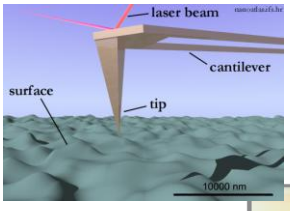
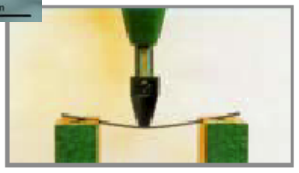
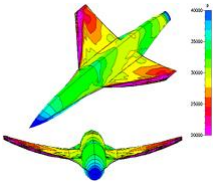
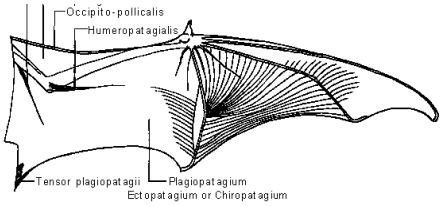
Beam Deflections by Energy Methods

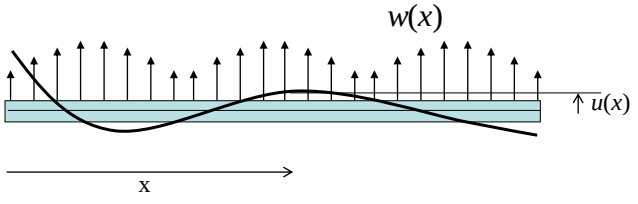


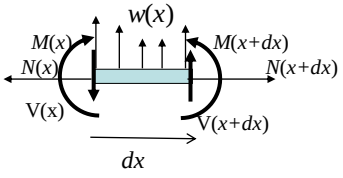






Deflection of a bending beam under loading $w(x)$





$$\frac{dN}{dx} = 0, \frac{dV}{dx} = -w, \frac{dM}{dx} = -V \Rightarrow \frac{d^2M}{dx^2} = w$$

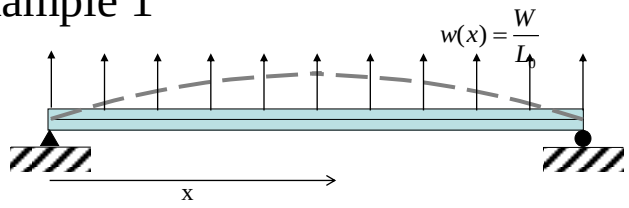
$M(x) = EI\kappa(x)$ Moment-curvature

$\kappa(x) = \frac{d^2u}{dx^2}$ Curvature-delection for $\left|\frac{du}{dx}\right| \ll 1$

$$\Rightarrow EI\frac{d^4u}{dx^4} = w(x)$$

+ BC

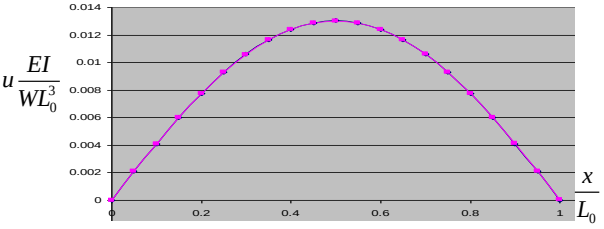
Example 1



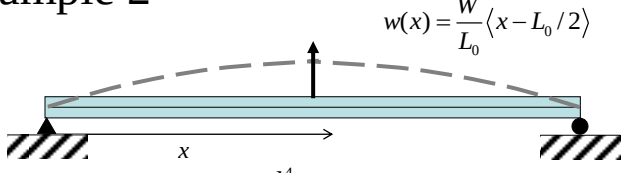
$$EI\frac{d^4u}{dx^4}=w(x)$$

$$u(0)=u(L_0)=0,\ M(0)=EIu''(0),\ M(L_0)=EIu''(L_0)=0$$

$$\frac{u}{L_0}=\frac{WL_0^2}{24EI}\frac{x}{L_0}\left[1-2\frac{x^2}{L_0^2}+\frac{x^3}{L_0^3}\right]$$



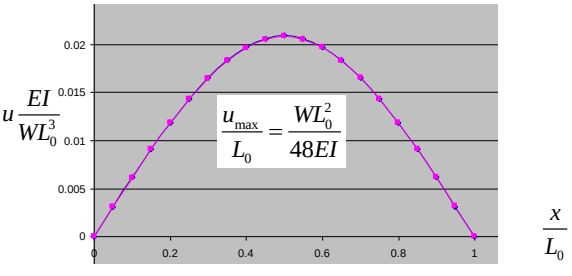
Example 2



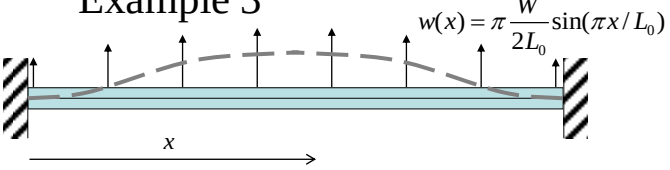
$$EI\frac{d^4u}{dx^4}=w(x)$$

$$u(0)=u(L_0)=0,\ M(0)=EIu''(0),\ M(L_0)=EIu''(L_0)=0$$

$$\frac{u}{L_0}=\frac{WL_0^2}{24EI}\frac{x}{L_0}\left[3-4\frac{x^2}{L_0^2}\right]\ 0\le\frac{x}{L_0}\le\frac{1}{2},\ \frac{u}{L_0}=\frac{WL_0^2}{48EI}\left(1-\frac{x}{L_0}\right)\left[3-4\left(\frac{x}{L_0}-1\right)^2\right]\ \frac{1}{2}\le\frac{x}{L_0}\le 1$$

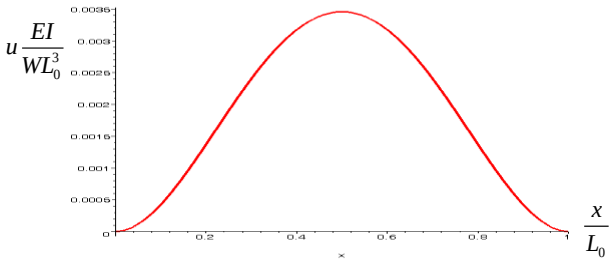


Example 3

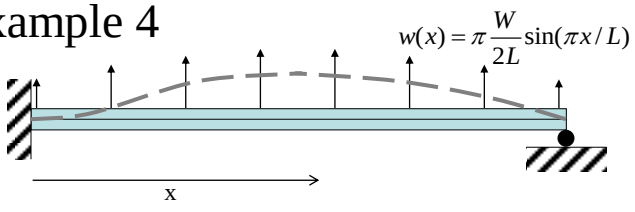


$$EI\frac{d^4u}{dx^4} = w(x),\ u(0) = u(L_0) = 0,\ u'(0) = u'(L_0) \Rightarrow$$

$$\frac{u}{L_0} = \frac{WL_0^2}{2EI\pi^3}\left[\sin\left(\frac{\pi x}{L_0}\right) + \pi\frac{x}{L_0}\left(1 - \frac{x}{L_0}\right)\right]$$



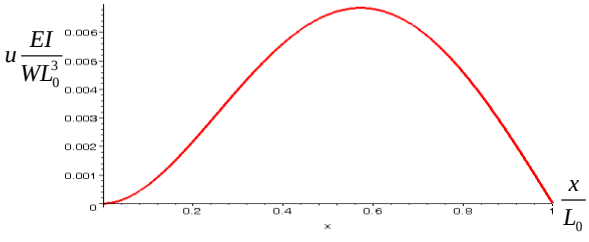
Example 4



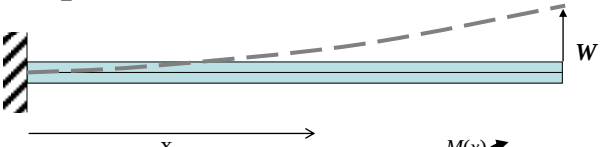
$$EI\frac{d^4u}{dx^4} = w(x)$$

$$u(0) = u(L_0) = 0,\ u'(0) = 0,\ M(L_0) = EIu''(L_0) = 0$$

$$\frac{u}{L_0} = \frac{WL_0^2}{2EI\pi^3}\left[\sin\left(\frac{\pi x}{L_0}\right) - \frac{\pi x}{2L_0}\left(2 - \frac{x}{L_0}\right)\left(1 - \frac{x}{L_0}\right)\right]$$



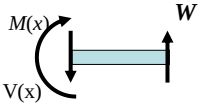
Example 5



$$EI\frac{d^4u}{dx^4} = w(x) = 0$$

$$u(0) = 0,\ u'(0) = 0$$

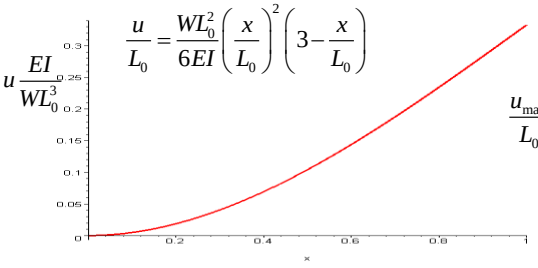
$$M(L_0) = EIu''(L_0) = 0, V(L_0) = -EIu'''(L_0) = W$$



$x=L_0$-dx $\quad$ $x=L_0$

$V(L_0-dx)=W$

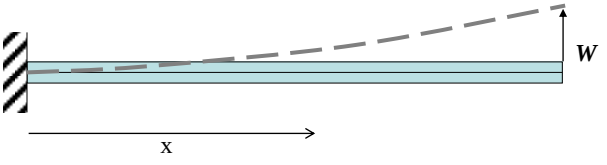
$$\frac{u}{L_0} = \frac{WL_0^2}{6EI}\left(\frac{x}{L_0}\right)^2\left(3-\frac{x}{L_0}\right)$$



$$\frac{u_{max}}{L_0} = \frac{WL_0^2}{3EI}$$

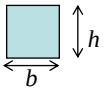
$$\frac{x}{L_0}$$

How Good is Beam Theory?



Acrylic Cantilever:

- L_0=1 meter,
- b=h=0.5 in=0.0127 meters,
- W=4.9 Newtons (0.5kg)
- E=1.5GPa
- I=$b^4/12=2.17\times10^{-9}m^4$



$$u(L_0) = \frac{WL_0^3}{3EI} = 0.5 \text{ meters}$$

MEASURED $u(L_0)$=.29 meters