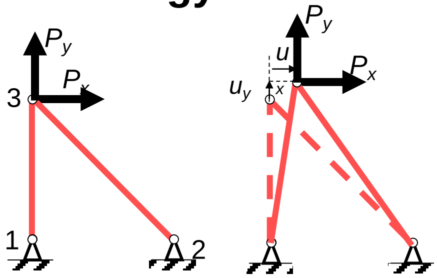
Potential energy of a structure



Elastic Potential energy of member ij $\quad V^{ij} = \frac{1}{2} K^{ij} (\delta^{ij})^2$

Potential energy of applied force at joint i $\quad V^{i}_{force} = -\mathbf{u}^{i} \cdot \mathbf{P}^{i}$

$$V = \sum_{members} \frac{1}{2} K^{ij} (\delta^{ij})^2 - \sum_{\text{joints j}} \mathbf{u}^{j} \cdot \mathbf{P}^{j} = \frac{1}{2} \underline{u} \cdot [K] \underline{u} - \underline{r} \cdot \underline{u}$$

Find the set of joint displacements that minimizes V.

These minimizing displacements are those attained by the structure in static equilibrium

$$[K]\underline{u} = \underline{r}$$

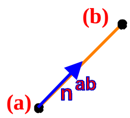
Strain Energy for a single member:

$$V^e = \frac{1}{2} \left(\frac{EA}{L} \right)^e \left(\delta^e \right)^2 = \frac{1}{2} \left(\frac{EA}{L} \right)^e \left((\mathbf{u}^{\text{b}} - \mathbf{u}^{\text{a}}) \cdot \mathbf{n}^{\text{ab}} \right)^2$$

$$V^e = \frac{1}{2} \left(\frac{EA}{L} \right)^e \begin{bmatrix} u_1^{\text{a}} & u_2^{\text{a}} & u_1^{\text{b}} & u_2^{\text{b}} \end{bmatrix} \cdot \begin{bmatrix} \left(n_1^{\text{ab}}\right)^2 & n_1^{\text{ab}} n_2^{\text{ab}} & -\left(n_1^{\text{ab}}\right)^2 & -n_1^{\text{ab}} n_2^{\text{ab}} \\ n_1^{\text{ab}} n_2^{\text{ab}} & \left(n_2^{\text{ab}}\right)^2 & -n_1^{\text{ab}} n_2^{\text{ab}} & -\left(n_2^{\text{ab}}\right)^2 \\ -\left(n_1^{\text{ab}}\right)^2 & -n_1^{\text{ab}} n_2^{\text{ab}} & \left(n_1^{\text{ab}}\right)^2 & n_1^{\text{ab}} n_2^{\text{ab}} \\ -n_1^{\text{ab}} n_2^{\text{ab}} & -\left(n_2^{\text{ab}}\right)^2 & n_1^{\text{ab}} n_2^{\text{ab}} & \left(n_2^{\text{ab}}\right)^2 \end{bmatrix} \begin{bmatrix} u_1^{\text{a}} \\ u_2^{\text{a}} \\ u_1^{\text{b}} \\ u_2^{\text{b}} \end{bmatrix}$$

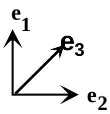
$$V^e = \frac{1}{2} \underline{u}^e \cdot \left[K^e \right] \underline{u}^e$$

K^e **and** $\underline{u}^e$ **are the element stiffness matrix and element displacement vector.**



In 3D

(Same as it ever was…)



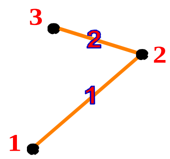
$$V^e = \frac{1}{2}\left(\frac{EA}{L}\right)^e \left(\delta^e\right)^2 = \frac{1}{2}\left(\frac{EA}{L}\right)^e \left(\left(\mathbf{u}^{\mathrm{b}} - \mathbf{u}^{\mathrm{a}}\right)\cdot \mathbf{n}^{\mathrm{ab}}\right)^2$$

$$V^e = \frac{1}{2}\underline{u}^e \cdot \left[K^e\right]\underline{u}^e$$

$$K^e = \left(\frac{EA}{L}\right)^e \begin{bmatrix} \left(n_1^{ab}\right)^2 & n_1^{ab}n_2^{ab} & n_1^{ab}n_3^{ab} & -\left(n_1^{ab}\right)^2 & -n_1^{ab}n_2^{ab} & -n_1^{ab}n_3^{ab} \\ & \left(n_2^{ab}\right)^2 & n_2^{ab}n_3^{ab} & -n_1^{ab}n_2^{ab} & -\left(n_2^{ab}\right)^2 & -n_2^{ab}n_3^{ab} \\ & & \left(n_3^{ab}\right)^2 & -n_1^{ab}n_3^{ab} & -n_2^{ab}n_3^{ab} & -\left(n_3^{ab}\right)^2 \\ & & & \left(n_1^{ab}\right)^2 & n_1^{ab}n_2^{ab} & n_1^{ab}n_3^{ab} \\ & & & & \left(n_2^{ab}\right)^2 & n_2^{ab}n_3^{ab} \\ & & & & & \left(n_3^{ab}\right)^2 \end{bmatrix}, \quad \underline{u}^e = \begin{bmatrix} u_1^a \\ u_2^a \\ u_3^a \\ u_1^b \\ u_2^b \\ u_3^b \end{bmatrix}$$

3

Better to write member energies in terms of the global displacement vector:



$$V^{(1)} = \frac{1}{2}\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} \cdot \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{12}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{13}^{(1)} & K_{23}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ K_{14}^{(1)} & K_{24}^{(1)} & K_{34}^{(1)} & K_{44}^{(1)} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} \qquad V^{(2)} = \frac{1}{2}\begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \end{bmatrix} \cdot \begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ K_{12}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ K_{13}^{(2)} & K_{23}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ K_{14}^{(2)} & K_{24}^{(2)} & K_{34}^{(2)} & K_{44}^{(2)} \end{bmatrix} \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \end{bmatrix}$$

$$V^{(1)} = \frac{1}{2}\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \end{bmatrix} \cdot \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} & 0 & 0 \\ & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} & 0 & 0 \\ & & K_{33}^{(1)} & K_{34}^{(1)} & 0 & 0 \\ & & & K_{44}^{(1)} & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \end{bmatrix} \qquad V^{(2)} = \frac{1}{2}\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ & & & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ & & & & K_{33}^{(2)} & K_{34}^{(2)} \\ & & & & & K_{44}^{(2)} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \end{bmatrix}$$

4

Total Elastic Energy is assembled easily

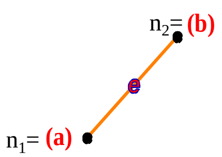
$$V = V^{(1)} + V^{(2)}$$

$$= \frac{1}{2}\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \end{bmatrix} \cdot \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} & 0 & 0 \\ & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} & 0 & 0 \\ & & K_{33}^{(1)} + K_{11}^{(2)} & K_{34}^{(1)} + K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ & & & K_{44}^{(1)} + K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ & & & & K_{33}^{(2)} & K_{34}^{(2)} \\ & & & & & K_{44}^{(2)} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \end{bmatrix}$$

$$V = \frac{1}{2}\underline{u} \cdot [K]\underline{u}$$

•**[*K*]** is the *Global Stiffness Matrix*. It is the sum of all the element stiffness matrices.

Positioning the element stiffness components in the global stiffness matrix



$$\begin{matrix} & u_1^{(a)} & u_2^{(a)} & u_1^{(b)} & u_2^{(b)} & \\ & K_{11}^{(e)} & K_{12}^{(e)} & K_{13}^{(e)} & K_{14}^{(e)} & u_1^{(a)} \\ & & K_{22}^{(e)} & K_{23}^{(e)} & K_{24}^{(e)} & u_2^{(a)} \\ & & & K_{33}^{(e)} & K_{34}^{(e)} & u_1^{(b)} \\ & & & & K_{44}^{(e)} & u_2^{(b)} \end{matrix}$$

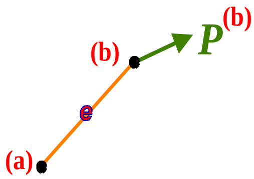
$$\begin{matrix} K_{11}^{(e)} \to K_{2a-1,2a-1}, & K_{12}^{(e)} \to K_{2a-1,2a}, & K_{13}^{(e)} \to K_{2a-1,2b-1} & K_{14}^{(e)} \to K_{2a-1,2b} \\ & K_{22}^{(e)} \to K_{2a,2a}, & K_{23}^{(e)} \to K_{2a,2b-1} & K_{24}^{(e)} \to K_{2a,2b} \\ & & K_{33}^{(e)} \to K_{2b-1,2b-1} & K_{34}^{(e)} \to K_{2b-1,2b} \\ & & & K_{44}^{(e)} \to K_{2b,2b} \end{matrix}$$

$$K_{2(i-1)+ii,2(j-1)+jj}^{(e)} \to K_{2(n_i-1)+ii,2(n_j-1)+jj}$$

$i = 1, 2$ and $j = 1, 2$ for the 1st and 2nd nodes for the member. Node #s n_i

$ii = 1, 2$ and $jj = 1, 2$ for the 1- and 2- directions of displacement

Energy Due to External Loads:



$$V_{loads} = \sum_{\substack{\text{Loaded}\\\text{Nodes}}} -\mathbf{P}^{(b)} \cdot \mathbf{u}^{(b)} = -\underline{r} \cdot \underline{u}$$

$$P_i^{(b)} \rightarrow r_{2(b-1)+i}$$

Total Potential Energy for the System:

$$\begin{aligned} V^{\text{Tot}}[\underline{u}] &= V[\underline{u}] + Q[\underline{u}] \\ &= \frac{1}{2}\underline{u} \cdot [K]\underline{u} - \underline{r} \cdot \underline{u} \\ &= \sum_{i=1}^{2N} u_i \sum_{j=1}^{2N} \frac{1}{2} K_{ij} u_j - \sum_{i=1}^{2N} r_i u_i \end{aligned}$$

7

Minimizing the Potential Energy

We have set up the following expression for the potential energy of a truss

$$\begin{aligned} V[\underline{u}] &= \frac{1}{2}\underline{u} \cdot [K]\underline{u} - \underline{r} \cdot \underline{u} \\ &= \sum_{i=1}^{2N} u_i \sum_{j=1}^{2N} \frac{1}{2} K_{ij} u_j - \sum_{i=1}^{2N} r_i u_i \end{aligned}$$

Now, minimize V :

$$\frac{\partial V^{\text{tot}}}{\partial u_k} = \frac{1}{2}\sum_{i=1}^{2N} u_i K_{ik} + \frac{1}{2}\sum_{j=1}^{2N} K_{kj} u_j - r_k = \sum_{j=1}^{2N} K_{kj} u_j - r_k = 0$$

or, in matrix notation: $[K]\underline{u} = \underline{r}$

9

We have not taken into account prescribed displacements! (so det[K]=0)

To enforce, say $u_2^{(1)} = \Delta$

we could modify the finite element equations as follows:

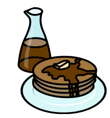
$$\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{12N} \\ k_{21} & k_{22} & & k_{22N} \\ \vdots & & \ddots & \\ k_{2N1} & k_{2N2} & & k_{2N2N} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ \vdots \\ u_2^{(N)} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_4 \end{bmatrix} \qquad \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1\,2N} \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ k_{2N\,1} & k_{2N\,2} & & k_{2N\,2N} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ \vdots \\ u_2^{(N)} \end{bmatrix} = \begin{bmatrix} r_1 \\ \Delta \\ \vdots \\ r_4 \end{bmatrix}$$

Better:
$$\begin{bmatrix} k_{11} & 0 & \cdots & k_{1\,2N} \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ k_{2N\,1} & 0 & & k_{2N\,2N} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ \vdots \\ u_2^{(N)} \end{bmatrix} = \begin{bmatrix} r_1 - k_{12}\Delta \\ \Delta \\ \vdots \\ r_4 - k_{2N\,2}\Delta \end{bmatrix}$$

Preserves the all-important symmetry of [*K*]!

10

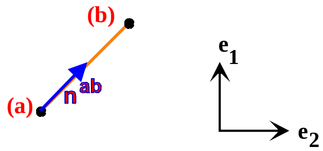
NOW SOLVE FOR *u*:



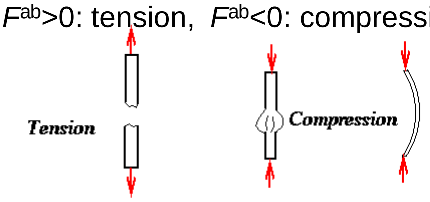
$$\underline{\mathbf{u}} = [\mathbf{K}]^{-1}\underline{\mathbf{r}}$$

Once you have *u*, the member strains and forces are

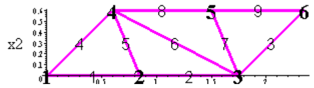
$$\varepsilon^{ab} = (\mathbf{u}^b - \mathbf{u}^a)\cdot \mathbf{n}^{ab} / L^{ab} \qquad F^{ab} = \left(\frac{EA}{L}\right)^{ab} (\mathbf{u}^b - \mathbf{u}^a)\cdot \mathbf{n}^{ab}$$



$F^{ab}>0$: tension, $F^{ab}<0$: compression

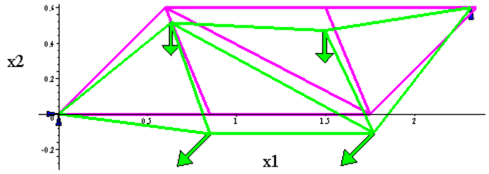


11



Maple Program Truss

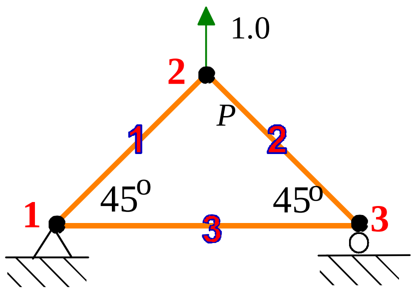
- Reads in geometric and material data describing a truss, its supports, and its loads from **an input file**
- Shows graphically the undeformed and deformed structure with member forces
- Writes nodal displacements and member forces to **an output file**



About the Input/Output files:

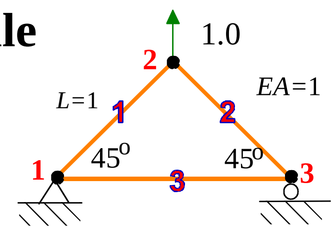
1. Both input and output files are text files, and are opened with the Notepad editor. You can just click on the file Icon.
2. If the input file is called **rats.inp.txt**, an output file called **rats.out.txt**. It will be in the same directory and folder as the input file.
3. To view the output file: Open it with Notepad.

Tutorial Truss



Let $EA=1$, $L_1=L_2=1$

Input File

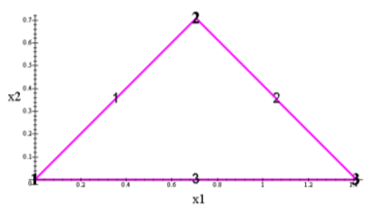


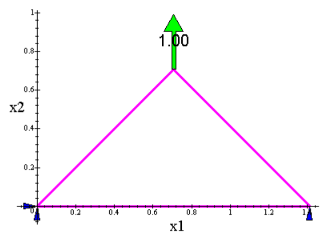
Tutorial	Title
3	Number of nodes
0.0 0.0	x1 y1 for node 1
.707 0.707	x1 y1 for node 2
1.41 0.0	x1 y1 for node 3
3	Number of elements
1 2 1.0	Node 1# node 2# *EA* for element 1
2 3 1.0	Node 1# node 2# *EA* for element 2
3 1 1.0	Node 1# node 2# *EA* for element 3
3	Number of constraints (fixed displacements)
1 1 0.0	Node # direction value of fixed displacement
1 2 0.0	Node # direction value of fixed displacement
3 2 0.0	Node # direction value of fixed displacement
1	Number of loaded nodes
2 0.0 1.0	Node # *P*1 *P*2

To Run:

```
> restart:
# Finte Element Truss Analysis Program
#
#********** GIVE THE NAME OF THE INPUT FILE IN THE LINE BELOW
***********
#                Use frontslash/ in place of backslash\
#                Enclose the name in single backquotes`
#                   End the line with a colon:
#
infilename:=`C:/Documents and Settings/laptop306/Desktop/tandem.inp.txt`:
#
#************************** THEN HIT RETURN*************************
```

Pictures of the loaded, constrained truss appear on screen

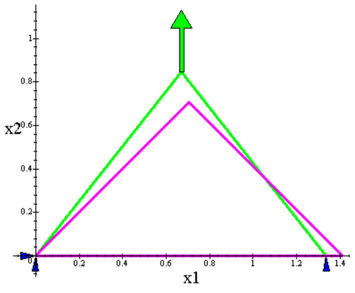




Hit Enter again if these look right to you.

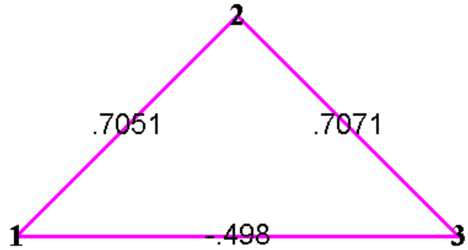
And that's all!

Screen Output: Deformation



Displacements scaled by: , .1047461753

Screen Output: Member Forces



Output File: Problem Statement

Tutorial
Number of Nodes and Elements: 3 3

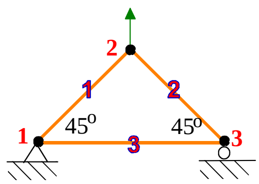
Node#	x1	x2
1	0.0000	0.0000
2	.7070	.7070
3	1.4100	0.0000

El#	Node 1	Node 2	EA
1	1	2	1.0000
2	2	3	1.0000
3	3	1	1.0000

Fixed Node#	DOF	Value
1	1	0.0000
1	2	0.0000
3	2	0.0000

Loaded Node#	P1	P2
2	0.0000	1.0000

Output File: Results



Nodal Displacements:

Node	u1	u2
1	0.0000	0.0000
2	-.3490	1.3461
3	-.7029	0.0000

Strains and Forces:

Element	Strain	Force
1	.7051	.7051
2	.7071	.7071
3	-.4985	-.4985

Computing procedure

1. Read input data
2. Loop over elements, calculate element stiffness mx, and add components to the global stiffness mx [K]
3. Assemble the residual r vector from the applied forces
4. Modify [K] and r to account for prescribed displacements
5. Solve for the global displacements u= $[K]^{-1}\underline{r}$
6. Calculate element strains and forces.
7. Print results to a file.

Maple Code: Input Variables

Variable	Type	Description
Title	String	Run title
nnode	integer	# of nodes
coord(i,j)	Array: nnode x 2	coord(i,1)=x coord of node # i coord(i,2)=y coord of node # i
nelem	integer	# of elements
connect	array: nelem x 3	connect(i,1)=1^{st} node no. of element i connect(i,2)=2^{nd} node no. of element i connect(i,3)=EA for element i
nfix	integer	# of prescribed displacements
fixnodes	array: nfix x 3	fixnodes(i,1)=node number with the i^{th} fixed displ fixnodes(i,2)=direction (1 or 2) of the i^{th} fixed displ fixnodes(i,3)=value of the i^{th} fixed displ
nload	integer	# of nodes with external loads
loads	array: nloadx3	loads(i,1)=node number of the i^{th} loaded node loads(i,2)=force on node # loads(i,1) in the x-direction loads(i,3)= force on node # loads(i,1) in the y-direction