34}^{(e)} \to K_{2b-1,2b} \\ & & & K_{44}^{(e)} \to K_{2b,2b} \end{array}$$

$$K_{2(i-1)+ii,2(j-1)+jj}^{(e)} \to K_{2(a-1)+ii,2(b-1)+jj}$$

$i = 1, 2$ and $j = 1, 2$ (for the 1st and 2nd nodes for the member)

$ii = 1, 2$ and $jj = 1, 2$ (for the 1- and 2- directions of displacement)

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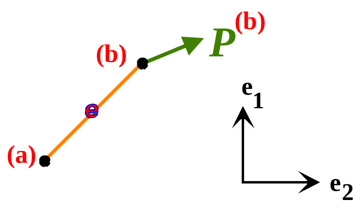
Externally Applied Forces

These are specified by prescribing the node on which the loading acts and the force vector ***P***.



Potential energy due this load is then $V_{load}^{(b)} = -\mathbf{P}^{(b)} \cdot \mathbf{u}^{(b)}$

The total contribution to the potential energy due to prescribed forces on all of the loaded nodes is

$$V_{load} = \sum_{\substack{\text{Loaded} \\ \text{Nodes}}} -\mathbf{P}^{(b)} \cdot \mathbf{u}^{(b)}$$

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Global residual force vector: $\underline{r}$

But of course, we have to write V_{loads} in terms of the global force vector $\underline{u}$. $V_{loads} = -\underline{r} \cdot \underline{u}$

***Recall the degree of freedom number assigned to the displacement of node b* in the *i-direction* is *(2(b*-1)+*i*).**

$$P_1^{(b)} \rightarrow r_{2b-1} \quad P_2^{(b)} \rightarrow r_{2b}$$

3D degree of freedom number assigned to the displacement of node *b* in the *i*-direction is (3(*b*-1)+*i*).

$$P_1^{(b)} \rightarrow r_{3b-2} \quad P_2^{(b)} \rightarrow r_{3b-1} \quad P_3^{(b)} \rightarrow r_{3b}$$

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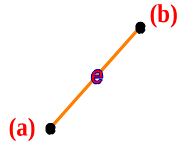
Where Are We So Far?



Elastic energy in an element (*e*): $V^e = \frac{1}{2}\underline{u}^e \cdot \left[K^e\right]\underline{u}^e$

$$K^e = \left(\frac{EA}{L}\right)^e \begin{bmatrix} \left(n_1^{ab}\right)^2 & n_1^{ab}n_2^{ab} & -\left(n_1^{ab}\right)^2 & -n_1^{ab}n_2^{ab} \\ n_1^{ab}n_2^{ab} & \left(n_2^{ab}\right)^2 & -n_1^{ab}n_2^{ab} & -\left(n_2^{ab}\right)^2 \\ -\left(n_1^{ab}\right)^2 & -n_1^{ab}n_2^{ab} & \left(n_1^{ab}\right)^2 & n_1^{ab}n_2^{ab} \\ -n_1^{ab}n_2^{ab} & -\left(n_2^{ab}\right)^2 & n_1^{ab}n_2^{ab} & \left(n_2^{ab}\right)^2 \end{bmatrix}, \quad \underline{u}^e = \begin{bmatrix} u_1^a \\ u_2^a \\ u_1^b \\ u_2^b \end{bmatrix}$$

$$K_{11}^{(e)} \rightarrow K_{2a-1,2a-1}, \quad K_{12}^{(e)} \rightarrow K_{2a-1,2a}, \quad K_{13}^{(e)} \rightarrow K_{2a-1,2b-1} \quad K_{14}^{(e)} \rightarrow K_{2a-1,2b}$$

$$K_{22}^{(e)} \rightarrow K_{2a,2a}, \quad K_{23}^{(e)} \rightarrow K_{2a,2b-1} \quad K_{24}^{(e)} \rightarrow K_{2a,2b}$$

$$K_{33}^{(e)} \rightarrow K_{2b-1,2b-1} \quad K_{34}^{(e)} \rightarrow K_{2b-1,2b}$$

$$K_{44}^{(e)} \rightarrow K_{2b,2b}$$

Total Energy:

$$V = \sum_{e=1}^{\#elements} V^e + \sum_{Nodes} -\mathbf{P}^{(b)} \cdot \mathbf{u}^{(b)} = \sum_{e=1}^{\#elements} \frac{1}{2}\underline{u}^e \cdot \left[K^e\right]\underline{u}^e = \frac{1}{2}\underline{u} \cdot [K]\underline{u} - \underline{r} \cdot \underline{u}$$

$$u_i^{(b)} = u_{2(b-1)+i} \qquad P_i^{(b)} \rightarrow r_{2(b-1)+i}$$

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