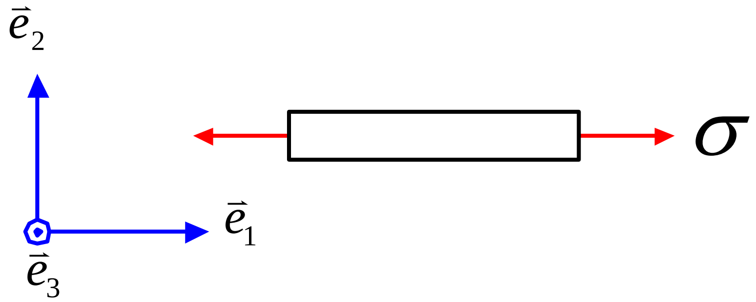
<u>Stress in a solid</u>

Examples of common stress states:

1. Uniaxial stress

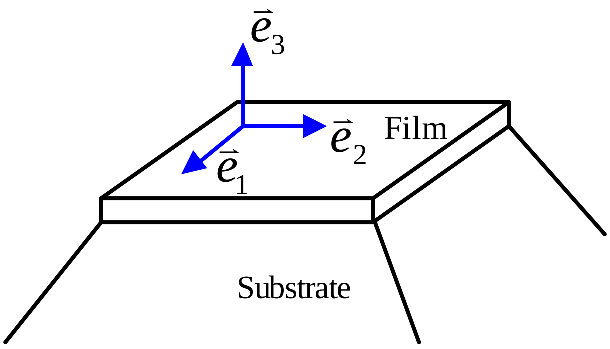


$$\underline{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Principal stresses: $\sigma_I = \sigma$, $\sigma_{II} = \sigma_{III} = 0$

Principal directions: $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ ($\vec{e}_2$, $\vec{e}_3$ can be replaced by any direction in 2-3 plane)

2. Biaxial stress

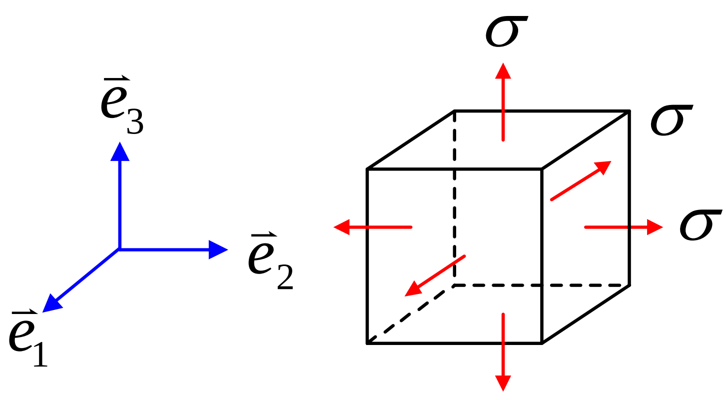


$$\underline{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Principal stresses: $\sigma_I = \sigma_{II} = \sigma$, $\sigma_{III} = 0$

Principal directions: $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ ($\vec{e}_1$, $\vec{e}_2$ can be replaced by any direction in 1-2 plane)

3. Hydrostatic stress

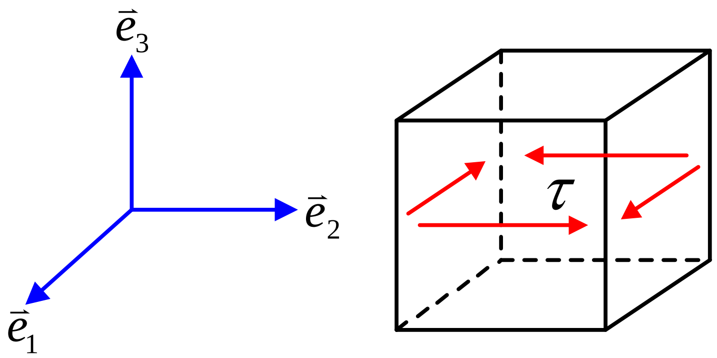


$$\underline{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} = \sigma \underline{I}$$

Principal stresses: $\sigma_I = \sigma_{II} = \sigma_{III} = \sigma$

Principal directions: any direction (since there is no shear stress on any cut plane)

4. Pure shear



$$\underline{\sigma} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \tau\, \vec{e}_1 \otimes \vec{e}_2 + \tau\, \vec{e}_2 \otimes \vec{e}_1$$

Principal stresses:

$$\begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \lambda \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \Rightarrow \begin{bmatrix} -\lambda & \tau & 0 \\ \tau & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

For nontrivial solution of n_i,

$$\det\begin{bmatrix} -\lambda & \tau & 0 \\ \tau & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = 0,$$

which indicates

$$-\lambda\left(\lambda^2 - \tau^2\right) = 0 \Rightarrow \lambda = \tau, 0, -\tau$$

Following the convention $\sigma_I > \sigma_{II} > \sigma_{III}$, we can write

$$\sigma_I = \tau, \quad \sigma_{II} = 0, \quad \sigma_{III} = -\tau$$

<u>Principal directions:</u>

Principal direction I ($\sigma_I = \tau$):

$$\begin{bmatrix} -\tau & \tau & 0 \\ \tau & -\tau & 0 \\ 0 & 0 & -\tau \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

That is

$$n_1 = n_2, \quad n_3 = 0$$

In addition, $n_1^2 + n_2^2 + n_3^2 = 1$ (because $\vec{n}$ is a unit vector).

$$\begin{cases} n_1 = \pm\dfrac{1}{\sqrt{2}} \\ n_2 = \pm\dfrac{1}{\sqrt{2}} \\ n_3 = 0 \end{cases}$$

Therefore: $\vec{n}_I = \pm\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, 0\right)$

Principal direction II ($\sigma_{II} = 0$):

$$\begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

That is

$$n_1 = n_2 = 0$$

In addition, $n_1^2 + n_2^2 + n_3^2 = 1$ ($\vec{n}$ is a unit vector).

$$\begin{cases} n_1 = 0 \\ n_2 = 0 \\ n_3 = \pm 1 \end{cases}$$

Therefore: $\vec{n}_{II} = \pm(0, 0, 1)$

Principal direction III ($\sigma_{III} = -\tau$):

$$\begin{bmatrix} \tau & \tau & 0 \\ \tau & \tau & 0 \\ 0 & 0 & \tau \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

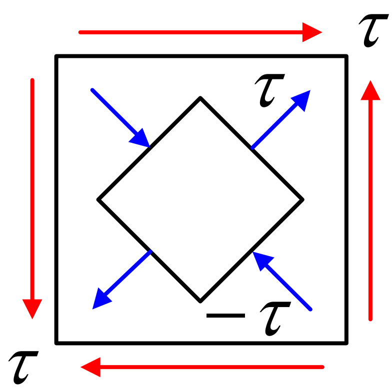
That is

$$n_1 = -n_2, \quad n_3 = 0$$

In addition, $n_1^2 + n_2^2 + n_3^2 = 1$ ($\vec{n}$ is a unit vector).

$$\begin{cases} n_1 = \pm \dfrac{1}{\sqrt{2}} \\ n_2 = \mp \dfrac{1}{\sqrt{2}} \\ n_3 = 0 \end{cases}$$

Therefore: $\vec{n}_{III} = \pm\left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}, 0\right)$



Deviatoric stress tensor

An arbitrary stress tensor can be written in

$$\sigma_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} + \sigma'_{ij}$$

Define $\sigma_h = \frac{1}{3}\sigma_{kk} = \frac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III})$, which represents the hydrostatic part of the stress.

$$\sigma'_{ij} = \sigma_{ij} - \sigma_h\delta_{ij}$$

Let $i \to j$,

$$\sigma'_{jj} = \sigma_{jj} - \sigma_h\delta_{jj} = \sigma_{jj} - 3\sigma_h = 0$$

i.e. $\text{trace}(\sigma'_{ij}) = \sigma'_{11} + \sigma'_{22} + \sigma'_{33} = 0$.

σ'_{ij} has 3 invariants: $\sigma'_{kk} = 0$, $\sigma'_{ij}\sigma'_{ij}$, $\det(\sigma'_{ij})$.

Among the 3 invariants of σ'_{ij}, $\sigma'_{ij}\sigma'_{ij}$ is related to the von Mises stress which plays an important role in modeling plastic yielding).

von Mises stress

Consider the case of uniaxial tension with principal stresses:

$$\sigma_I = \sigma, \quad \sigma_{II} = \sigma_{III} = 0$$

The stress tensor in the principal directions is

$$\underline{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The principal stresses of the deviatoric stress tensor are

$$\sigma'_I = \frac{2}{3}\sigma, \quad \sigma'_{II} = \sigma'_{III} = -\frac{1}{3}\sigma$$

Therefore

$$\sigma'_{ij}\sigma'_{ij} = \sigma_I'^{\,2} + \sigma_{II}'^{\,2} + \sigma_{III}'^{\,2} = \frac{2}{3}\sigma^2$$

i.e.

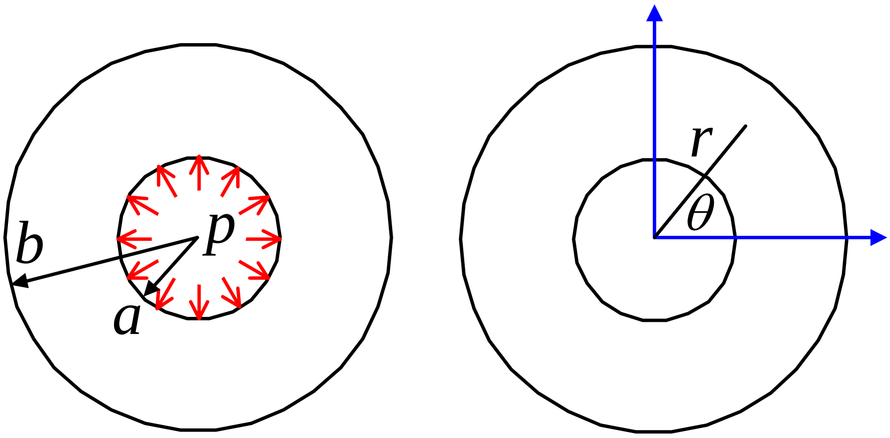
$$\sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} = \sigma$$

von Mises stress is defined as $\sigma_e = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}}$ such that $\sigma_e = \sigma$ in the case of uniaxial tension.

<u>Traction BC's for BVPs in solid mechanics</u>

The Cauchy relationship between stress and traction serves as a basis to express traction boundary conditions in terms of the relevant stress components. We consider two examples

Example 1:



$\vec{t} = \underline{\sigma}\,\vec{e}_r = 0$ @ $r = b$ (traction free at $r = b$)

$\vec{t} = \underline{\sigma}\left(-\vec{e}_r\right) = p\,\vec{e}_r$ @ $r = a$

In Polar coordinates,

@ $r = b$:

$\sigma_{rr} = \vec{e}_r \cdot \underline{\sigma}\,\vec{e}_r = 0$, $\sigma_{r\theta} = \vec{e}_\theta \cdot \underline{\sigma}\,\vec{e}_r = 0$

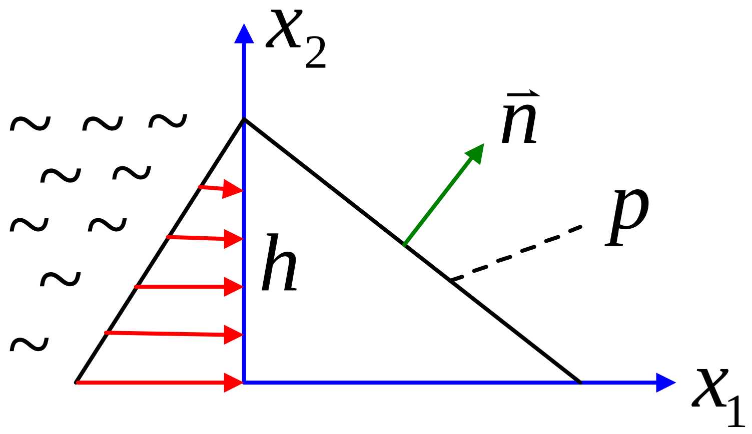
@ $r = a$:

$\sigma_{rr} = \vec{e}_r \cdot \underline{\sigma}\,\vec{e}_r = -p$, $\sigma_{r\theta} = \vec{e}_\theta \cdot \underline{\sigma}\,\vec{e}_r = 0$

Therefore, the BCs for the above problem are often expressed as

$$\begin{cases} \sigma_{rr} = -p & \sigma_{r\theta} = 0 \quad r = a \\ \sigma_{rr} = 0 & \sigma_{r\theta} = 0 \quad r = a \end{cases}$$

Example 2:



$$\vec{t} = \underline{\sigma}(-\vec{e}_1) = \rho g (h - x_2)\vec{e}_1 \quad @ \quad x_1 = 0$$

$$\vec{t} = \underline{\sigma}\vec{n} = 0 \quad @ \quad p \text{ plane}$$

$@ \quad x_1 = 0$

$$\sigma_{11} = \vec{e}_1 \cdot \underline{\sigma}\,\vec{e}_1 = -\rho g (h - x_2)$$

$$\sigma_{12} = \vec{e}_2 \cdot \underline{\sigma}\,\vec{e}_1 = 0$$

Maximum and minimum normal & shear stresses in a solid

We want to show that, given principal stresses $\sigma_I > \sigma_{II} > \sigma_{III}$, the maximum and minimum normal and shear stresses in a solid have the following simple results:

$$\max(\sigma_n) = \sigma_I, \quad \min(\sigma_n) = \sigma_{III}$$

$$\max(\sigma_s) = \frac{\sigma_I - \sigma_{III}}{2}, \quad \min(\sigma_s) = 0$$

Proof:

Choose base vectors in the principal directions,

$$\underline{\sigma} = \begin{bmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{bmatrix}$$

The traction on an arbitrary plane with normal $\vec{n}$ (note that $n_1^2 + n_2^2 + n_3^2 = 1$) is

$$\vec{t} = \underline{\sigma}\,\vec{n}$$

The normal stress on that plane is

$$\sigma_n = \vec{n}\cdot\vec{t} = \vec{n}\cdot\underline{\sigma}\,\vec{n} = \sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_{III} n_3^2$$

Introduce Lagrangian multiplier λ,

$$F = \sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_{III} n_3^2 - \lambda\left(n_1^2 + n_2^2 + n_3^2\right)$$

$$\frac{\partial F}{\partial n_1} = 0 \Rightarrow 2\sigma_I n_1 - 2\lambda n_1 = 0 \qquad (1)$$

$$\frac{\partial F}{\partial n_2} = 0 \Rightarrow 2\sigma_{II} n_2 - 2\lambda n_2 = 0 \qquad (2)$$

$$\frac{\partial F}{\partial n_3} = 0 \Rightarrow 2\sigma_{III} n_3 - 2\lambda n_3 = 0 \qquad (3)$$

Case 1: $n_1 \neq 0$, $n_2 \neq 0$, $n_3 \neq 0$

From the equations (1)-(3):

$\sigma_I = \sigma_{II} = \sigma_{III} = \lambda$,

which is contradictory to our assumption $\sigma_I > \sigma_{II} > \sigma_{III}$. (The cases when some of the principal stresses are equal are quite simple and can be treated separately.)

Case 2: 2 of n_1, n_2, n_3 are not zero

If $n_1 = 0$, $n_2 \neq 0$, $n_3 \neq 0$

$$2\sigma_{II} n_2 - 2\lambda n_2 = 0$$

$$2\sigma_{III} n_3 - 2\lambda n_3 = 0$$

$$n_2^2 + n_3^2 = 1$$

The solution is $\sigma_{II} = \sigma_{III} = \lambda$, again contradictory to our assumption $\sigma_I > \sigma_{II} > \sigma_{III}$.

Similarly, ($n_2 = 0$, $n_1 \neq 0$, $n_3 \neq 0$) and ($n_3 = 0$, $n_1 \neq 0$, $n_2 \neq 0$) also lead to solutions contradictory to our assumption.

Case 3: 2 of n_1, n_2, n_3 are zero

If $n_1 \neq 0$, $n_2 = n_3 = 0$, $\vec{n} = \pm(1, 0, 0)$

If $n_2 \neq 0$, $n_1 = n_3 = 0$, $\vec{n} = \pm(0, 1, 0)$

If $n_3 \neq 0$, $n_1 = n_2 = 0$, $\vec{n} = \pm(0, 0, 1)$

These are just principal directions. Therefore,

$$\max(\sigma_n) = \sigma_I, \quad \min(\sigma_n) = \sigma_{III} \quad (\sigma_I > \sigma_{II} > \sigma_{III})$$