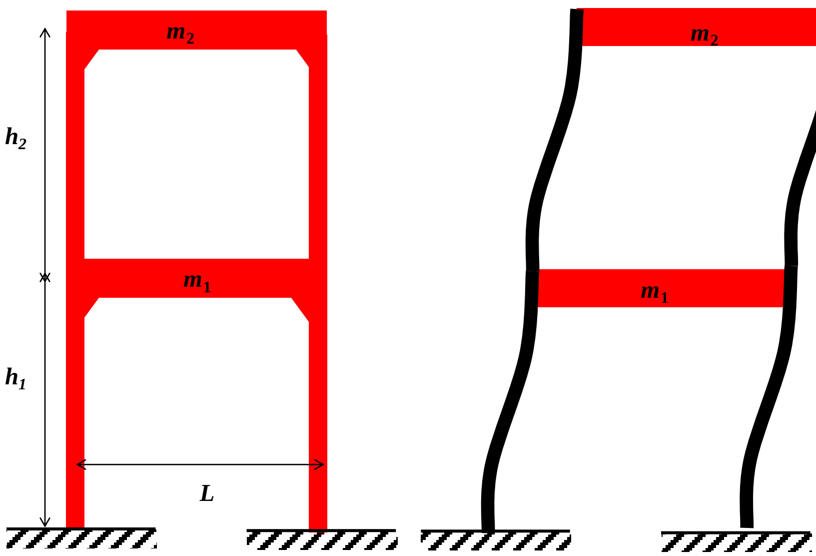
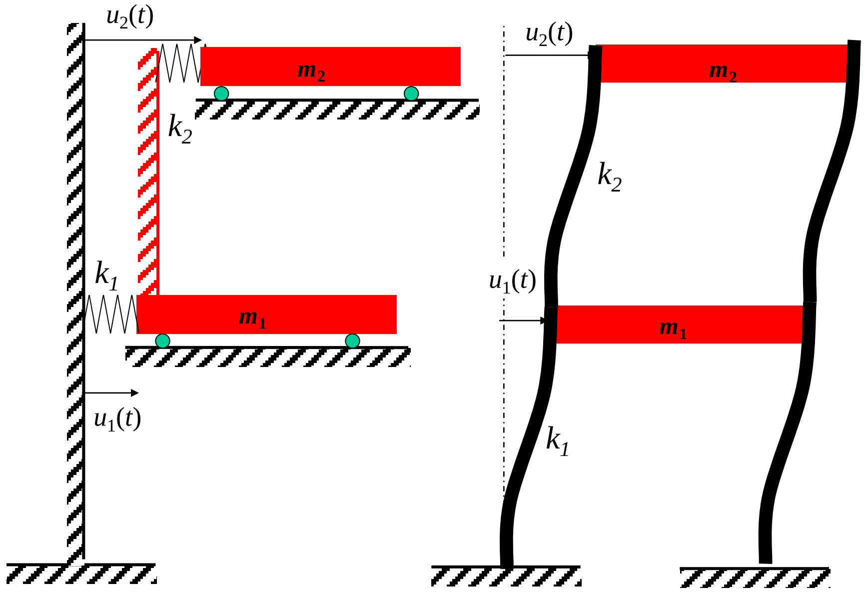
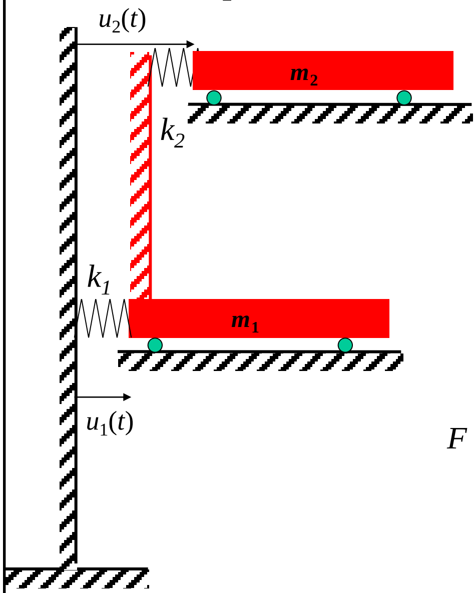
Free Vibration of a Multi-Degree of freedom structure



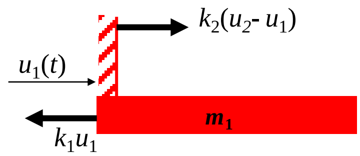
Represent walls as lateral springs, stiffness k_i



Equations of Motion: Free Vibration



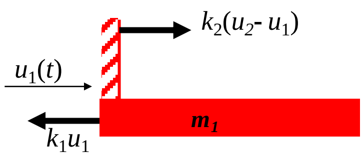
$$F = ma \Rightarrow m_2\ddot{u}_2 + k_2\left(u_2 - u_1\right) = 0$$



$$F = ma \Rightarrow m_1\ddot{u}_1 + k_1u_1 - k_2\left(u_2 - u_1\right) = 0$$

Equations of Motion: Free Vibration

$$m_2\ddot{u}_2 + k_2\left(u_2 - u_1\right) = 0$$



$$m_1\ddot{u}_1 + k_1u_1 - k_2\left(u_2 - u_1\right) = 0$$

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[M\right]\underline{\ddot{u}} + \left[K\right]\underline{u} = \underline{0}$$

Free Vibration: Find ω_n, A_1 and A_2

Try $u_1 = A_1 \sin(\omega_n t + \phi)$, $u_2 = A_2 \sin(\omega_n t + \phi)$

Both floors move in phase at frequency ω, amplitudes A_1 and A_2

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-\omega_n^2 \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\left[-\omega_n^2 \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \right]\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Free Vibration: Find ω_n, A_1 and A_2

Try $u_1 = A_1 \sin(\omega_n t + \phi)$, $u_2 = A_2 \sin(\omega_n t + \phi)$

$$\left[-\omega_n^2 \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \right]\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[-\omega_n^2 [M] + [K] \right] \underline{A} = \underline{0}$$

Eigenvalue/Eigenvector Problem

$$\left[-\omega_n^2 [1] + \left[M^{-1} K \right] \right] \underline{A} = \underline{0}$$

Example $m_1=m_2$, $k_1=k_2$

Try

$u_1=A_1\sin(\omega t+\phi)$,

$u_2=A_2\sin(\omega t+\phi)$

$$m\ddot{u}_1 + 2ku_1 - ku_2 = 0$$

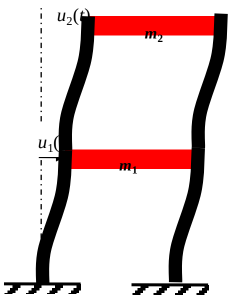
$$m\ddot{u}_2 + ku_2 - ku_1 = 0$$

$$-\omega_n^2\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} 2k-\omega_n^2 m & -k \\ -k & k-\omega_n^2 m \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2k/m-\omega_n^2 & -k/m \\ -k/m & k/m-\omega_n^2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



Solve

$$\begin{pmatrix} 2k/m-\omega_n^2 & -k/m \\ -k/m & k/m-\omega_n^2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

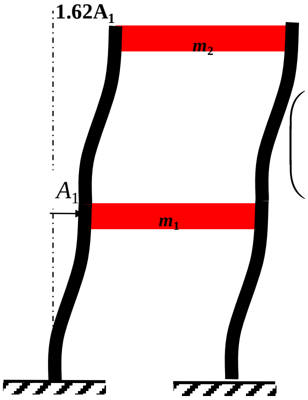
$$\det\begin{pmatrix} 2k/m-\omega_n^2 & -k/m \\ -k/m & k/m-\omega_n^2 \end{pmatrix} = 0$$

Two Natural Frequencies:

$$\omega_1^2 = (3-\sqrt{5})\frac{k}{2m},\ \omega_2^2 = (3+\sqrt{5})\frac{k}{2m}$$

Mode I

Mode Shape



$$\omega_1^2 = (3-\sqrt{5})\frac{k}{2m} \Rightarrow \omega_1 = 0.62\sqrt{\frac{k}{m}}$$

$$\begin{pmatrix} 2k/m-\omega_1^2 & -k/m \\ -k/m & k/m-\omega_1^2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

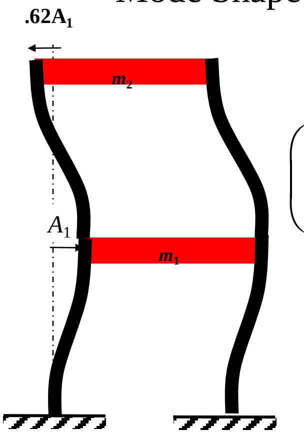
$$\Rightarrow A_2 = \frac{1+\sqrt{5}}{2}A_1 = 1.62A_1$$

$$u_1(t) = A_1 \sin\left(\omega_1 t + \phi\right)$$

$$u_2(t) = A_1\frac{1+\sqrt{5}}{2}\sin\left(\omega_1 t + \phi\right) = 1.62u_1(t)$$

Mode II

Mode Shape



$$\omega^2 = \omega_2^2 = (3+\sqrt{5})\frac{k}{2m} \Rightarrow \omega_2 = 1.62\sqrt{\frac{k}{m}}$$

$$\begin{pmatrix} 2k/m-\omega_2^2 & -k/m \\ -k/m & k/m-\omega_2^2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

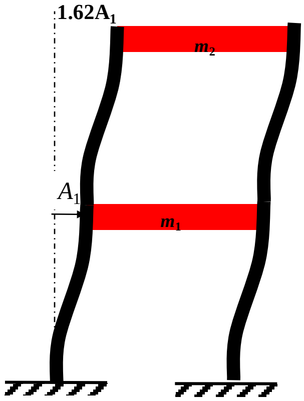
$$\Rightarrow A_2 = \frac{1-\sqrt{5}}{2}A_1 = 0.62A_1$$

$$u_1(t) = A_1 \sin\left(\omega_2 t + \phi\right)$$

$$u_2(t) = A_1\frac{1-\sqrt{5}}{2}\sin\left(\omega_2 t + \phi\right) = 0.62u_1(t)$$

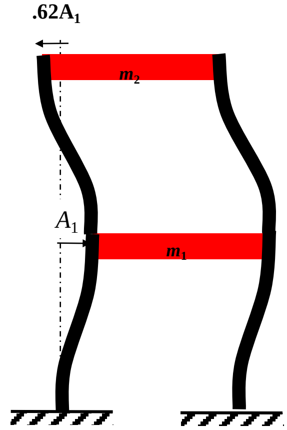
Summary

Mode I



$$\omega_1 = 0.62\sqrt{\frac{k}{m}}$$

Mode II

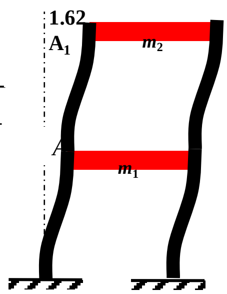


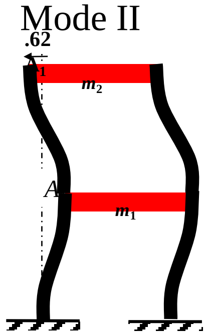
$$\omega_2 = 1.62\sqrt{\frac{k}{m}}$$

General Free Vibration Response

Mode I + Mode II

$$\omega_1 = 0.62\sqrt{\frac{k}{m}}$$





$$\omega_2 = 1.62\sqrt{\frac{k}{m}}$$

$$u_1(t) = A_1^I \sin\left(\omega_1 t + \phi_1\right) + A_1^{II} \sin\left(\omega_2 t + \phi_2\right)$$

$$u_2(t) = A_1^I\, 0.62 \sin\left(\omega_1 t + \phi_1\right) - A_1^{II}\, 1.62 \sin\left(\omega_2 t + \phi_2\right)$$

4 unknowns:

$$A_1^I, A_1^{II}, \phi_1, \phi_2$$

4 IC

$$u_1(0) = u_{10}, \dot{u}_1(0) = v_{10}, u_2(0) = u_{20}, \dot{u}_2(0) = v_{20}$$