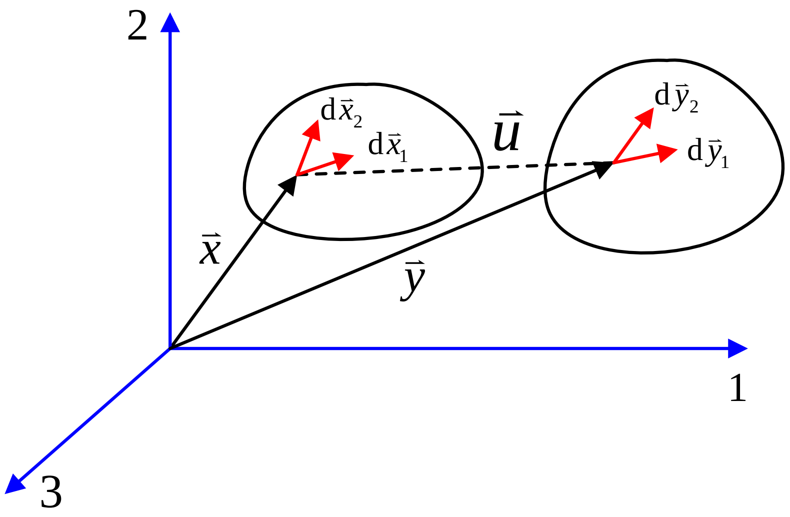
Strain in a solid



Consider an arbitrary fiber within the elastic body,

In the undeformed configuration, we can represent the fiber as a small vetor: $\mathrm{d}\vec{x} = \vec{m}\mathrm{d}l_0$ where $\mathrm{d}l_0$ is the length and $\vec{m}$ is the unit vector along the fiber direction (orientation of the fiber).

In the deformed configuration, the same fiber is represented as $\mathrm{d}\vec{y} = \vec{n}\mathrm{d}l$. Write the deformed position of a particle as

$$\vec{y} = \vec{y}(x_1, x_2, x_3) = \vec{x} + \vec{u}(x_1, x_2, x_3)$$

where $\vec{u}(x_1, x_2, x_3)$ is clearly the displacement vector. We can write a differential segment $\mathrm{d}\vec{y}$ as

$$\mathrm{d}\vec{y} = \underline{F}\mathrm{d}\vec{x}$$

where

$$F_{ij} = \frac{\partial y_i}{\partial x_j}$$

is called the deform gradient tensor. This suggests that

$$\vec{n}\mathrm{d}l = \underline{F}\vec{m}\mathrm{d}l_0$$

The ratio between the deformed length to undeformed length: $\lambda = \dfrac{\mathrm{d}l}{\mathrm{d}l_0} = 1 + \varepsilon$ (ε: strain) is defined as stretch. Therefore,

$$\underline{F}\vec{m} = \lambda\vec{n}$$

$\underline{F}$ contains information about both stretch and rotation (change of orientation from $\vec{m}$ to $\vec{n}$).

In order to separate stretch from rigid body rotation, consider the dot product of two fibers,

$$\mathrm{d}\vec{y}_1 \cdot \mathrm{d}\vec{y}_2 = \underline{F}\mathrm{d}\vec{x}_1 \cdot \underline{F}\mathrm{d}\vec{x}_2 = \mathrm{d}\vec{x}_1 \cdot \underline{F}^T \underline{F}\mathrm{d}\vec{x}_2$$

where $\underline{C} = \underline{F}^T \underline{F}$ is the right Cauchy-Green strain. Since the dot product only depends on the relative angle between the two vectors, rigid body rotation has been effectively "filtered" out of $\underline{F}$.

We can easily see that $\underline{C}$ is a symmetric tensor because

$$\underline{C}^T = \left(\underline{F}^T \underline{F}\right)^T = \underline{F}^T \underline{F} = \underline{C}$$

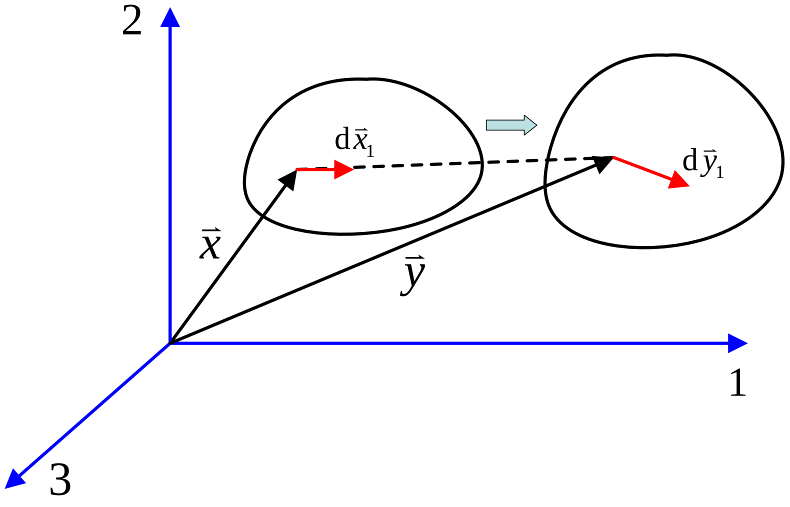
Write $\underline{C}$ in general matrix form,

$$\underline{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

There exist 3 principal values/directions of $\underline{C}$. Assume the 3 principal directions are ($\vec{m}_I$, $\vec{m}_{II}$, $\vec{m}_{III}$). If we choose ($\vec{m}_I$, $\vec{m}_{II}$, $\vec{m}_{III}$) as the base vectors, then

$$\underline{C} = \begin{bmatrix} C_I & 0 & 0 \\ 0 & C_{II} & 0 \\ 0 & 0 & C_{III} \end{bmatrix}, \quad \vec{m}_I \perp \vec{m}_{II} \perp \vec{m}_{III}$$

Physical/Geometrical Interpretations of $\underline{C}$



Considering a fiber initially along one of the base vectors, say $\vec{e}_1$ direction,

$$\mathrm{d}\vec{x}_1 = \mathrm{d}l_{10}\vec{e}_1, \quad \mathrm{d}\vec{y}_1 = \mathrm{d}l_1\vec{n}$$

$$\mathrm{d}l_1^2 = \mathrm{d}\vec{y}_1 \cdot \mathrm{d}\vec{y}_1 = \underline{F}\mathrm{d}\vec{x}_1 \cdot \underline{F}\mathrm{d}\vec{x}_1 = \mathrm{d}l_{10}\vec{e}_1 \cdot \underline{C}\mathrm{d}l_{10}\vec{e}_1 = \mathrm{d}l_{10}^2 C_{11}$$

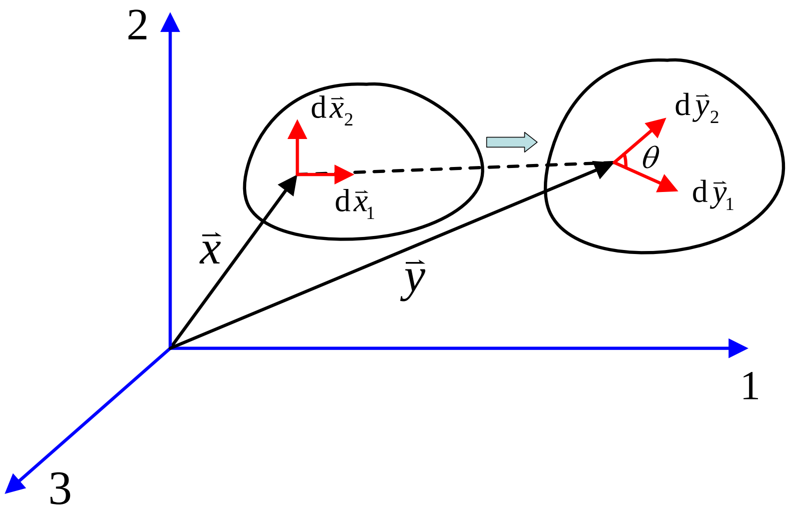
$$C_{11} = \frac{\mathrm{d}l_1^2}{\mathrm{d}l_{10}^2} = \lambda_1^2$$

Therefore, the geometrical interpretation of C_{11} is that it is the stretch of a fiber initially aligned in the $\vec{e}_1$ -direction.

Similarly,

$C_{22} = \lambda_2^2$ is the stretch of a fiber initially aligned in the $\vec{e}_2$ -direction)

$C_{33} = \lambda_3^2$ is the stretch of a fiber initially aligned in the $\vec{e}_3$-direction)



To understand the off-diagonal terms of $\underline{C}$, let us consider two fibers initially aligned in the $\vec{e}_1$ and $\vec{e}_2$ directions, respectively,

$$d\vec{x}_1 = dl_{10}\vec{e}_1, \quad d\vec{x}_2 = dl_{20}\vec{e}_2$$

$$d\vec{y}_1 = dl_1\vec{n}_1, \quad d\vec{y}_2 = dl_2\vec{n}_2$$

$$d\vec{y}_1 \cdot d\vec{y}_2 = \underline{F}d\vec{x}_1 \cdot \underline{F}d\vec{x}_2 = d\vec{x}_1 \cdot \underline{C}d\vec{x}_2 \Rightarrow dl_1 dl_2 \cos\theta_{12} = dl_{10}\vec{e}_1 \cdot \underline{C}dl_{20}\vec{e}_2$$

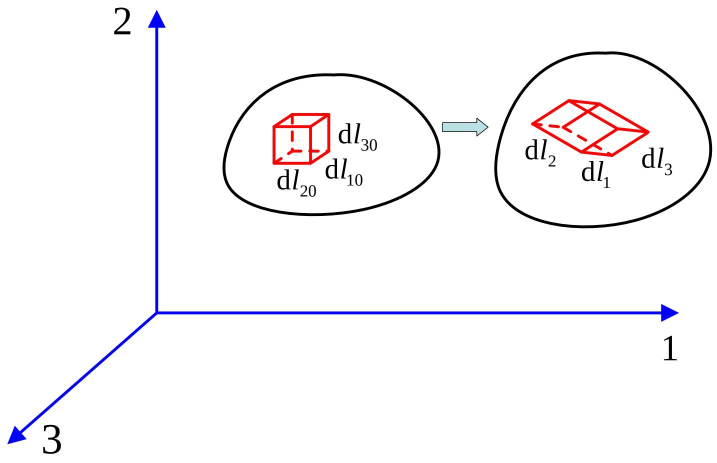
$$\cos\theta_{12} = \frac{C_{12}}{\lambda_1\lambda_2} = \frac{C_{12}}{\sqrt{C_{11}C_{22}}}$$

Therefore, the geometrical interpretation of C_{12} is that it is a measure of the angle between two fibers initially aligned in the $\vec{e}_1$ and $\vec{e}_2$ directions.

The matrix

$$\underline{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

gives away the information how a small block of material deforms.

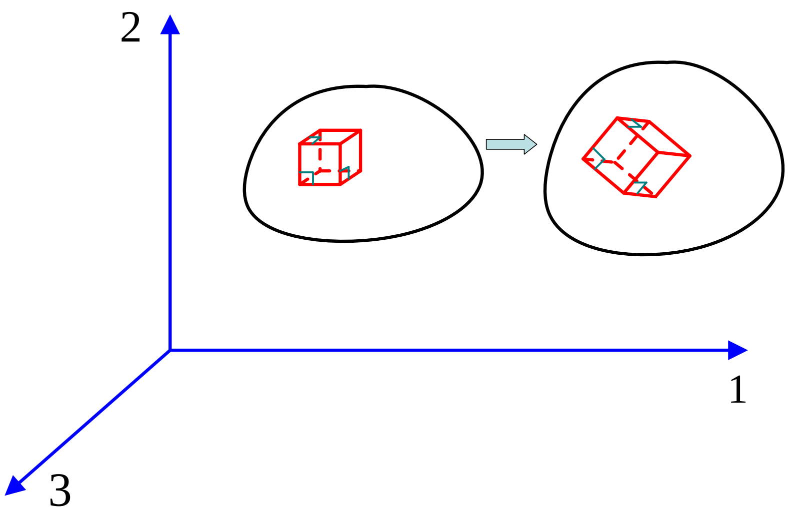


$$dl_1 = \sqrt{C_{11}} dl_{10}, \quad dl_2 = \sqrt{C_{22}} dl_{20}, \quad dl_3 = \sqrt{C_{33}} dl_{30}$$

$$\cos\theta_{12} = \frac{C_{12}}{\sqrt{C_{11}C_{22}}}, \quad \cos\theta_{23} = \frac{C_{23}}{\sqrt{C_{22}C_{33}}}, \quad \cos\theta_{13} = \frac{C_{13}}{\sqrt{C_{11}C_{33}}}$$

In the principal coordinates, the above geometrical interpretations suggest

$$\underline{C} = \begin{bmatrix} \lambda_I^2 & 0 & 0 \\ 0 & \lambda_{II}^2 & 0 \\ 0 & 0 & \lambda_{III}^2 \end{bmatrix}$$

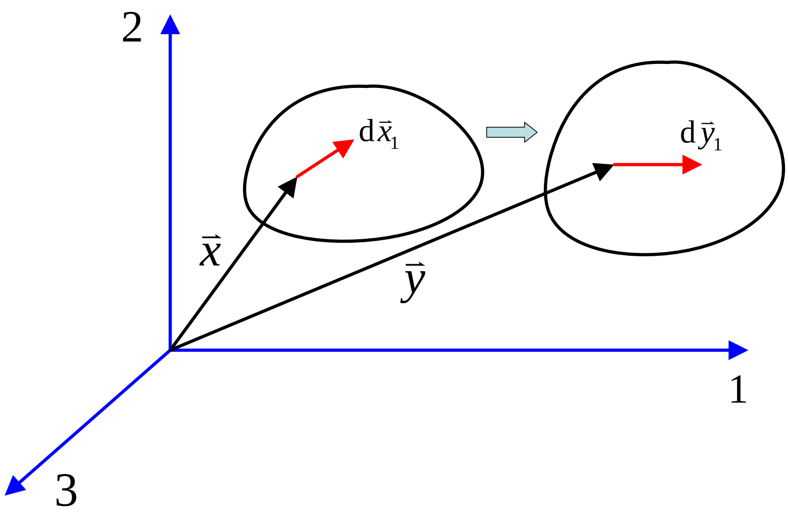


where λ_I, λ_{II}, λ_{III} are principal stretches.

$$\underline{C}\vec{m} = \lambda^2 \vec{m} \quad \Rightarrow \quad \vec{m}_I \perp \vec{m}_{II} \perp \vec{m}_{III}$$

This indicates that the fibers along the principal directions will remain perpendicular to each other after deformation.

The above formulation of strain has been focused on making predictions about the deformed configuration (called Eulerian) based on know information in the undeformed (called Lagrangian) configuration. Alternatively, we could reverse the direction of analysis. We could start from the deformed configuration and try to predict the undeformed configuration. For example, consider a fiber $\mathrm{d}\vec{y}_1$ in aligned in the $\vec{e}_1$ direction after deformation, what is the stretch that has happened to this fiber?



Assume we know $\mathrm{d}\vec{y}_1 = \mathrm{d}l\vec{e}_1$, we need to calculate $\mathrm{d}\vec{x}_1 = \mathrm{d}l_{10}\vec{m}$

$$\mathrm{d}\vec{y} = \underline{F}\mathrm{d}\vec{x} \Rightarrow \mathrm{d}\vec{x} = \underline{F}^{-1}\mathrm{d}\vec{y}$$

$$\mathrm{d}\vec{x}_1 \cdot \mathrm{d}\vec{x}_1 = \mathrm{d}l_{10}^2 = F^{-1}\mathrm{d}\vec{y}_1 \cdot F^{-1}\mathrm{d}\vec{y}_1 = \mathrm{d}\vec{y}_1 \cdot \left(FF^T\right)^{-1}\mathrm{d}\vec{y}_1 = \mathrm{d}\vec{y}_1 \cdot \underline{B}^{-1}\mathrm{d}\vec{y}_1$$

$\underline{B} = \underline{F}\,\underline{F}^T$ is called the left Cauchy-Green strain.

$$\mathrm{d}l_{10}^2 = \mathrm{d}l_1\vec{e}_1 \cdot \underline{B}^{-1}\mathrm{d}l_1\vec{e}_1 \Rightarrow B_{11}^{-1} = \lambda_1^{-2}$$

Therefore, the stretch that has happened to a fiber aligned in the $\vec{e}_1$ direction after deformation is given by B_{11}^{-1}.

Similarly,

$$B_{22}^{-1} = \lambda_2^{-2}$$

$$B_{33}^{-1} = \lambda_3^{-2}$$

Write $\underline{B}$ and $\underline{B}^{-1}$ in matrix form,

$$\underline{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}, \quad \underline{B}^{-1} = \begin{bmatrix} B_{11}^{-1} & B_{12}^{-1} & B_{13}^{-1} \\ B_{21}^{-1} & B_{22}^{-1} & B_{23}^{-1} \\ B_{31}^{-1} & B_{32}^{-1} & B_{33}^{-1} \end{bmatrix}$$

The off-diagonal term has the following interpretation:

$$\mathrm{d}\bar{x}_1 \cdot \mathrm{d}\bar{x}_2 = \mathrm{d}l_{10}\mathrm{d}l_{20}\cos\alpha_{12} = \mathrm{d}l_1\mathrm{d}l_2\bar{e}_1 \cdot \underline{B}^{-1}\bar{e}_2 \Rightarrow \cos\alpha_{12} = \frac{B_{12}^{-1}}{\sqrt{B_{11}^{-1}B_{22}^{-1}}}$$

Clearly, B_{12}^{-1} gives the information about the original angle of two fibers that have become

aligned in the

Similarly,

$$\cos\alpha_{23} = \frac{B_{23}^{-1}}{\sqrt{B_{22}^{-1}B_{33}^{-1}}}, \quad \cos\alpha_{13} = \frac{B_{13}^{-1}}{\sqrt{B_{11}^{-1}B_{33}^{-1}}}$$

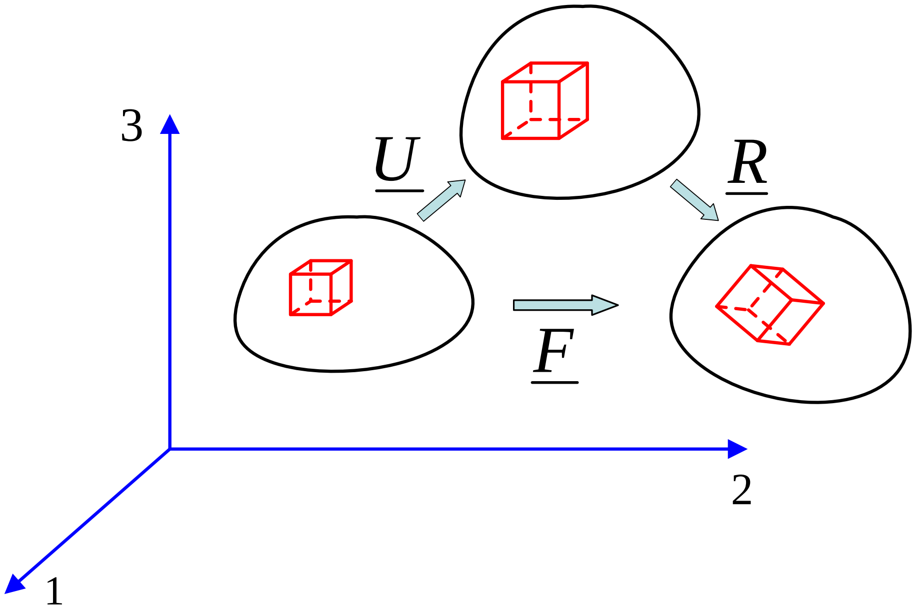
In principal coordinates, we can define

$$\underline{U} = \sqrt{\underline{C}} = \begin{bmatrix} \lambda_I & 0 & 0 \\ 0 & \lambda_{II} & 0 \\ 0 & 0 & \lambda_{III} \end{bmatrix}$$

Here $\underline{U}$ is called the right stretch tensor.

The deformation, $\underline{F}\mathrm{d}\bar{x} = \mathrm{d}\bar{y}$, can be generally described as stretch + rigid body rotation. If stretch happens first, rotation second, we can write

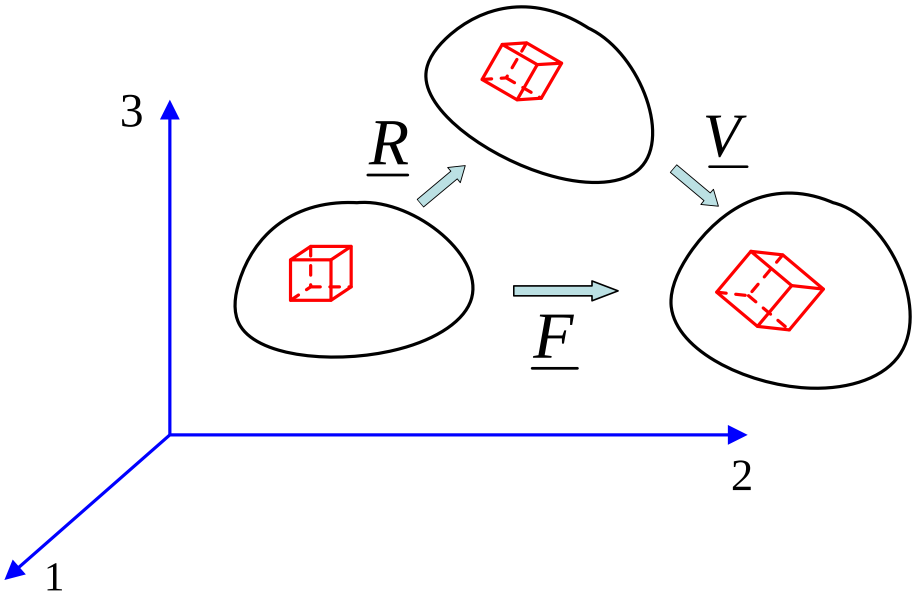
$$\mathrm{d}\bar{x} \to \mathrm{d}\bar{z} = \underline{U}\mathrm{d}\bar{x} \to \mathrm{d}\bar{y} = \underline{R}\mathrm{d}\bar{z} \Rightarrow \mathrm{d}\bar{y} = \underline{R}\underline{U}\mathrm{d}\bar{x} = \underline{F}\mathrm{d}\bar{x}$$



This suggests a decomposition $\underline{F} = \underline{R}\underline{U}$, which is called the right polar decomposition

Alternatively, if rigid body rotation happens first, then stretch,

$$\mathrm{d}\vec{x} \to \mathrm{d}\vec{z} = \underline{R}\mathrm{d}\vec{x} \to \mathrm{d}\vec{y} = \underline{V}\mathrm{d}\vec{z} \Rightarrow \mathrm{d}\vec{y} = \underline{V}\,\underline{R}\mathrm{d}\vec{x} = \underline{F}\mathrm{d}\vec{x}$$



We then have $\underline{F} = \underline{V}\,\underline{R}$, which is called the left polar decomposition.

$$\underline{C} = \underline{F}^T \underline{F} = \left(\underline{R}\underline{U}\right)^T \underline{R}\underline{U} = \underline{U}^T \underline{R}^T \underline{R}\underline{U} = \underline{U}^T \underline{U} = \underline{U}^2$$

$$\underline{B} = \underline{F}\,\underline{F}^T = \underline{V}\,\underline{R}\left(\underline{V}\,\underline{R}\right)^T = \underline{V}\,\underline{R}\,\underline{R}^T\,\underline{V}^T = \underline{V}\,\underline{V}^T = \underline{V}^2$$

We can easily show that the principal values and directions of $\underline{C}$ and $\underline{B}$ are related.