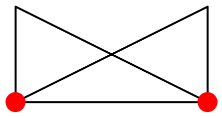
Lecture 4

Announcement: ABAQUS tutorial will be given between 6-8 PM on September 18 (Monday).

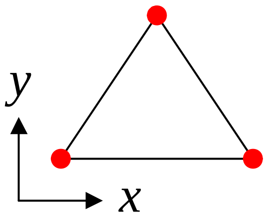
FEM interpolation functions:

1D FEM:



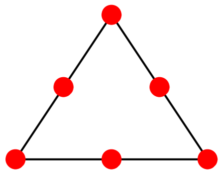
$w(x) = a_1 + a_2 x$ (linear interpolation)

Element types in 2D:



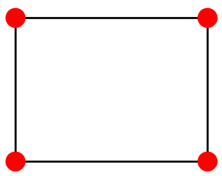
3 noded triangle (linear element)

$w(x, y) = a_1 + a_2 x + a_3 y$



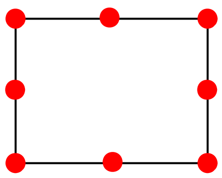
6 noded triangle

$w(x, y) = a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + a_6 y^2$



4 noded quadrilateral

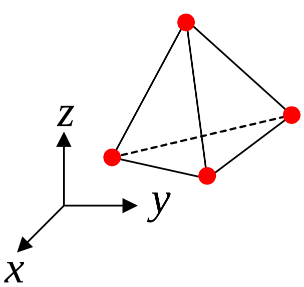
$w(x, y) = a_1 + a_2 x + a_3 y + a_4 xy$



8 noded quadrilateral

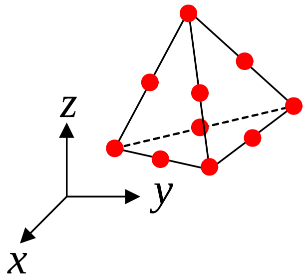
$w(x, y) = a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + a_6 y^2 + a_7 x^2 y + a_8 xy^2$

Element types in 3D:



4 noded tetrahedron (linear element)

$$w(x,\, y,\, z) = a_1 + a_2 x + a_3 y + a_4 z$$

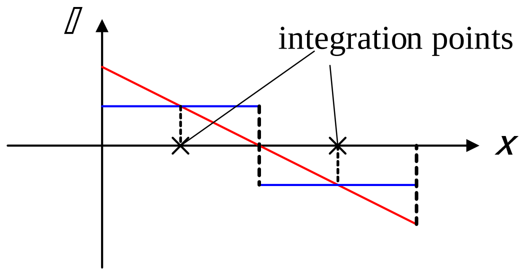


10 noded tetrahedron

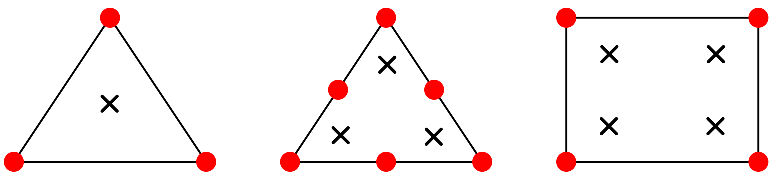
$$w(x,\, y,\, z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 xy + a_6 yz + a_7 xz + a_8 x^2 + a_9 y^2 + a_{10} z^2$$

Integration points for stress:

1D:

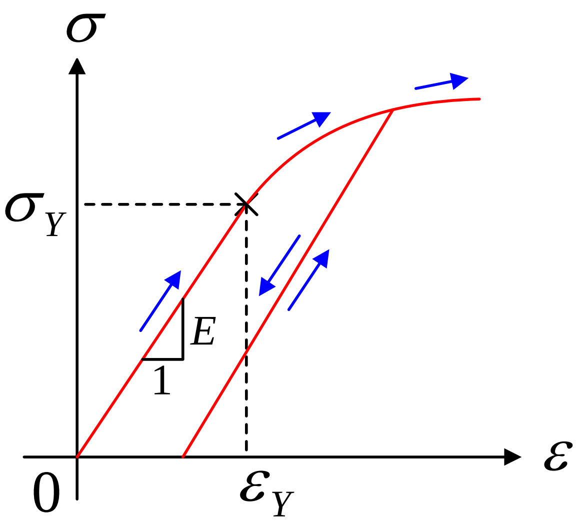


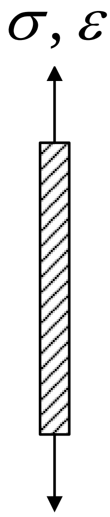
2D:



Material behavior

$\sigma = E\varepsilon$ (Hooke's law is valid before materials yield at a critical strain beyond which irreversible plastic deformation begins to accumulate)





Typically:

Yield strain: $\varepsilon_Y \cong 0.2\%$

Young's modulus: $E \prec 10^{11}\ \mathrm{N/m^2} = 100\ \mathrm{GPa}$ (210 GPa for Steel)

Possion's ratio: $\nu \prec 1/4 \sim 1/3$ (1/3 for steel)

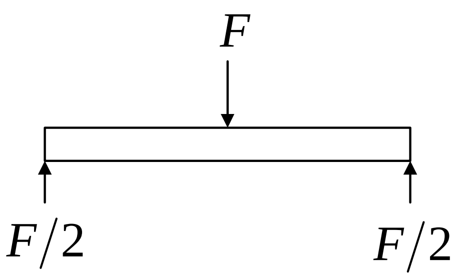
$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{KU} = \mathbf{F}$$

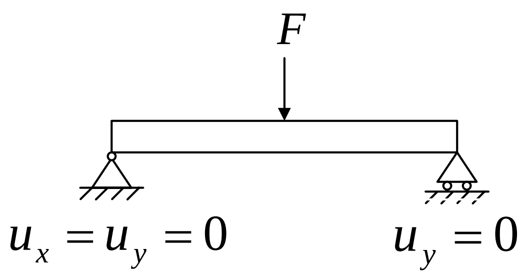
$\mathbf{M}$, $\mathbf{K}$ are positive definite matrices. In numerical calculations such as FEM, it is very important to eliminate rigid body motion. Otherwise, the stiffness matrix will have zero eigenvalues and becomes non-positive definite. This is especially problematic for static problems since we are solving

$$\mathbf{KU} = \mathbf{F}$$

(Can you explain why?)

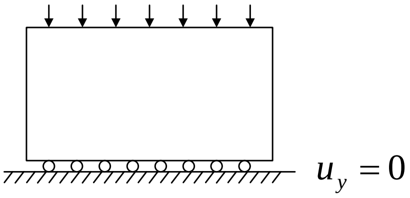
Example 1:



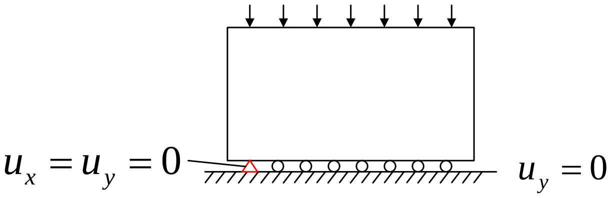


(ill-defined BC) (well-defined)

Example 2:



(ill-defined)



(well-defined)

(End of FEM introduction)

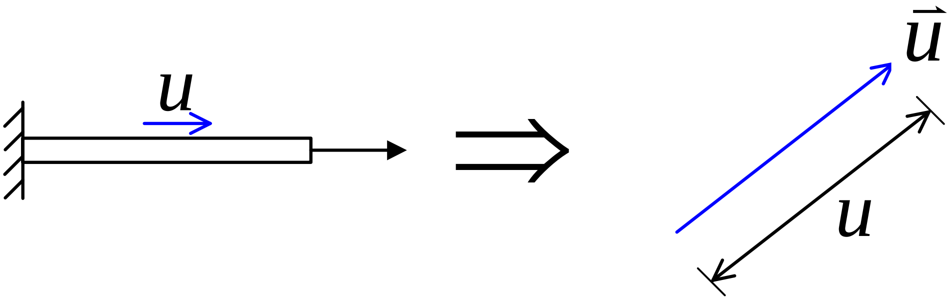
2. Mathematical Background

Displacements

Stresses

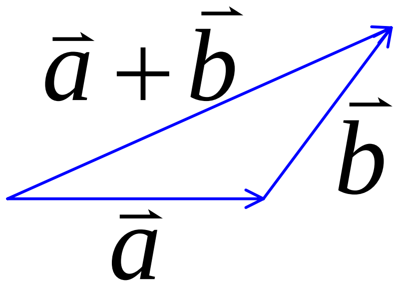
Strains

1D to 3D:

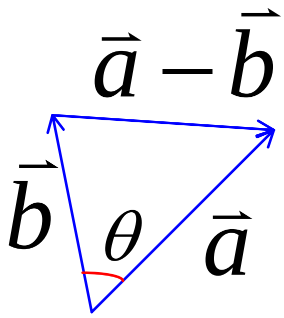


In 3D problems, $\vec{u}$ is a vector (magnitude $|\vec{u}| = u$ plus direction)

Summation of vectors:



Subtraction of vectors:



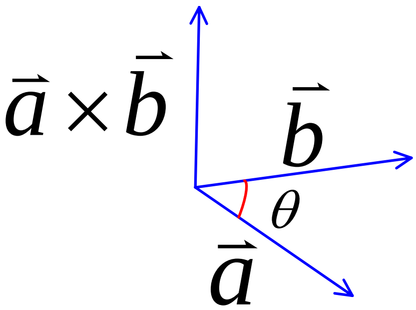
Dot product (or scalar product) of vectors:

$$\vec{a}\cdot\vec{b} = ab\cos\theta$$

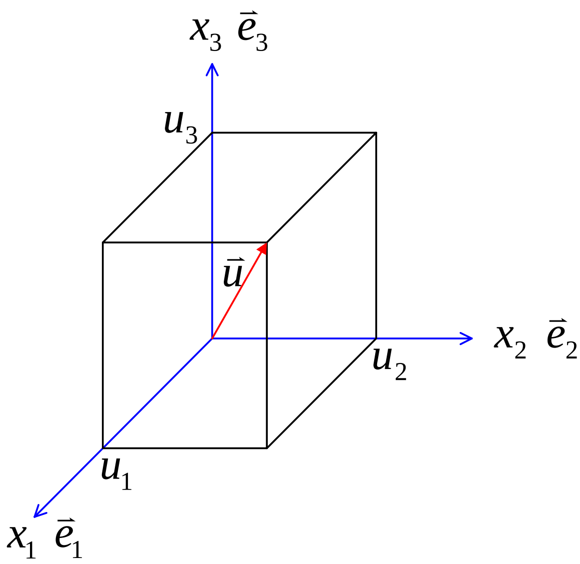
$$\text{If }\ \vec{a}\cdot\vec{b}=0,\ \ \vec{a}\ \text{ is }\ \perp\ \text{ to }\ \vec{b}\quad(\theta=90^\circ)$$

Cross product (or vector product) of vectors:

$$\left|\vec{a}\times\vec{b}\right| = ab\sin\theta$$



To define a vector, we need a coordinate.



$\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ are defined as base vectors.

$\left|\vec{e}_1\right| = \left|\vec{e}_2\right| = \left|\vec{e}_3\right| = 1.$

$$\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3 = \sum_{k=1}^{3} u_k\vec{e}_k = u_k\vec{e}_k$$ (**Einstein Summation Convention**)

Note that the index *k* is completely arbitrary. It is just a notation for summation over 1, 2, 3. One could alternatively write

$$\vec{u} = u_k\vec{e}_k = u_i\vec{e}_i = u_j\vec{e}_j = \cdots$$

	Index notation	Matrix notation
Vectors:	u_i	$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$
Matrices:	A_{ij}	$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

For base vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$

$$\vec{e}_1 \cdot \vec{e}_1 = 1,\ \vec{e}_1 \cdot \vec{e}_2 = 0,\ \vec{e}_1 \cdot \vec{e}_3 = 0$$

$$\vec{e}_2 \cdot \vec{e}_1 = 0,\ \vec{e}_2 \cdot \vec{e}_2 = 1,\ \vec{e}_2 \cdot \vec{e}_3 = 0$$

$$\vec{e}_3 \cdot \vec{e}_1 = 0,\ \vec{e}_3 \cdot \vec{e}_2 = 0,\ \vec{e}_3 \cdot \vec{e}_3 = 1$$

in more concise form

$$\vec{e}_i \cdot \vec{e}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} = \delta_{ij}$$

where δ_{ij} is called **Kronecker Delta**.

In matrix notation

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is called the **Identity Matrix** because

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

We can also perform the calculation by using index notation:

$$\delta_{ij}u_i = \sum_{i=1}^{3}\delta_{ij}u_i = u_j$$

Dot product of $\vec{u} = u_i\vec{e}_i$ and $\vec{v} = v_j\vec{e}_j$:

By index notation:

$$\vec{u}\cdot\vec{v} = (u_i\vec{e}_i)\cdot(v_j\vec{e}_j) = u_iv_j(\vec{e}_i\cdot\vec{e}_j) = u_iv_j\delta_{ij} = u_iv_i = u_1v_1 + u_2v_2 + u_3v_3$$

By matrix notation:

$$\vec{u}\cdot\vec{v} \to \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3$$

More examples of calculations using index notation:

1) $\delta_{ii} = \sum_{i=1}^{3}\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

2) $\delta_{ij}\delta_{ij} = \delta_{ii} = 3$

3) $\delta_{ij}\delta_{jk} = \delta_{ik}$

4) $A_{ij}B_{jk} = \sum_{i=1}^{3}A_{ij}B_{jk}$ is the index notation of $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$.