Today we introduce the application of index notations in vector and tensor algebra.

Index notation:

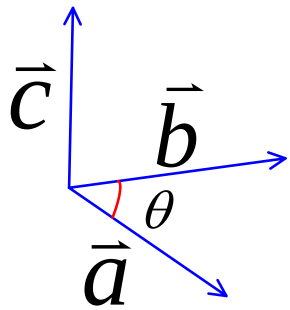
e.g. Kronecker delta: $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$. In matrix form: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Dot product of vectors: $\vec{a} \cdot \vec{b} = (a_i \vec{e}_i) \cdot (b_j \vec{e}_j) = a_i b_j (\vec{e}_i \cdot \vec{e}_j) = a_i b_j \delta_{ij} = a_i b_i$

Cross product of vectors: $\vec{c} = \vec{a} \times \vec{b}$,

The magnitude of cross product is: $|\vec{c}| = ab \sin\theta$

The direction of cross product is:



For base vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$:

$$\vec{e}_1 \times \vec{e}_1 = 0, \quad \vec{e}_1 \times \vec{e}_2 = \vec{e}_3, \quad \vec{e}_1 \times \vec{e}_3 = -\vec{e}_2$$

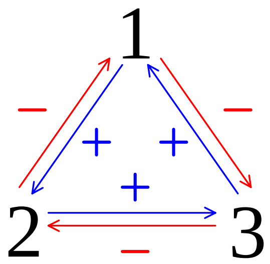
$$\vec{e}_2 \times \vec{e}_1 = -\vec{e}_3, \quad \vec{e}_2 \times \vec{e}_2 = 0, \quad \vec{e}_2 \times \vec{e}_3 = \vec{e}_1$$

$$\vec{e}_3 \times \vec{e}_1 = \vec{e}_2, \quad \vec{e}_3 \times \vec{e}_2 = -\vec{e}_1, \quad \vec{e}_3 \times \vec{e}_3 = 0$$

Or we can write the nine equations in a concise form as:

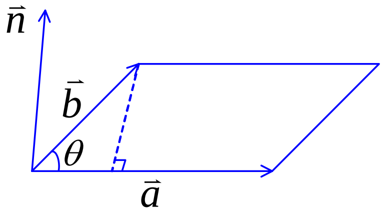
$$\vec{e}_i \times \vec{e}_j = \varepsilon_{ijk} \vec{e}_k$$

where $\varepsilon_{ijk} = \begin{cases} 1, & \text{ijk} = 123, 231, 312 \\ -1, & \text{ijk} = 213, 132, 321 \\ 0, & \text{otherwise} \end{cases}$ (called permutation symbol)



Vector operations are useful in representing areas and volumes in a solid body.

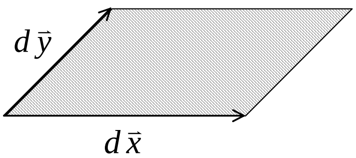
<u>Area of a parallelogram</u>:



($\vec{n} \perp$ to the plane of $\vec{a}$ and $\vec{b}$)

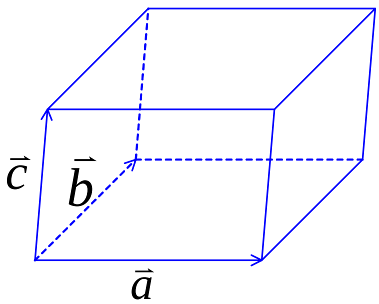
$$A = ab\sin\theta = \left|\vec{a}\times\vec{b}\right|$$

$$\vec{A} = A\vec{n} = \vec{a}\times\vec{b}$$

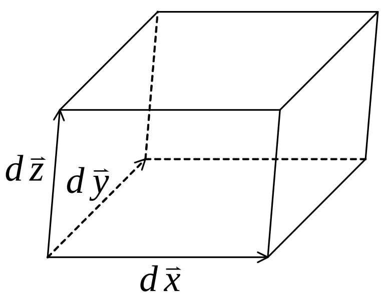


$$\mathrm{d}\vec{A} = \mathrm{d}\vec{x}\times\mathrm{d}\vec{y}$$

<u>Volume of a parallelepiped</u>:



$$V = A(\vec{c}\cdot\vec{n}) = \vec{c}\cdot\vec{A} = \vec{c}\cdot(\vec{a}\times\vec{b})$$



$$\mathrm{d}V = \mathrm{d}\vec{x}_3\cdot(\mathrm{d}\vec{x}_1\times\mathrm{d}\vec{x}_2) = \mathrm{d}\vec{x}_1\cdot(\mathrm{d}\vec{x}_2\times\mathrm{d}\vec{x}_3) = \mathrm{d}\vec{x}_2\cdot(\mathrm{d}\vec{x}_3\times\mathrm{d}\vec{x}_1)$$

In index notation, we can express volume of a parallelepiped as

$$V = \vec{c} \cdot \left(\vec{a} \times \vec{b}\right) = c_i \vec{e}_i \cdot \left(a_j \vec{e}_j \times b_k \vec{e}_k\right) = c_i a_j b_k \vec{e}_i \cdot \left(\vec{e}_j \times \vec{e}_k\right) = \varepsilon_{ijk} c_i a_j b_k = \varepsilon_{ijk} a_i b_j c_k \,,$$

We could also write this in matrix notation as

$$V = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\left(b_2 c_3 - b_3 c_2\right) - a_2\left(b_1 c_3 - b_3 c_1\right) + a_3\left(b_1 c_2 - b_2 c_1\right) = \varepsilon_{ijk} a_i b_j c_k$$

For an arbitrary matrix $\underline{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$, its determinant is $\det\left(\underline{A}\right) = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$.

More generally,

$$\varepsilon_{ijk} A_{pi} A_{qj} A_{rk} = \det\left(\underline{A}\right) \varepsilon_{pqr}$$

<u>$\varepsilon - \delta$ relation</u>:

$$\varepsilon_{ijk} \varepsilon_{pqr} = \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix}$$

Proof:

For any matrix,

$$\det\left(\underline{A}\right) = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

For any change in rows, the calculation rule of a determinant obeys

$$\varepsilon_{ijk} \det\left(\underline{A}\right) = \begin{vmatrix} A_{i1} & A_{i2} & A_{i3} \\ A_{j1} & A_{j2} & A_{j3} \\ A_{k1} & A_{k2} & A_{k3} \end{vmatrix}$$

Similarly, for any change in column,

$$\varepsilon_{ijk} \varepsilon_{pqr} \det\left(\underline{A}\right) = \begin{vmatrix} A_{ip} & A_{iq} & A_{ir} \\ A_{jp} & A_{jq} & A_{jr} \\ A_{kp} & A_{kq} & A_{kr} \end{vmatrix}$$

Set $A_{ij} = \delta_{ij}$ ($\underline{A} = \underline{\delta}$), $\det\left(\underline{I}\right) = 1$,

$$\varepsilon_{ijk}\varepsilon_{pqr} = \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix}$$

In particular,

$$\varepsilon_{ijk}\varepsilon_{iqr} = \begin{bmatrix} \delta_{ii} & \delta_{iq} & \delta_{ir} \\ \delta_{ji} & \delta_{jq} & \delta_{jr} \\ \delta_{ki} & \delta_{kq} & \delta_{kr} \end{bmatrix} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}$$

The above are called $\varepsilon - \delta$ relations.

By using the $\varepsilon - \delta$ relationship, we can calculate

$$\begin{aligned} \vec{e}_i \times \vec{e}_j \times \vec{e}_k &= \varepsilon_{ijq}\vec{e}_q \times \vec{e}_k = \varepsilon_{ijq}\varepsilon_{qkr}\vec{e}_r = \varepsilon_{qij}\varepsilon_{qkr}\vec{e}_r \\ &= \left(\delta_{ik}\delta_{jr} - \delta_{ir}\delta_{jk}\right)\vec{e}_r \\ &= \delta_{ik}\vec{e}_j - \delta_{jk}\vec{e}_i \end{aligned}$$

$$\begin{aligned} \vec{a} \times \vec{b} \times \vec{c} &= a_i b_j c_k \vec{e}_i \times \vec{e}_j \times \vec{e}_k = a_i b_j c_k \left(\delta_{ik}\vec{e}_j - \delta_{jk}\vec{e}_i\right) \\ &= a_i c_i b_j \vec{e}_j - b_j c_j a_i \vec{e}_i \\ &= \left(\vec{a}\cdot\vec{c}\right)\vec{b} - \left(\vec{b}\cdot\vec{c}\right)\vec{a} \end{aligned}$$

Gradient of a scalar field $T(x, y, z)$

$$\nabla = \vec{e}_1 \frac{\partial}{\partial x_1} + \vec{e}_2 \frac{\partial}{\partial x_2} + \vec{e}_3 \frac{\partial}{\partial x_3}$$

$$\nabla T = \vec{e}_1 \frac{\partial T}{\partial x_1} + \vec{e}_2 \frac{\partial T}{\partial x_2} + \vec{e}_3 \frac{\partial T}{\partial x_3} = \vec{e}_i \frac{\partial T}{\partial x_i} = \vec{e}_i T_{,i}$$

Divergence of a vector field $\vec{u}(x, y, z)$**:**

$$\begin{aligned} \nabla \cdot \vec{u} &= \vec{e}_i \frac{\partial}{\partial x_i} \cdot u_j \vec{e}_j = \frac{\partial u_j}{\partial x_i}\delta_{ij} = \frac{\partial u_i}{\partial x_i} \\ &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = u_{i,i} \end{aligned}$$

e.g.

$$\nabla \cdot \underline{\sigma} + \vec{f} = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$$

$$\nabla \cdot \underline{\sigma} = \sigma_{ij,j}$$

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i$$

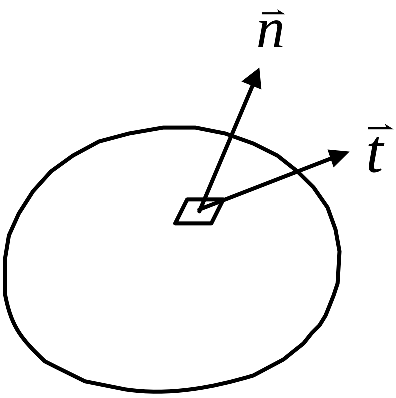
Curl of a vector field $\vec{u}(x, y, z)$**:**

$$\nabla \times \vec{u} = \vec{e}_i \frac{\partial}{\partial x_i} \times u_j \vec{e}_j = u_{j,i} \varepsilon_{ijk} \vec{e}_k$$

Tensors:

(Second rank) tensors are linear transformations that map one vector into another vector. Higher order tensors can map one tensor into another tensor. One can also say that vectors are first rank tensors and scalar are zeroth rank tensors.

For example,



If $\vec{n}$ is the normal vector of a tiny area on the surface, $\vec{t}$ is surface traction, we will show that there exists a tensor which maps $\vec{n}$ to $\vec{t}$,

$$\underline{\sigma} \vec{n} = \vec{t}$$

where $\underline{\sigma}$ will turn out to be the stress tensor (discussed later).

An example of higher order tensor is the elastic modulus tensor. If $\underline{\sigma}$ is stress tensor and $\underline{\varepsilon}$ is strain tensor, then we need a 4th order tensor to relate them to each other:

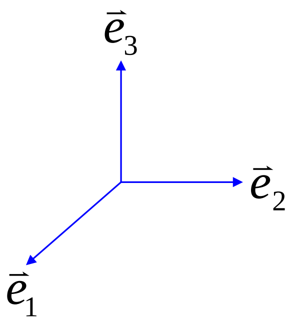
$$\underline{\sigma} = \underline{C} \underline{\varepsilon}$$

where $\underline{C}$ is called the modulus tensor (to be discussed later).

This relationship, called the generalized Hooke's law, is the 3D generalization of the 1D Hooke's law,

$$\sigma = E\varepsilon \, .$$

<u>Base tensors:</u>



$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{e}_1 \otimes \vec{e}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; \vec{e}_1 \otimes \vec{e}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; \vec{e}_1 \otimes \vec{e}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ldots$$

where "$\otimes$" is the tensor product (or dyad product) symbol.

$$\vec{e}_i \otimes \vec{e}_j = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix} \quad \begin{matrix} \text{ith row} \\ \text{jth column} \end{matrix}$$