Example of a Low-Pass Filter System Design

Some years ago, a Brown faculty member in Geological Sciences asked me to design and build a set of geomagnetic sensors. The sensors were used to measure the fluctuation in the Earth's local magnetic field caused by distant meteorological disturbances, particularly disturbances from distant lightning. Information on the local conductivity of the Earth's crust can be derived from this data and used to find some kinds of mineral deposits (ones with exceptionally high or low conductivity – water, ores and oil) and to interpret crustal structure. Most of the data is near the resonant frequencies of the cavity formed by the ionosphere and the surface of the core of the Earth. These frequencies are around 7 Hz and below, with some information up to about 20 Hz. The earliest satisfactory sensors were simple coils with long stick-shaped cores of laminated transformer iron. These systems have severe problems with interference from magnetic fields of 60 Hz power lines and their harmonics, no matter where the measurements are made. There is essentially no place on Earth, not even in the desert wilderness, that is completely free of fields from electric power distribution.

The system used filters to remove out-of-band noise, particularly the noise at 60 and 120 Hz. The filters needed to have some modest gain to help bring the sensor signal levels up to usable voltages without introducing noise or dynamic range problems. Consider the design of a low-pass filter for such an application with a cutoff at 20 Hz and a gain at DC of 20 DB. To meet the requirement for the suppression of 60 Hz, we ask that its gain at 60 Hz be at least 35 Db less than at DC. Since the sensors had roughly 2K source impedance, we required that the filter have at least 10 K ohms input impedance. For convenience, we will assume that a Butterworth filter will be adequate. (The sensor output goes to an ADC converter, so some compromise between sharp cutoff and phase linearity is required. Butterworth filters were common practice at the time to meet this tradeoff. The system also used notch filters at 60 and 120 Hz to remove more power line interference.)

The first thing we need to know is what order of Butterworth filter will be adequate for the 60 Hz rejection specification. The magnitude of the response function for a 20 Hz cutoff filter of order n is:

$$\left| H_n\left(\frac{s}{\omega_c}\right) \right| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}} = \frac{1}{\sqrt{1+\left(\frac{f}{20}\right)^{2n}}} \tag{1.1}$$

Thirty five decibels is a ratio of 56.2:1. Therefore, $\left| H\left(f=60\right) \le 0.018 \right|$, and

$\sqrt{1+9^n} \ge 56.2$. The order n can be determined as the first integer larger than the solution to this equation. It is straightforward to check that $n = 4$ satisfies the requirement.

Since a fourth order filter is required, it can be made from two quadratic sections. For the first stage, we will use the circuit shown in Fig. 1, which also gives the principal design equations. All the system gain will be included in this stage in order to have the

best noise performance. The factored form of the Butterworth filter function of fourth order with a 1 rad/sec cutoff is:

$$H_4(s) = \frac{1}{\left(s^2 + 1.848s + 1\right)\left(s^2 + 0.767s + 1\right)} \tag{1.2}$$

We will assign the first factor to the first stage because it has the lowest Q requirement. Scaling that factor for frequency and gain implies that the transfer function of the first stage should be:

$$H_{1st}(s) = \frac{-10}{\left(\dfrac{s}{\omega_c}\right)^2 + \dfrac{1.848s}{\omega_c} + 1} \tag{1.3}$$

where $\omega_c = 2\pi f_c = 125.6$ rad/sec. By comparison to the design equations of Fig. 1, we have the following equations for component values:

$$\frac{R2}{2R1} = 10$$

$$2R3 \cdot C2 = \frac{1.848}{\omega_c} = 1.47 \cdot 10^{-2}$$

$$R2 \cdot R3 \cdot C2^2 = \frac{1}{\omega_c^2} = 6.33 \cdot 10^{-5}$$

$$R1 \cdot C1 = 4R3 \cdot C2$$

 These are four equations in five unknowns, namely R1, R2, R3, C1 and C2. One component value is free to be chosen arbitrarily within the constraint of meeting the requirement that z_{in} be at least 10 K ohms. For signal frequencies near DC, $z_{in} = 2 \cdot R1$, while at frequencies above 20 Hz, the impedance approaches $z_{in} = R1$. Therefore, we will choose R1 = 10 K ohms, keeping in mind that the choice might be revised upward if the capacitors turn out to be inconveniently large. With this choice of R1, the solutions to the equations in the sequence in which they are found are:

R2 = 200 KΩ
C2 = $4.3 \cdot 10^{-8}\ fd. = 0.043\ \mu$fd.
R3 = 170 KΩ
C1 = $2.94 \cdot 10^{-6}\ fd. = 2.94\ \mu$fd.

Figure 2 shows a Sallen-Key filter with unity gain that we will use for the second stage. Scaling the second quadratic factor in the Butterworth filter function for the cutoff frequency of 20 Hz and comparing it to the transfer function in Fig. 2 leads to the component equations:

$$R^2 C_1 C_2 = \frac{1}{\omega_c^2} = 6.33 \cdot 10^{-5}$$

and

$$2RC_1 = \frac{0.767}{\omega_c} = 6.1 \cdot 10^{-3}.$$

Again, there is one fewer design equation than the number of unknown components. This is quite generally true in opamp circuits since the performance of a circuit with ideal operational amplifiers depends only on ratios of component values rather than on their absolute values. If one chooses R = 10 K ohms, then the capacitors are C1 = 0.305 µfd and C2 = 2.08 µfd.

While this completes a design, there are still some practical details to consider. All the capacitors need to have good tolerances. They cannot be electrolytic capacitors because the voltages across them may be both positive and negative. Accurate, non-polarized capacitors tend to be both expensive and physically large. Moreover, we will need to use standard values, and capacitors are not available in as many values as resistors. The largest standard value I feel is acceptable is 1 µfd. Therefore, I will scale up all resistors in the first stage by a factor of three and all capacitors down by a factor of 2.94. (That shifts the cutoff up by a factor of 3/2.94 = 1.02. Since capacitor tolerances are 5%, this error is not significant.) This makes the two capacitor values 1 µfd and 0.015 µfd, which are both standard values. The 30 K resistors are also standard and the 500 K and 588 K values are available in 1 % tolerance. Similarly, the second stage has to be scaled and its capacitors rounded to standard values. I scaled the first stage by a minimum factor so that its input bias current would not become a consideration in offset stability. This is not a consideration in the second stage because of the gain in the first. For the latter stage, a larger scaling factor of 10 is justified. The capacitors become 0.2 µfd and 0.03 µfd. The final system is shown in Figure 3 with all the scaled values. As a final check, one might want to use SPICE to confirm that the effects of component tolerance and operational amplifier bandwidth will still always leave the system with acceptable frequency response.
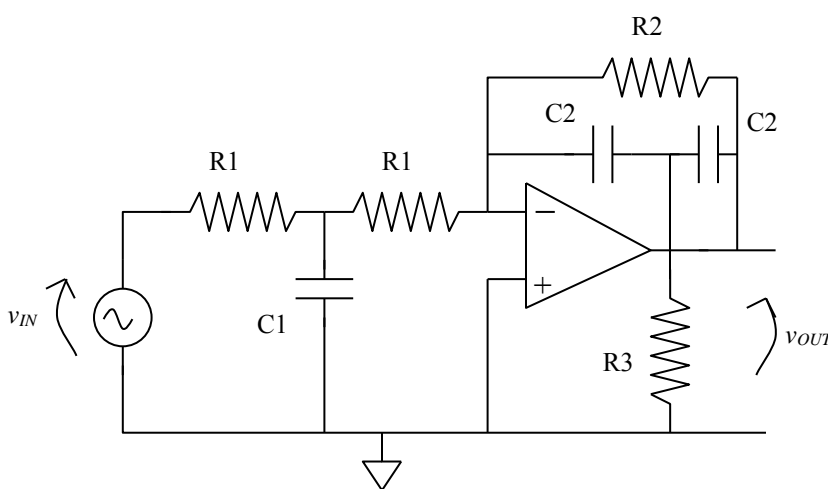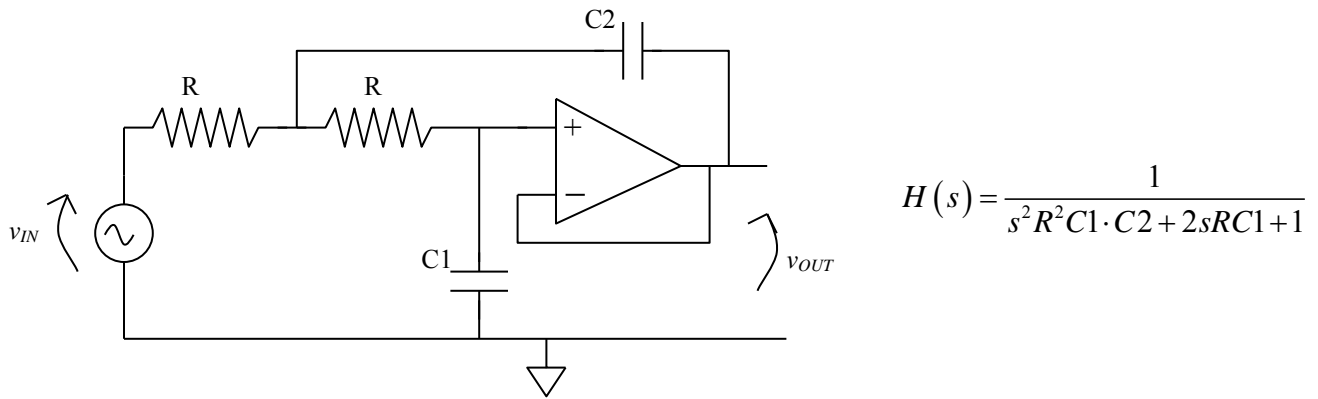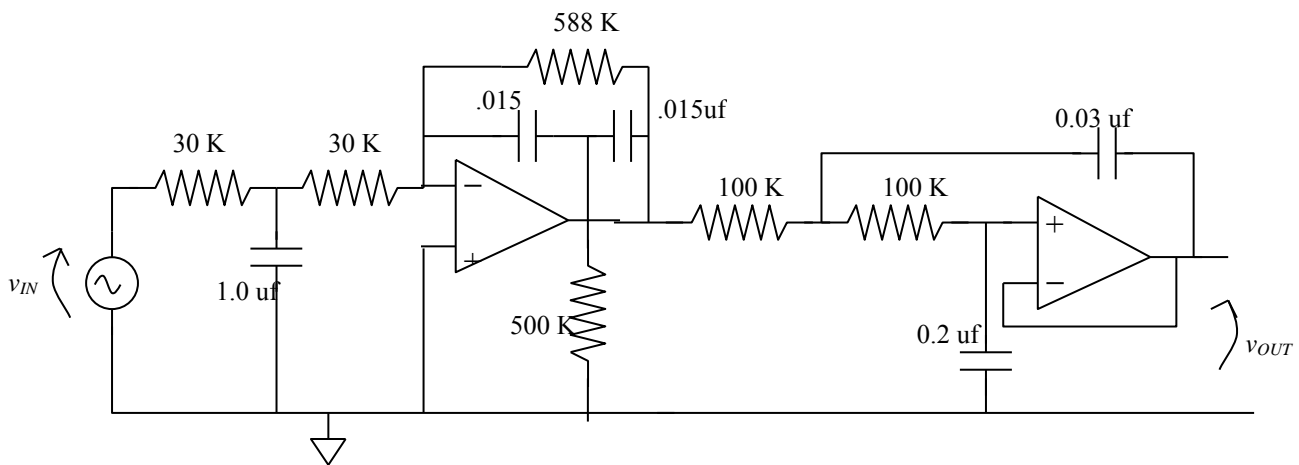

Design constraint:
$$R1 \cdot C1 = 4R3 \cdot C2$$

Transfer function:
$$H(s) = \frac{-\left(\dfrac{R2}{2R1}\right)}{s^2 R2 \cdot R3 \cdot C2^2 + 2sR3 \cdot C2 + 1}$$

Figure 1: Low-pass (all pole) quadratic section with gain.

$$H(s) = \frac{1}{s^2 R^2 C1 \cdot C2 + 2sRC1 + 1}$$

Figure 2: Unity gain, Sallen-Key low pass filter and its transfer function.

Figure 3: Final system Design – Component values have been scaled for smaller capacitors with standard values.