We continue on the mathematical background.

<u>Base tensors:</u> $\vec{e}_i \otimes \vec{e}_j$ (dyadic form)

$$\vec{e}_1 \otimes \vec{e}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{e}_1 \otimes \vec{e}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{e}_1 \otimes \vec{e}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ldots$$

Note that: $\vec{e}_i \otimes \vec{e}_j \neq \vec{e}_j \otimes \vec{e}_i$, $\vec{e}_i \otimes \vec{e}_j \neq \left(\vec{e}_j \otimes \vec{e}_i\right)^T$

Rules of operation:

$$\left(\vec{e}_i \otimes \vec{e}_j\right)\vec{e}_k = \vec{e}_i\left(\vec{e}_j \cdot \vec{e}_k\right) = \delta_{jk}\vec{e}_i$$

$$\vec{e}_k \cdot \left(\vec{e}_i \otimes \vec{e}_j\right) = \left(\vec{e}_k \cdot \vec{e}_i\right)\vec{e}_j = \delta_{ki}\vec{e}_j$$

Tensor product of vectors (also called tensor dyad):

$$\vec{a} \otimes \vec{b} = \left(a_i \vec{e}_i\right) \otimes \left(b_j \vec{e}_j\right) = a_i b_j \left(\vec{e}_i \otimes \vec{e}_j\right) = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

We can compare the matrix forms of different vector products:

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\vec{a} \otimes \vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

In general, $\underline{T} = T_{ij}\, \vec{e}_i \otimes \vec{e}_j$

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = T_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + T_{12} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + T_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \cdots$$

$$T_{ij} = \vec{e}_i \cdot \underline{T}\, \vec{e}_j$$

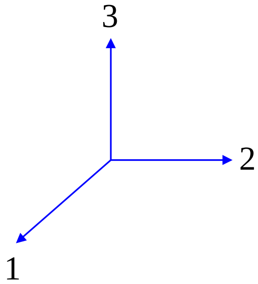
For example,

$$T_{11} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

One can also verify this using index notation:

$$T_{ij} = \vec{e}_i \cdot \left(T_{pq} \vec{e}_p \otimes \vec{e}_q \right) \vec{e}_j = T_{pq} \left(\vec{e}_i \cdot \vec{e}_p \right) \left(\vec{e}_q \cdot \vec{e}_j \right) = T_{pq} \delta_{ip} \delta_{qj} = T_{ij}$$

This definition is very useful in practice in deriving the components of a tensor from its physical descriptions.



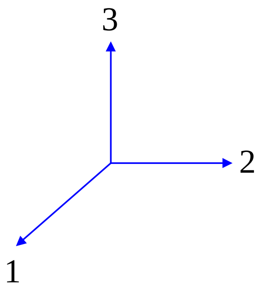
Example 1: Find the tensor $\underline{R}$ that reflects any vectors with respect to the 2-3 plane.

Solution: According to the definition of reflection, we have

$$\underline{R}\,\vec{e}_1 = -\vec{e}_1, \quad \underline{R}\,\vec{e}_2 = \vec{e}_2, \quad \underline{R}\,\vec{e}_3 = \vec{e}_3$$

Therefore:

$$R_{ij} = \vec{e}_i \cdot \underline{R}\,\vec{e}_j = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Example 2: Find the tensor $\underline{Q}$ that rotates any vector about the $\underline{x_3}$ axis by 90° clockwise.

$$\underline{Q}\,\vec{e}_1 = -\vec{e}_2, \quad \underline{Q}\,\vec{e}_2 = \vec{e}_1, \quad \underline{Q}\,\vec{e}_3 = \vec{e}_3$$

Therefore:

$$Q_{ij} = \vec{e}_i \cdot \underline{Q}\,\vec{e}_j = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetric tensors:

$$\underline{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} = \underline{T}^T$$

Antisymmetric tensors:

$$\underline{A} = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix} = -\underline{A}^T$$

An arbitrary tensor $\underline{P}$ can be formed into a sum of a symmetric tensor and an antisymmetric tensor:

$$\underline{P} = \frac{\underline{P} + \underline{P}^T}{2} + \frac{\underline{P} - \underline{P}^T}{2} = \underline{P}^S + \underline{P}^A$$

Eigenvalues of a tensor:

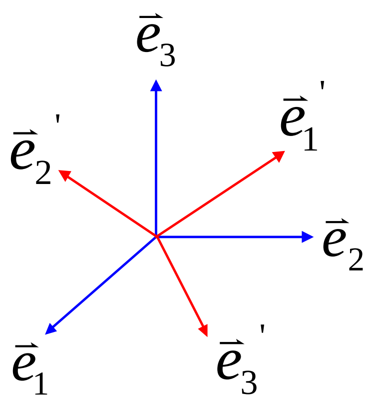
$\underline{T}\,\vec{u} = \lambda\,\vec{u}$

where $\vec{u}$ is called the eigenvectors of $\underline{T}$. λ is the eigenvalue.

$$\det\left(\underline{T} - \lambda \underline{I}\right) = 0$$

For a second rank tensor in 3 dimension, there are 3 eigenvalues and 3 eigenvectors.

Tensor transformation under change of coordinates:



In different coordinates,

$$\underline{T} = T_{ij}\vec{e}_i \otimes \vec{e}_j = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

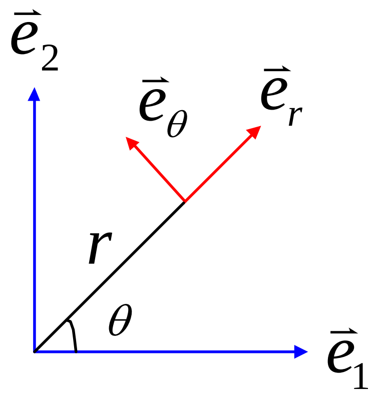
$$\underline{T} = T'_{pq}\vec{e}'_p \otimes \vec{e}'_q = \begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{bmatrix}$$

How are T_{ij} and T'_{ij} related?

$$\underline{T} = T_{ij}\vec{e}_i \otimes \vec{e}_j = T'_{pq}\vec{e}'_p \otimes \vec{e}'_q$$

$$T'_{pq} = \vec{e}'_p \cdot \underline{T}\,\vec{e}'_q = \vec{e}'_p \cdot \left(T_{ij}\vec{e}_i \otimes \vec{e}_j\right)\vec{e}'_q = T_{ij}\left(\vec{e}'_p \cdot \vec{e}_i\right)\left(\vec{e}'_q \cdot \vec{e}_j\right)$$

Example: 2D Cartesian coordinate and Polar coordinate.



$$\begin{aligned}\underline{T} &= T_{11}\vec{e}_1 \otimes \vec{e}_1 + T_{12}\vec{e}_1 \otimes \vec{e}_2 + T_{21}\vec{e}_2 \otimes \vec{e}_1 + T_{22}\vec{e}_2 \otimes \vec{e}_2 \\ &= T_{rr}\vec{e}_r \otimes \vec{e}_r + T_{r\theta}\vec{e}_r \otimes \vec{e}_\theta + T_{\theta r}\vec{e}_\theta \otimes \vec{e}_r + T_{\theta\theta}\vec{e}_\theta \otimes \vec{e}_\theta\end{aligned}$$

$$\begin{aligned}T_{11} &= \vec{e}_1 \cdot \underline{T}\,\vec{e}_1 = \vec{e}_1 \cdot \left(T_{rr}\vec{e}_r \otimes \vec{e}_r + T_{r\theta}\vec{e}_r \otimes \vec{e}_\theta + T_{\theta r}\vec{e}_\theta \otimes \vec{e}_r + T_{\theta\theta}\vec{e}_\theta \otimes \vec{e}_\theta\right)\vec{e}_1 \\ &= T_{rr}\cos^2\theta - T_{r\theta}\cos\theta\sin\theta - T_{\theta r}\cos\theta\sin\theta + T_{\theta\theta}\sin^2\theta\end{aligned}$$

$$T_{12} = T_{rr}\cos\theta\sin\theta + T_{r\theta}\cos^2\theta - T_{\theta r}\sin^2\theta - T_{\theta\theta}\cos\theta\sin\theta$$

<u>Tensor equations:</u>

Example: Generalized Hooke's Law

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

Inverting this equation by letting $j \to i$,

$$\varepsilon_{ii} = \frac{1+\nu}{E}\sigma_{ii} - \frac{\nu}{E}\sigma_{kk}\delta_{ii} = \frac{1-2\nu}{E}\sigma_{ii}$$

i.e.

$$\sigma_{kk} = \frac{E}{1-2\nu}\varepsilon_{kk}$$

Substituting σ_{kk} into the original equation,

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\frac{E}{1-2\nu}\varepsilon_{kk}\delta_{ij}$$

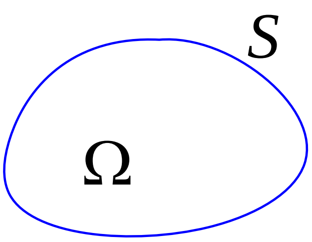
$$\frac{1+\nu}{E}\sigma_{ij} = \varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}$$

$$\begin{aligned}\sigma_{ij} &= \frac{E}{1+\nu}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij} \\ &= 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}\end{aligned}$$

E, ν : Young's modulus and Poisson's ratio

$\mu = \frac{E}{2(1+\nu)}$, $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ are called Lamé constants.

<u>Divergence theorem:</u>



$$\int_\Omega (\nabla \cdot \vec{v}) \mathrm{d}v = \oint_S \vec{v} \cdot \vec{n}\, \mathrm{d}S$$

Generalization:

$$\int_{\Omega} (\cdot)_{,i} \, \mathrm{d}v = \oint_{S} (\cdot) n_i \, \mathrm{d}S$$

$(\cdot)$ can be any vector or tensor field.