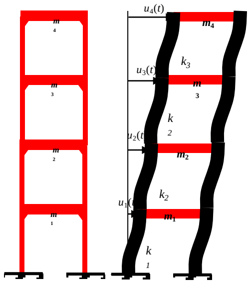
Find 4 natural frequencies, Mode shapes

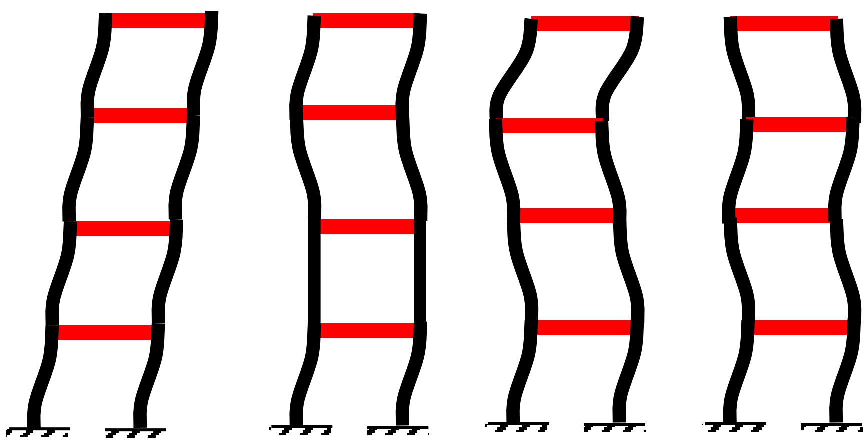


$$\underbrace{\begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & m_3 & \\ & & & m_4 \end{bmatrix}}_{\text{M: Mass Matrix}} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 & & \\ -k_2 & k_2+k_3 & -k_3 & \\ & -k_3 & k_3+k_4 & -k_4 \\ & & -k_4 & k_4 \end{bmatrix}}_{\text{K: Stiffness Matrix}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\det\left[-\omega^2\mathbf{M}+\mathbf{K}\right]=0, \quad \left[-\omega^2\mathbf{M}+\mathbf{K}\right]\underline{A}=\underline{0}$$

$$\det\left[-\omega^2\mathbf{1}+\mathbf{H}\right]=0, \quad \left[-\omega^2\mathbf{1}+\mathbf{H}\right]\underline{A}=\underline{0}$$

They look like this (equal *k*’s, equal *m*’s)



$\omega_1 = 0.35\sqrt{\frac{k}{m}}$ $\omega_2 = \sqrt{\frac{k}{m}}$ $\omega_3 = 1.5\sqrt{\frac{k}{m}}$ $\omega_4 = 1.9\sqrt{\frac{k}{m}}$

Mode Shape Orthogonality

$$-\omega_i^2 \mathbf{M}\underline{A}_i + \mathbf{K}\underline{A}_i = \underline{0}$$

$$-\omega_i^2 \underline{A}_j \bullet \mathbf{M}\underline{A}_i + \underline{A}_j \bullet \mathbf{K}\underline{A}_i = 0$$

$$-\omega_i^2 \underline{A}_i \bullet \mathbf{M}^T \underline{A}_j + \underline{A}_i \bullet \mathbf{K}^T \underline{A}_j = 0$$

$$-\omega_i^2 \underline{A}_i \bullet \mathbf{M}\underline{A}_j + \underline{A}_i \bullet \mathbf{K}\underline{A}_j = 0 \quad (1)$$

$$-\omega_j^2 \mathbf{M}\underline{A}_j + \mathbf{K}\underline{A}_j = \underline{0}$$

$$-\omega_j^2 \underline{A}_i \bullet \mathbf{M}\underline{A}_j + \underline{A}_i \bullet \mathbf{K}\underline{A}_j = 0 \quad (2)$$

$$\left(\omega_i^2 - \omega_j^2\right) \underline{A}_j \bullet \mathbf{M}\underline{A}_i = 0$$

$$\underline{A}_j \bullet \mathbf{M}\underline{A}_i = 0 \;\; i \neq j$$

Initial Conditions $\underline{u}(0)=\underline{u}_0$, $\dot{\underline{u}}(0)=\underline{v}_0$

$$\underline{u}(t) = \sum \left[a_i \cos \omega_i t + b_i \sin \omega_i t \right] \underline{A}_i$$

$$\underline{u}(0) = \sum a_i \underline{A}_i = \underline{u}_0, \; \dot{\underline{u}}(0) = \sum \omega_i b_i \underline{A}_i = \underline{v}_0$$

$$\sum a_i \underline{A}_j \bullet \mathbf{M}\underline{A}_i = a_j \underline{A}_j \bullet \mathbf{M}\underline{A}_j = \underline{A}_j \bullet \mathbf{M}\underline{u}_0$$

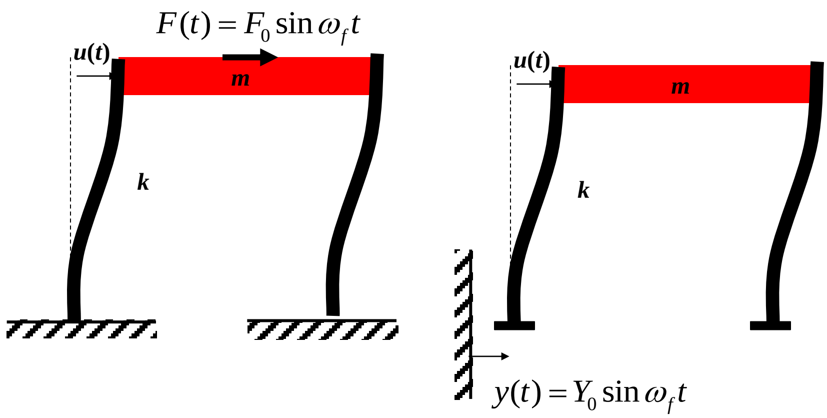
$$\Rightarrow a_j = \frac{\underline{A}_j \bullet \mathbf{M}\underline{u}_0}{\underline{A}_j \bullet \mathbf{M}\underline{A}_j}$$

$$\sum \omega_i b_i \underline{A}_j \bullet \mathbf{M}\underline{A}_i = \omega_j b_j \underline{A}_j \bullet \mathbf{M}\underline{A}_j = \underline{A}_j \bullet \mathbf{M}\underline{v}_0$$

$$\Rightarrow b_j = \frac{\underline{A}_j \bullet \mathbf{M}\underline{v}_0}{\omega_j \underline{A}_j \bullet \mathbf{M}\underline{A}_j}$$

$$\underline{A}_j \bullet \mathbf{M}\underline{A}_i = 0 \;\; i \neq j$$

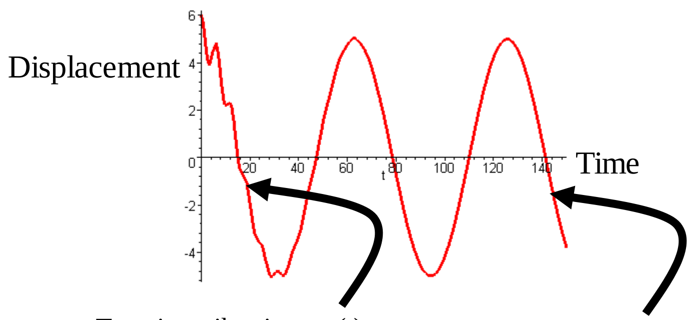
Forced Vibration (Periodic)



External Forcing

Base Excitation

Typical Response



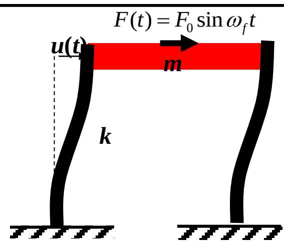
Transient vibrations $u_h(t)$ at the natural frequency. Depend on I.C. These decay exponentially with time.

Steady-state Vibrations $u_p(t)$ at the forcing frequency Independent of I.C. These do not decay.

Total response: $u(t)=u_h(t)+u_p(t)$

Amplitude of the steady-state vibrations is (very) large if the forcing frequency is at or near the system natural frequency.

The Steady State Response: Undamped



$$\frac{d^2u}{dt^2} + \omega_n^2 u = \frac{F_o}{k}\sin(\omega t)$$

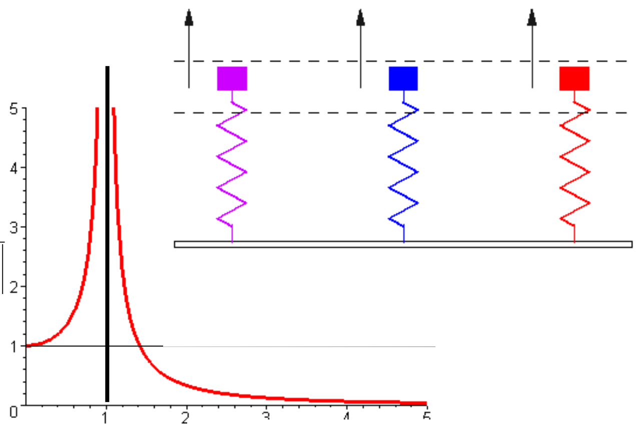
$$u(t) = X \sin(\omega t) \qquad X = \frac{F_0/k}{\left|1-\omega^2/\omega_n^2\right|}$$

Amplitude $X \to \infty$ as $\omega \to \omega_n$

The Steady State Response

Amplitude / static disp

$$\frac{X}{F_0/k} = \frac{1}{\left|1-\omega^2/\omega_o^2\right|}$$



forcing freq/nat. freq: ω/ω_O