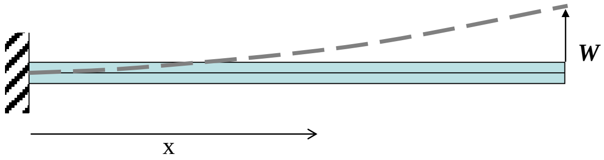
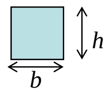
How Good is Beam Theory?



Acrylic Cantilever:

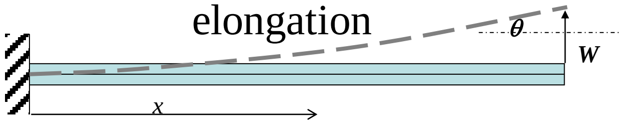


- L_0=1 meter,
- b=h=0.5 in=0.0127 meters,
- W=4.9 Newtons (0.5kg)
- E=1.5GPa
- I=b^4/12=$2.17\times10^{-9}m^4$

$$u(L_0)=\frac{WL_0^3}{3EI}=0.5 \text{ meters}$$

- MEASRED u(L)=.29 meters

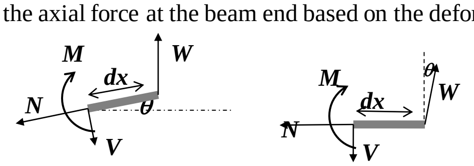
Beam Theory predicts no axial force or elongation



- Beam theory gives

$$u(L_0)=\frac{WL_0^3}{3EI},\ \theta(L_0)=\frac{WL_0^2}{2EI}=\frac{3}{2}\frac{u(L_0)}{L_0}$$

- Estimate the axial force at the beam end based on the deformed beam geometry



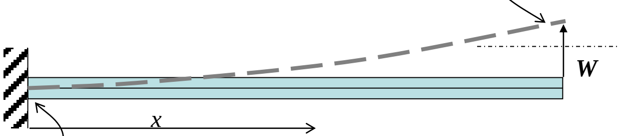
$$N(L_0)=W\sin\theta(L_0),\quad V(L_0)=W\cos\theta(L_0)$$

$$\Rightarrow N(L_0)\approx W\frac{3u(L_0)}{2L_0},\ V(L_0)\approx W\left[1-\frac{1}{2}\left(\frac{3u(L_0)}{2L_0}\right)^2\right]$$

***N* is of the order *Wu*(L_0)/L_0**

If the axial force is $N \approx W(3u_{max}/2L_0)$:

- Axial stress at $x=L_0$ is $\sigma_x(L_0) = \frac{N}{A} \approx \frac{W}{A}\frac{u_{L_0}}{3L_0}$



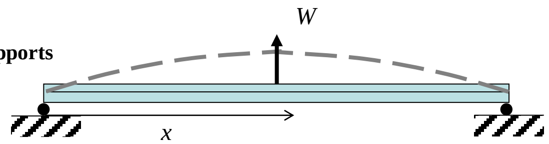
- Axial stress (due to bending) at $x=0$ is

$$\sigma_x(0, y=-h/2) = \frac{Mh}{2I} \approx \frac{WL_0 h}{2I} = \frac{WL_0 h}{2h^4/12} = 6\frac{W}{A}\frac{L_0}{h}$$

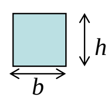
Bending dominates the stresses in this first order correction to small strain beam theory

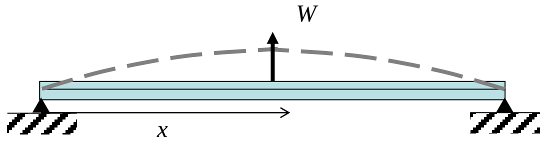
Which case would be better predicted by small deflection beam theory?

I: axial deflection permitted at the supports



II : no axial deflection permitted at the supports

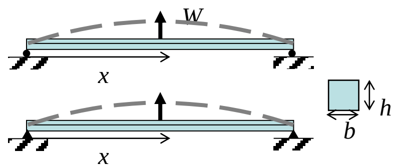


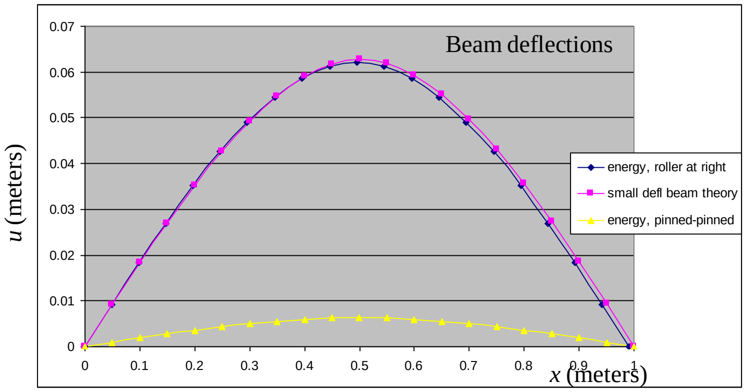


Beam theory: $u(L_0/2) = \frac{WL_0^3}{48EI}$

Energy minimization (no small deflection assumptions)

- L_0=1 meter, b=h=0.5 in=0.0127 meters,
- W=9.8Newtons (1kg), E=1.5GPa (acrylic)

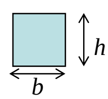


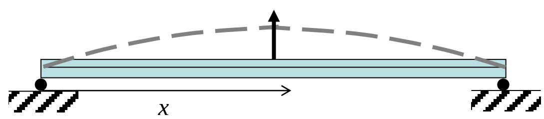


Beam theory overestimates the deflection but is closest in case I

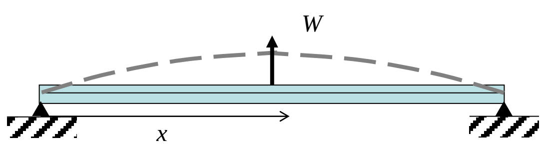
$$\text{Beam theory: } u(L_0/2)=\frac{WL_0^3}{48EI}=6.3\text{cm}$$

I: Energy calculation: $u(L_0/2)$=6.2 mm=99% of u_{beam}

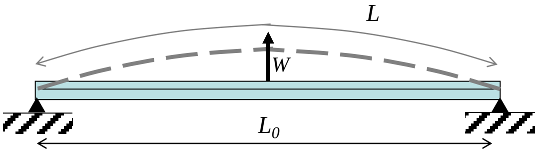




II: Energy calculation: $u(L_0/2)$=6 mm=10% of u_{beam}



To do the calculation for case II, we must allow the beam to elongate (or no vertical deflection can occur):



- Beam original length is L_0;
- Beam length after deflection is L.
- Strain energy of the beam due to elongation is

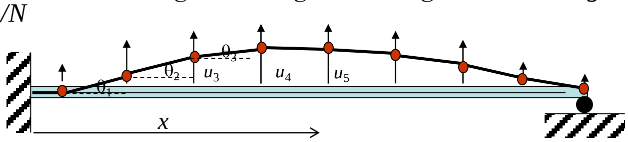
$$V_e = \frac{1}{2}\frac{EA}{L_0}\left(L - L_0\right)^2$$

- Total energy:

$$V[u] = \int_0^L \frac{1}{2} EI\left(\kappa(x)\right)^2 dx - \int_0^L w(x)u(x)dx + \frac{1}{2}\frac{EA}{L_0}\left(L - L_0\right)^2$$

Excel

- Divide the beam into N segments, each of original length $s=L_0/N$. *Now allow the beam's length to change! New length is L, new seg length is S=L/N*

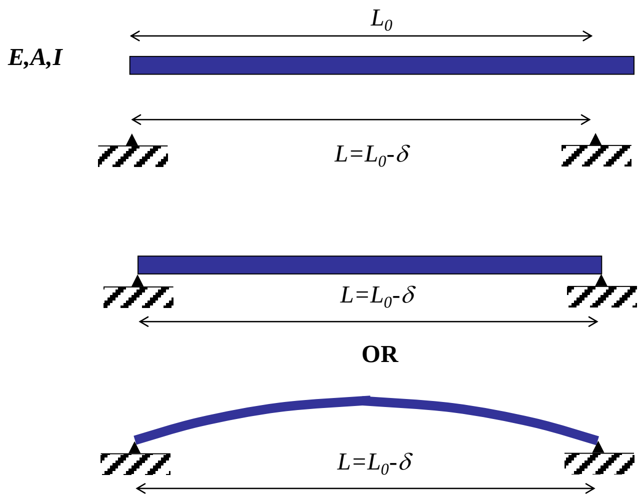


- Beam deflections are defined by the new segment length S and the orientation angles θ_i

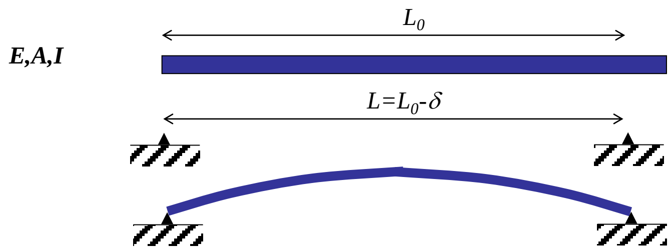
$$x_1 = S\cos\theta_0, \quad x_2 = x_1 + S\cos\theta_1 \; .... \; x_i = x_{i-1} + S\cos\theta_{i-1}$$
$$u_1 = S\sin\theta_0, \quad u_2 = u_1 + S\sin\theta_1 \; .... \; u_i = u_{i-1} + S\sin\theta_{i-1}$$

- Curvature is $\kappa_i = d\theta/ds$. At the i[th] segment, the curvature is $\kappa_i=(\theta_i-\theta_{i-1})/s$
- Elastic energy due to bending is $V_i = \frac{S}{2} EI\kappa_i^2$
- Load is $W_i=w(x_i)s$ (i=1,2,…N-1), $W_0=w(0)s/2$, $W_N=w(L_0)s/2$
- Total energy is $V = \frac{1}{2}EIs\sum_{i=1}^{N}\frac{\left(\theta_i-\theta_{i-1}\right)^2}{s^2} - \sum_{i=0}^{N} w(x_i)su_i + \frac{1}{2}\frac{EA}{L_0}\left(L-L_0\right)^2$
- Minimize V, varying the angles θ_i, and length S subject to any constraints from the BC.

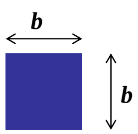
Buckling via energy



Critical δ by Euler



$$P_{CR} = \pi^2 \frac{EI}{L_0^2} = EA\frac{\delta_{CR}}{L_0} \Rightarrow \delta_{CR} = \pi^2 \frac{I}{AL_0}$$



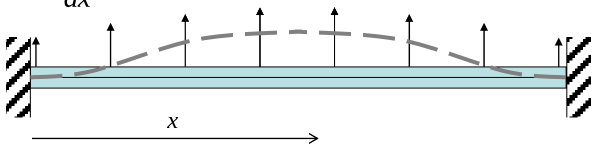
$I = b^4/12 = A^2/12$

$$\delta_{CR} = \pi^2 \frac{A^2}{12AL_0} = \pi^2 \frac{A}{12L_0}$$

Would I Lie to You?

Let $u(x)$ be the solution to the BVP

$$EI\frac{d^4u}{dx^4}=w(x),\ u(0)=u(L_0)=0,\ \ u'(0)=u'(L_0)$$



Then, among all functions that satisfy the boundary conditions, the solution u minimizes the potential energy.

$$V[u]=\int_0^{L_0}\frac{1}{2}EI\left(\frac{d^2u}{dx^2}\right)^2 dx-w(x)u(x)$$

Prove it All Night

Let $u(x)$ be the solution to the BVP.

$$u^{IV}(x)=w(x)/EI,\ u(0)=u(L_0)=0,\ \ u'(0)=u'(L_0)=0$$

Let $\delta(x)$ be any old function of x that satisfies the Boundary conditions

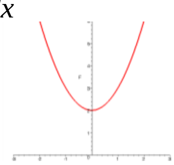
$$\delta(0)=\delta(L_0)=0,\ \ \delta'(0)=\delta'(L_0)=0$$

The deflection defined by $v(x)=u(x)+a\,\delta(x)$ satisfies the same BC for any value of a

$$v(0)=v(L_0)=0,\ \ v'(0)=v'(L_0)=0$$

Define a function $F(a)$, which is the PE of the deflection v:

$$F(a)=V[v]=V[u+a\delta]=\int_0^{L_0}\frac{1}{2}EI\left(\frac{d^2v}{dx^2}\right)^2 dx-\int_0^{L_0}w(x)v(x)dx$$



Will show $dF/da=0$ when $a=0$.
$F(0)$ gives the energy $V[u]$

Differentiate

$$F(a) = V[v] = V[u + a\delta] = \int_0^{L_0} \frac{1}{2} EI \left(\frac{d^2 v}{dx^2} \right)^2 dx - \int_0^{L_0} w(x)v(x)dx$$

$$F(a) = \int_0^{L_0} \frac{1}{2} EI \left(u'' + a\delta'' \right)^2 dx - \int_0^{L_0} w(u + a\delta)dx$$

$$\frac{dF}{da}(a) = \int_0^{L_0} EI \left(u'' + a\delta'' \right) \delta'' dx - \int_0^{L_0} w\delta dx$$

$$\frac{dF}{da}(0) = \int_0^{L_0} EIu''\delta'' dx - \int_0^{L_0} w\delta dx$$

Integrate by parts

$$\int_0^{L_0} EIu''\delta'' dx = \underbrace{EIu''\delta'\Big|_0^{L_0}}_{0} - \int_0^{L_0} EIu'''\delta' dx = -\underbrace{EIu'''\delta\Big|_0^{L_0}}_{0} + \int_0^{L_0} EI \frac{d^4 u}{dx^4} \delta dx$$

So far....

$$F(a) = V[v] = V[u + a\delta] = \int_0^{L_0} \frac{1}{2} EI \left(\frac{d^2 v}{dx^2} \right)^2 dx - \int_0^{L_0} w(x)v(x)dx$$

$$\frac{dF}{da}(0) = \int_0^{L_0} EIu''\delta'' dx - \int_0^{L_0} w\delta dx$$

$$\int_0^{L_0} EIu''\delta'' dx = \int_0^{L_0} EI \frac{d^4 u}{dx^4} \delta dx$$

So good!

$$\frac{dF}{da}(0) = \int_0^{L} EI \frac{d^4 u}{dx^4} \delta dx - \int_0^{L_0} w\delta dx = \int_0^{L_0} \left(EI \frac{d^4 u}{dx^4} - w \right) \delta dx = 0$$

$$\text{since } EI \frac{d^4 u}{dx^4} = w$$

The PE is a minimum for the deflection *u* satisfying the DEQ and the BC