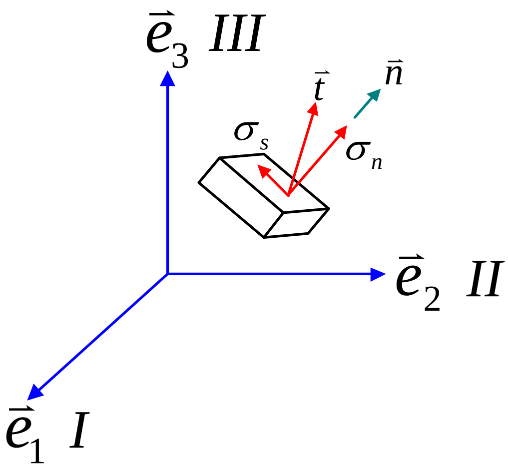
Maximum and minimum shear stresses in a solid

An general stress state

$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix},$$

when expressed in the principal directions, becomes diagonalized as

$$\underline{\sigma} = \begin{bmatrix} \sigma_{I} & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{bmatrix}$$



The traction on an arbitrary plane with normal $\vec{n}$ is

$$\vec{t} = \underline{\sigma}\,\vec{n} = \sigma_I n_1 \vec{e}_1 + \sigma_{II} n_2 \vec{e}_2 + \sigma_{III} n_3 \vec{e}_3$$

The magnitude of $\vec{t}$ is therefore

$$|\vec{t}| = \left(\sigma_I^2 n_1^2 + \sigma_{II}^2 n_2^2 + \sigma_{III}^2 n_3^2\right)^{1/2}$$

The normal stress on the plane is

$$\sigma_n = \vec{n}\cdot\vec{t} = \vec{n}\cdot\underline{\sigma}\,\vec{n} = \sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_{III} n_3^2$$

Note that $\sigma_s^2 + \sigma_n^2 = |\vec{t}|^2$, therefore

$$\sigma_s^2 = \sigma_I^2 n_1^2 + \sigma_{II}^2 n_2^2 + \sigma_{III}^2 n_3^2 - \left(\sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_{III} n_3^2\right)^2$$

We wish to find the maximum/minimum values of σ_s subject to constraint $n_1^2 + n_2^2 + n_3^2 = 1$ (because $\vec{n}$ is a unit vector).

Introduce Lagrangian multiplier λ,

$$F = \sigma_S^2 - \lambda n_i n_i$$

i.e.

$$F = \sigma_I^2 n_1^2 + \sigma_{II}^2 n_2^2 + \sigma_{III}^2 n_3^2 - \left(\sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_{III} n_3^2\right)^2 - \lambda\left(n_1^2 + n_2^2 + n_3^2\right)$$

Letting $\dfrac{\partial F}{\partial n_j} = 0 \quad \left(j = 1, 2, 3\right)$, we have

$n_1\left(\sigma_I^2 - 2\sigma_I\sigma_n - \lambda\right) = 0$ (1)

$n_2\left(\sigma_{II}^2 - 2\sigma_{II}\sigma_n - \lambda\right) = 0$ (2)

$n_3\left(\sigma_{III}^2 - 2\sigma_{III}\sigma_n - \lambda\right) = 0$ (3)

Case 1: $n_1 \neq 0$, $n_2 \neq 0$, $n_3 \neq 0$

$$\sigma_I^2 - 2\sigma_I\sigma_n - \lambda = 0$$

$$\sigma_{II}^2 - 2\sigma_{II}\sigma_n - \lambda = 0$$

$$\sigma_{III}^2 - 2\sigma_{III}\sigma_n - \lambda = 0$$

Eliminating σ_n from the above,

$$\lambda = -\sigma_I\sigma_{II} = -\sigma_{II}\sigma_{III} = -\sigma_I\sigma_{III} \Rightarrow \sigma_I = \sigma_{II} = \sigma_{III}$$

which is contradictory to our assumption $\sigma_I > \sigma_{II} > \sigma_{III}$.

Case 2: 2 of n_1, n_2, n_3 are not zero

If $n_3 = 0$, $n_1 \neq 0$, $n_2 \neq 0$,

$$\sigma_I^2 - 2\sigma_I\sigma_n - \lambda = 0$$

$$\sigma_{II}^2 - 2\sigma_{II}\sigma_n - \lambda = 0$$

$$n_1^2 + n_2^2 = 1$$

Eliminating σ_n from the first two equations results in

$$\lambda = -\sigma_I\sigma_{II}, \quad 2\sigma_n = \sigma_I + \sigma_{II}$$

Inserting $\sigma_n = \sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_{III} n_3^2 = \sigma_I n_1^2 + \sigma_{II}(1-n_1^2)$ (here $n_3 = 0$) into the first or second of the above equations will show,

$$n_1^2 = \frac{1}{2}\text{, hence } n_2^2 = 1 - n_1^2 = \frac{1}{2}$$

Therefore:

$$\vec{n} = \left(\pm\frac{1}{2}, \pm\frac{1}{2}, 0\right), \quad \sigma_s^2 = \frac{1}{2}\left(\sigma_I^2 + \sigma_{II}^2\right) - \frac{1}{4}\left(\sigma_I + \sigma_{II}\right)^2 = \frac{1}{4}\left(\sigma_I - \sigma_{II}\right)^2$$

The magnitude of this shear stress is

$$\sigma_s = \frac{1}{2}\left(\sigma_I - \sigma_{II}\right)$$

Similarly, If $n_1 = 0$, $n_2 \neq 0$, $n_3 \neq 0$,

$$\vec{n} = \left(0, \pm\frac{1}{2}, \pm\frac{1}{2}\right), \quad \sigma_s = \frac{1}{2}\left(\sigma_{II} - \sigma_{III}\right)$$

If $n_2 = 0$, $n_1 \neq 0$, $n_3 \neq 0$,

$$\vec{n} = \left(\pm\frac{1}{2}, 0, \pm\frac{1}{2}\right), \quad \sigma_s = \frac{1}{2}\left(\sigma_I - \sigma_{III}\right)$$

Following the convention $\sigma_I > \sigma_{II} > \sigma_{III}$, we find the maximum shear stress is

$$\max\left(\sigma_s\right) = \frac{\sigma_I - \sigma_{III}}{2}$$

Case 3: 2 of n_1, n_2, n_3 are zero

If $n_1 = n_2 = 0$, $n_3 \neq 0$, $n_3 = \pm 1$ & $\sigma_s = 0$

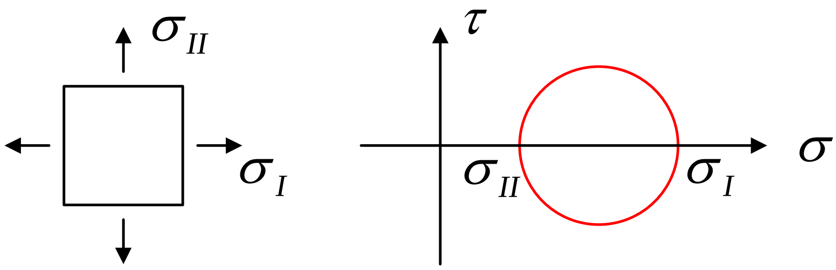
If $n_1 = n_3 = 0$, $n_2 \neq 0$, $n_2 = \pm 1$ & $\sigma_s = 0$

If $n_2 = n_3 = 0$, $n_1 \neq 0$, $n_1 = \pm 1$ & $\sigma_s = 0$

The above analysis thus indicates that $\max\left(\sigma_s\right) = \dfrac{\sigma_I - \sigma_{III}}{2}$ and $\min\left(\sigma_s\right) = 0$.

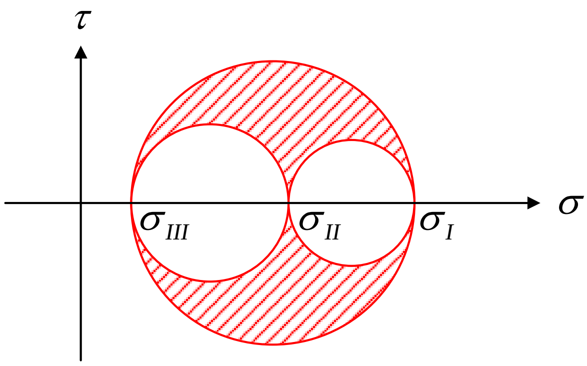
<u>Mohr circle (3D)</u>

2D Mohr circle



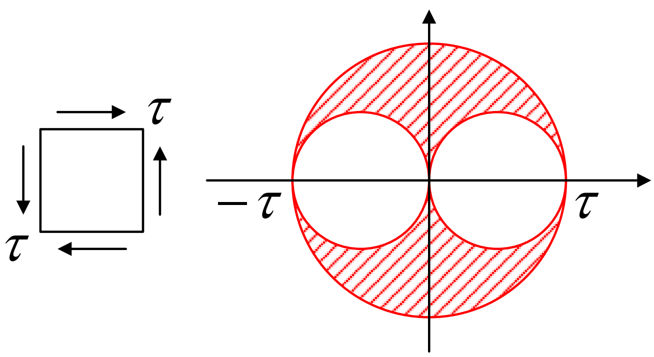
3D Mohr circle

For $\sigma_I > \sigma_{II} > \sigma_{III} > 0$, all accessible stress states lie within the shaded region bounded by 3 Mohr's circles. (The stress states within a plane containing two principle directions lie along the corresponding circle.)

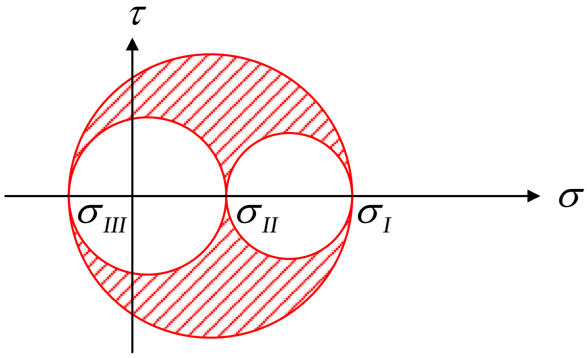


Examples of Mohr circle representations:

Pure shear $\sigma_I = \tau$, $\sigma_{II} = 0$, $\sigma_{III} = -\tau$

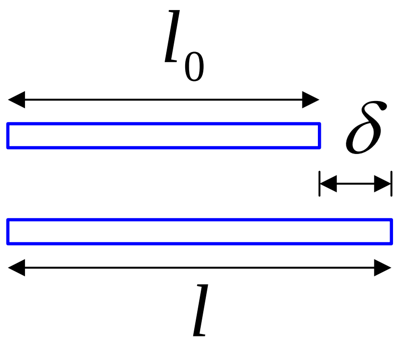


If $\sigma_I > \sigma_{II} > 0 > \sigma_{III}$, there exists a plane with zero normal stress and only shear stress.



Chap. 3 Strain in a solid

Engineering concept of strain



Elongation: δ

Percentage of elongation: $\dfrac{\delta}{l_0} = \varepsilon$

Stretch: $\lambda = \dfrac{l}{l_0} = 1 + \varepsilon$

Different measures of strain:

$$e_1 = \frac{\delta}{l_0}$$

$$e_2 = \frac{\delta}{l} = \frac{\delta}{l_0 + \delta} = \frac{\delta}{l_0} - \left(\frac{\delta}{l_0}\right)^2 + \cdots \cong \frac{\delta}{l_0}$$

$$e_3 = \frac{l^2 - l_0^2}{2l_0^2} = \frac{\delta}{l_0} + \frac{\delta^2}{2l_0^2} \cong \frac{\delta}{l_0}$$

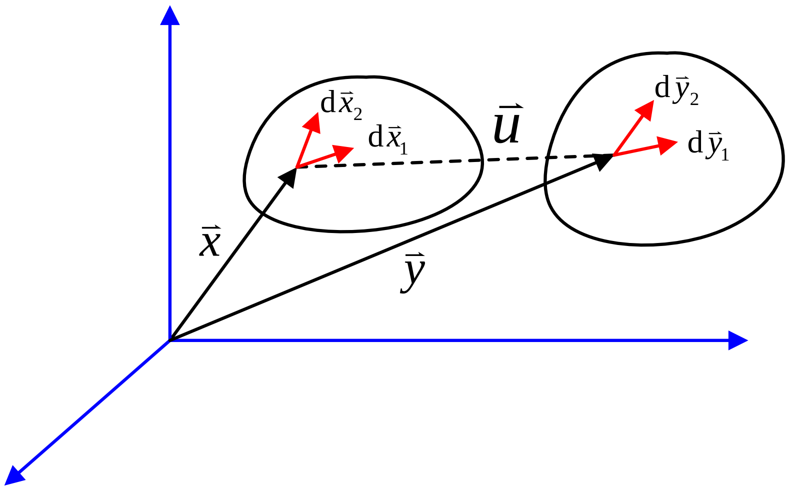
$$e_4 = \frac{l^2 - l_0^2}{2l^2} \cong \frac{\delta}{l_0}$$

$$e_5 = \ln\left(\frac{l}{l_0}\right) = \ln\left(1 + \frac{\delta}{l_0}\right) \cong \frac{\delta}{l_0}$$

…

All these measures are equivalent for small elongation and thus equivalent from an engineering point of view.

How to generalize these to 3D?



$\vec{u} = \vec{y} - \vec{x}$ (displacement vector)

$$\vec{y} = \vec{y}(\vec{x}, t)$$

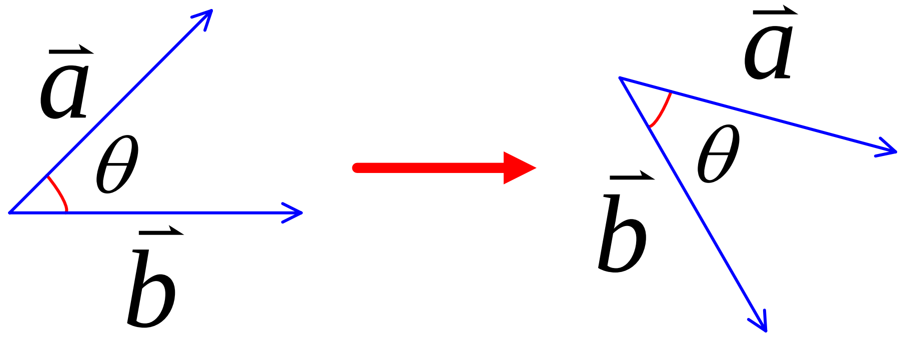
$$y_i = y_i(x_1, x_2, x_3, t)$$

$$\mathrm{d}y_i = \frac{\partial y_i}{\partial x_j}\mathrm{d}x_j = F_{ij}\mathrm{d}x_j$$

where

$$F_{ij} = \frac{\partial y_i}{\partial x_j} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}$$

is called the deformation gradient. We wish to find a measure of strain that is independent of rigid body rotation. However, information about the deformed position of one vector alone contains both strain and rotation. Strain is a relative measure of how material points move with respect to each other. We recall that the dot product of two vectors only depends on the magnitudes and the relative angle between the two.



$$\vec{a} \cdot \vec{b} = ab\cos\theta$$

This suggests that the dot product of two vectors is independent of rigid body motion and rotation. Therefore, we consider dot product of two differential segments in the deformed and undeformed configurations:

$$\mathrm{d}\vec{y}_1 \cdot \mathrm{d}\vec{y}_2 = \underline{F}\mathrm{d}\vec{x}_1 \cdot \underline{F}\mathrm{d}\vec{x}_2 = \mathrm{d}\vec{x}_1 \cdot \underline{F}^{\mathrm{T}}\underline{F}\mathrm{d}x_2$$

$$\mathrm{d}\vec{x}_1 \cdot \mathrm{d}\vec{x}_2 = \underline{F}^{-1}\mathrm{d}\vec{y}_1 \cdot \underline{F}^{-1}\mathrm{d}\vec{y}_2 = \mathrm{d}\vec{y}_1 \cdot \underline{F}^{-\mathrm{T}}\underline{F}^{-1}\mathrm{d}\vec{y}_2 = \mathrm{d}\vec{y}_1 \cdot \left(\underline{F}\,\underline{F}^{\mathrm{T}}\right)^{-1}\mathrm{d}\vec{y}_2$$

where $\underline{C} = \underline{F}^{\mathrm{T}}\underline{F}$ is called the right Cauchy-Green strain tensor and $\underline{B} = \underline{F}\,\underline{F}^{\mathrm{T}}$ the left Cauchy-Green Strain tensor.