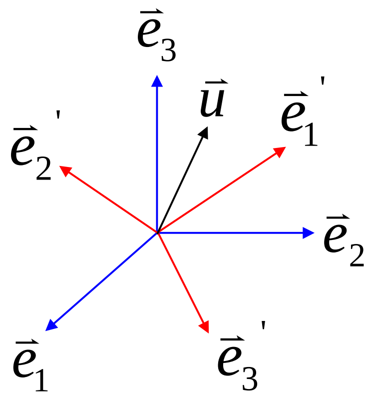
Review on coordinate transformation (change of basis) for tensors.



$$\vec{u} = u_i \vec{e}_i = u'_p \vec{e}'_p$$

$$u'_p = \vec{u} \cdot \vec{e}'_p = u_i \left(\vec{e}_i \cdot \vec{e}'_p \right)$$

$$u_i = \vec{u} \cdot \vec{e}_i = u'_q \left(\vec{e}'_q \cdot \vec{e}_i \right)$$

Combining the above 2 equations yields $u'_p = u'_q \left(\vec{e}_i \cdot \vec{e}'_q \right)\left(\vec{e}_i \cdot \vec{e}'_p \right) = u'_q Q_{iq} Q_{ip}$

Therefore $Q_{iq} Q_{ip} \equiv \delta_{pq}$

Similarly, we can show $Q_{ip} Q_{jp} \equiv \delta_{ij}$

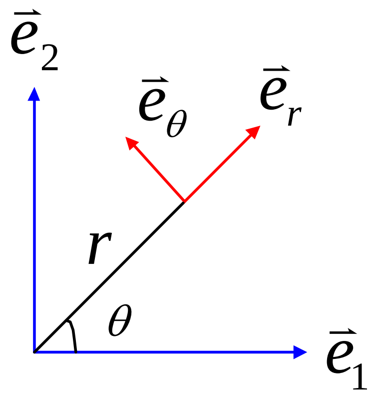
(Hint: using $u_i = u_j \left(\vec{e}_j \cdot \vec{e}'_p \right)\left(\vec{e}_i \cdot \vec{e}'_p \right) = u_j Q_{ip} Q_{jp}$)

In matrix form:

$$\underline{Q}\,\underline{Q}^T = \underline{I}\,, \quad \underline{Q}^T \underline{Q} = \underline{I}$$

Such matrices/tensors are called orthogonal matrices/tensors.

Example: Transformation from 2D Cartesian coordinate to 2D Polar coordinate.



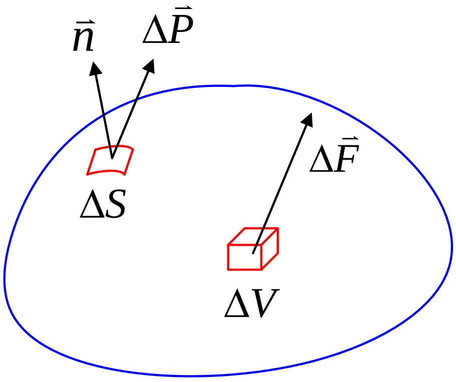
$$\underline{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \underline{Q}\,\underline{Q}^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Chap 3. Stress in a solid

Continuum —— Continuous media ignoring the atomic and other discreteness of matters.

Density —— $\rho = \dfrac{\Delta M}{\Delta V} \quad (\Delta V \to 0)$.

(ΔV is smaller than all important dimensions but still contains sufficient number of atoms)



Homogeneity —— All points have the same material properties.

(Opposite term: Heterogeneity)

Isotropy —— Material properties are the same in different directions.

(Opposite term: Anisotropy)

Forces in a continuum:

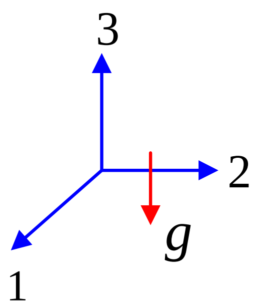
External (applied) forces

Body (volume) forces

$\Delta\vec{F}$ —— total body force on ΔV (e.g. gravity).

$$\vec{f} = \lim_{\Delta V \to 0} \left(\frac{\Delta\vec{F}}{\Delta V} \right)$$

For example: The body force due to gravity can be written as



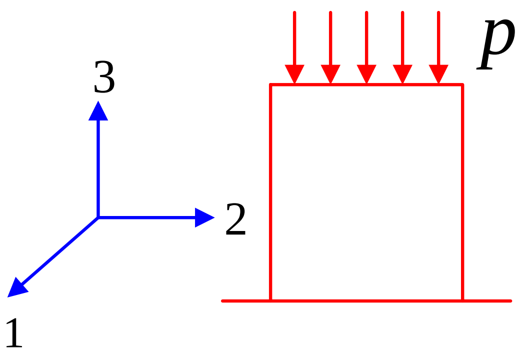
$$\vec{f} = -\rho g \vec{e}_3$$

Surface forces

$\Delta\vec{P}$ —— total surface forces on ΔS .

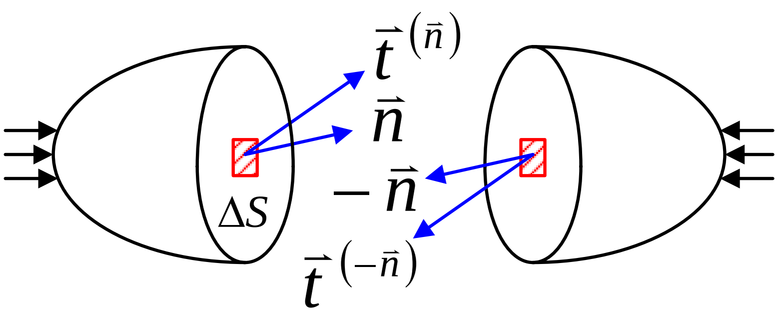
$$\vec{t} = \lim_{\Delta S \to 0}\left(\frac{\Delta\vec{P}}{\Delta S}\right)$$

For example: A uniform pressure on top of a block can be written as



$$\vec{t} = -p\vec{e}_3$$

Internal forces

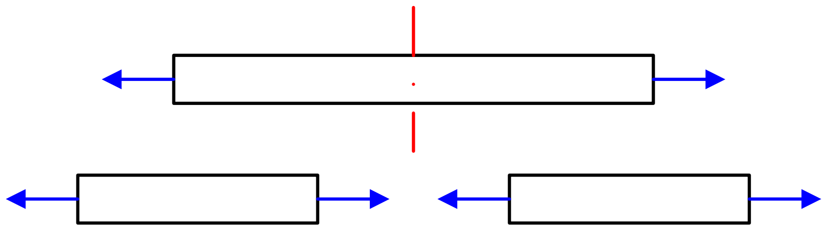


$\vec{t}^{(\vec{n})}$: traction vector at a point.

According to Newton's action-reaction law,

$$\vec{t}^{(\vec{n})} = -\vec{t}^{(-\vec{n})}$$

Simple example for 1D:



Let us put these concepts in terms of base vectors:

$$\vec{t}^{(\vec{e}_1)} = \vec{t}_1^{(\vec{e}_1)}\vec{e}_1 + \vec{t}_2^{(\vec{e}_1)}\vec{e}_2 + \vec{t}_3^{(\vec{e}_1)}\vec{e}_3$$

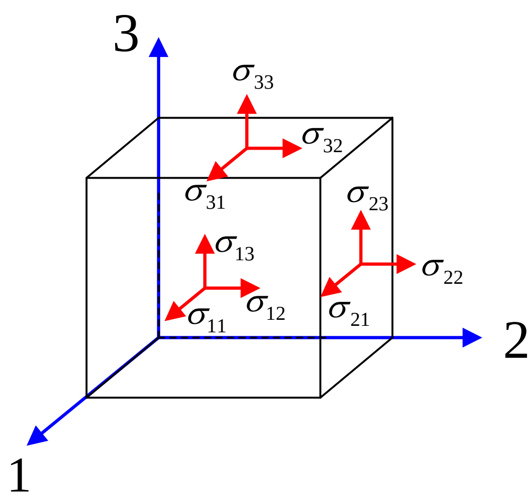
$$\vec{t}^{(\vec{e}_2)} = \vec{t}_1^{(\vec{e}_2)}\vec{e}_1 + \vec{t}_2^{(\vec{e}_2)}\vec{e}_2 + \vec{t}_3^{(\vec{e}_2)}\vec{e}_3$$

$$\vec{t}^{(\vec{e}_3)} = \vec{t}_1^{(\vec{e}_3)}\vec{e}_1 + \vec{t}_2^{(\vec{e}_3)}\vec{e}_2 + \vec{t}_3^{(\vec{e}_3)}\vec{e}_3$$

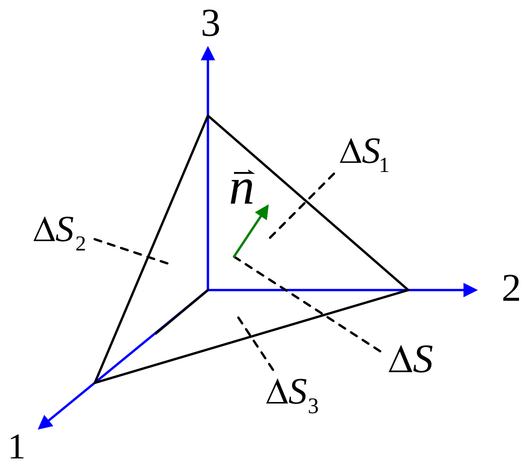
We can define

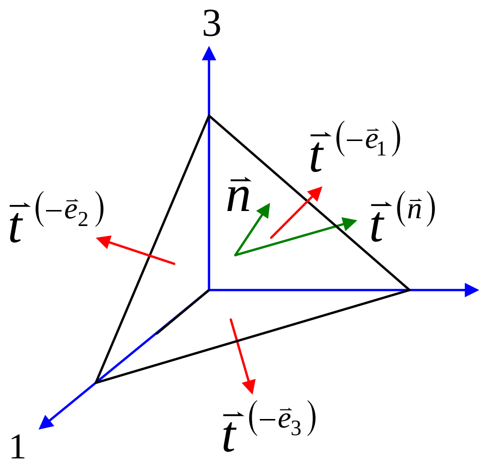
$$\sigma_{ij} = \vec{t}_j^{(\vec{e}_i)},$$

which is the so-called Cauchy stress. The subscript i denotes the direction of plane normal and j denotes the direction of force.



<u>Traction on an arbitrary plane with normal vector $\vec{n}$</u>





Consider force equilibrium on the tetrahedron shown above:

$$\vec{t}^{(\vec{n})}\Delta S + \vec{t}^{(-\vec{e}_1)}\Delta S_1 + \vec{t}^{(-\vec{e}_2)}\Delta S_2 + \vec{t}^{(-\vec{e}_3)}\Delta S_3 + \vec{f}\,\Delta V = 0,$$

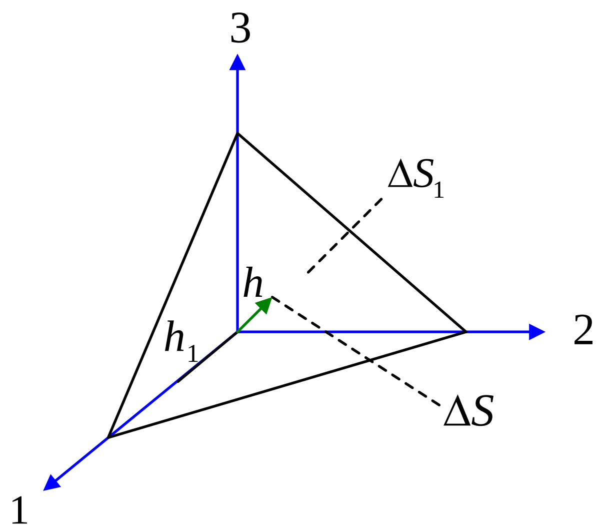
where $\vec{f}$ is the body force (which can also include inertia forces). Let $\Delta S \to 0$,

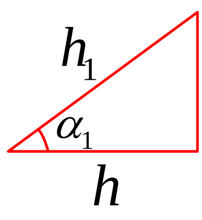
$$\vec{t}^{(\vec{n})} + \vec{t}^{(-\vec{e}_1)}\frac{\Delta S_1}{\Delta S} + \vec{t}^{(-\vec{e}_2)}\frac{\Delta S_2}{\Delta S} + \vec{t}^{(-\vec{e}_3)}\frac{\Delta S_3}{\Delta S} + \vec{f}\,\frac{\Delta V}{\Delta S} = 0$$

$\frac{\Delta V}{\Delta S} \sim \Delta S^{\frac{1}{2}} \to 0$ as $\Delta S \to 0$ (This is the same as saying that the surface-to-volume ratio becomes very large as the volume shrinks to zero.)

Consider the volume of the tetrahedron, $V = \frac{1}{3}\Delta S \cdot h = \frac{1}{3}\Delta S_1 \cdot h_1$

$$\Rightarrow \frac{\Delta S_1}{\Delta S} = \frac{h}{h_1} = \cos\alpha_1 = n \cdot \vec{e}_1 = n_1$$





Similarly,

$$\frac{\Delta S_2}{\Delta S} = n_2\,, \quad \frac{\Delta S_3}{\Delta S} = n_3$$

where n_1, n_2, n_3 are the components of the normal vector $\vec{n}$ ($\vec{n} = n_1\vec{e}_1 + n_2\vec{e}_2 + n_3\vec{e}_3$).

Hence,

$$\vec{t}^{\,(\vec{n})} = \vec{t}^{\,(\vec{e}_1)} n_1 + \vec{t}^{\,(\vec{e}_2)} n_2 + \vec{t}^{\,(\vec{e}_3)} n_3$$

In index notation,

$$\vec{t}_j^{\,(\vec{n})} = \vec{t}_j^{\,(\vec{e}_i)} n_i = \sigma_{ij} n_i$$

i.e.

$$\vec{t}^{\,(\vec{n})} = \underline{\sigma}^T\, \vec{n}$$

Special cases:

$$\vec{t}^{\,(\vec{e}_1)} = \underline{\sigma}\,\vec{e}_1 = \sigma_{ij}\left(\vec{e}_i \otimes \vec{e}_j\right)\vec{e}_i = \sigma_{1j}\vec{e}_j$$

$$\vec{t}^{\,(\vec{e}_2)} = \cdots = \sigma_{2j}\vec{e}_j$$

$$\vec{t}^{\,(\vec{e}_3)} = \cdots = \sigma_{3j}\vec{e}_j$$

σ_{11}, σ_{22}, σ_{33} are called the normal stresses (on a base plane).

σ_{12}, σ_{13}, σ_{23} etc. are called the shear stresses (on a base plane).