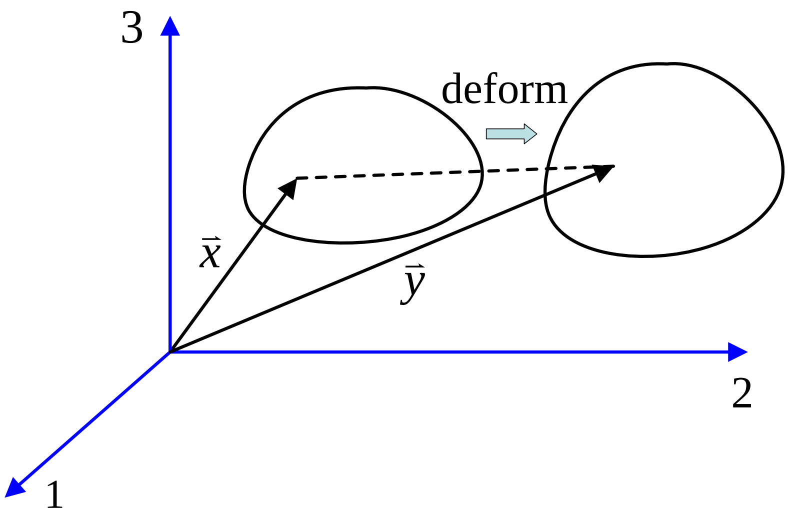
Announcements:

Mid-term (Tuesday, Oct 31) (1 page double sided notes allowed)
Proposal/team for ABAQUS project (Thursday, Nov 9)

<u>Review of strain concepts</u>



$\vec{y} = \vec{y}(\vec{x}, t), \quad y_i = y_i(x_1, x_2, x_3, t) = x_i + u_i$

Deformation gradient: $d\vec{y} = \underline{F} d\vec{x}, \quad F_{ij} = \dfrac{\partial y_i}{\partial x_j}$

Cauchy-Green strains:

$\underline{C} = \underline{F}^T \underline{F}, \quad C_{ij} = F_{ki} F_{kj}$

$\underline{B} = \underline{F}\,\underline{F}^T, \quad B_{ij} = F_{ik} F_{jk}$

Stretch tensors: $\underline{U} = \sqrt{\underline{C}}, \quad \underline{V} = \sqrt{\underline{B}}$

$$\underline{C} = \begin{bmatrix} \lambda_I^2 & 0 & 0 \\ 0 & \lambda_{II}^2 & 0 \\ 0 & 0 & \lambda_{III}^2 \end{bmatrix}, \quad \underline{U} = \begin{bmatrix} \lambda_I & 0 & 0 \\ 0 & \lambda_{II} & 0 \\ 0 & 0 & \lambda_{III} \end{bmatrix} = \sqrt{\underline{C}}$$

Polar decomposition:

$\underline{F} = \underline{R}\underline{U} = \underline{V}\,\underline{R}$

Let us see how the principal values/directions of C-G strains are related:

The principal values/directions of $\underline{C}$ and $\underline{U}$ are

$$\underline{U}\vec{m} = \lambda_U \vec{m}\,,\ \ \underline{C}\vec{m} = \lambda_U^2 \vec{m}$$

Multiply the above equation by $\underline{R}$,

$$\underline{R}\underline{U}\vec{m} = \lambda_U \underline{R}\vec{m}$$

Since $\underline{R}\underline{U} = \underline{V}\,\underline{R} = \underline{F}$, we have

$$\underline{V}\left(\underline{R}\vec{m}\right) = \lambda_U \left(\underline{R}\vec{m}\right)$$

Comparing this with the principal value equations for $\underline{V}$ and $\underline{B}$,

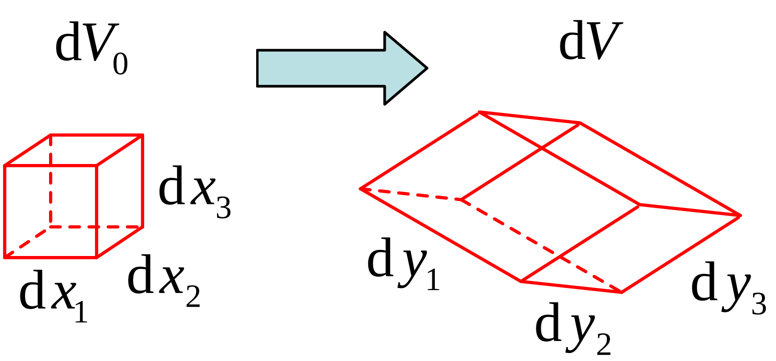
$$\underline{V}\vec{n} = \lambda_V \vec{n} \ \ (\text{and } \ \underline{B}\vec{n} = \lambda_V^2 \vec{n}\,)$$

we can see that

$$\lambda_U = \lambda_V = \lambda\,,\ \ \vec{n} = \underline{R}\vec{m}$$

Therefore, we have one set of principle values $\left(\lambda_I, \lambda_{II}, \lambda_{III}\right)$ and two sets of principle directions, $\left(\vec{m}_I, \vec{m}_{II}, \vec{m}_{III}\right)$ for $\underline{C}$, $\underline{U}$ and $\left(\vec{n}_I, \vec{n}_{II}, \vec{n}_{III}\right) = \underline{R}\left(\vec{m}_I, \vec{m}_{II}, \vec{m}_{III}\right)$ for $\underline{V}$ and $\underline{B}$.

Volume change



$\dfrac{\mathrm{d}V}{\mathrm{d}V_0}$: ratio of volume change (dilation) during deformation

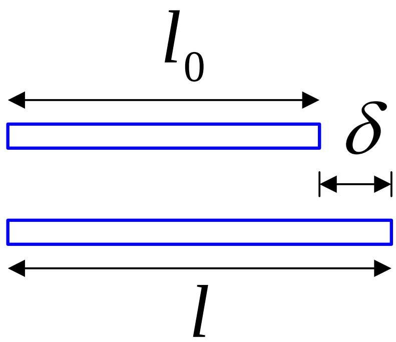
$$\begin{aligned} dV &= (dy_1 \times dy_2) \cdot dy_3 = \varepsilon_{ijk} dy_i dy_j dy_k \\ &= \varepsilon_{ijk} F_{ip} F_{jq} F_{kr} dx_p dx_q dx_r \\ &= \varepsilon_{pqr} \det(\underline{F}) dx_p dx_q dx_r \\ &= \det(\underline{F}) dV_0 \end{aligned}$$

Therefore: $\dfrac{dV}{dV_0} = \det(\underline{F}) = \det(\underline{U}) = \det(\underline{V}) = \sqrt{\det(\underline{B})} = \sqrt{\det(\underline{C})} = J$

J is called the Jacobian of deformation.

$$J = \begin{vmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \partial y_1/\partial x_3 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \partial y_2/\partial x_3 \\ \partial y_3/\partial x_1 & \partial y_3/\partial x_2 & \partial y_3/\partial x_3 \end{vmatrix}$$

Engineering concepts of strain



$$\varepsilon_{(1)} = \frac{\delta}{l_0} = \frac{l - l_0}{l_0} = \lambda - 1$$

$$\varepsilon_{(2)} = \frac{\delta}{l} = \frac{l - l_0}{l} = 1 - \lambda^{-1}$$

$$\varepsilon_{(3)} = \frac{l^2 - l_0^2}{2l_0^2} = \frac{1}{2}(\lambda^2 - 1)$$

$$\varepsilon_{(4)} = \frac{l^2 - l_0^2}{2l^2} = \frac{1}{2}(1 - \lambda^{-2})$$

For 3D, we can genenalize the corresponding definition as

(1): $(\underline{U} - \underline{I})$

(2): $(\underline{I} - \underline{V})$

$$(3):\ \frac{1}{2}\left(\underline{C}-\underline{I}\right)$$

$$(4):\ \frac{1}{2}\left(\underline{I}-\underline{B}^{-1}\right)$$

Among them,

$\underline{E}=\frac{1}{2}\left(\underline{C}-\underline{I}\right)$ is called the Lagrangian strain tensor.

$\underline{E}^{*}=\frac{1}{2}\left(\underline{I}-\underline{B}^{-1}\right)$ is called the Eulerian strain tensor.

Example 1: Pure dilation (no shear)

$$\underline{U}=\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}=\lambda\underline{I}$$

$$\underline{C}=\lambda^{2}\underline{I}$$

$$\underline{F}=\underline{R}\underline{U}=\lambda\underline{R}$$

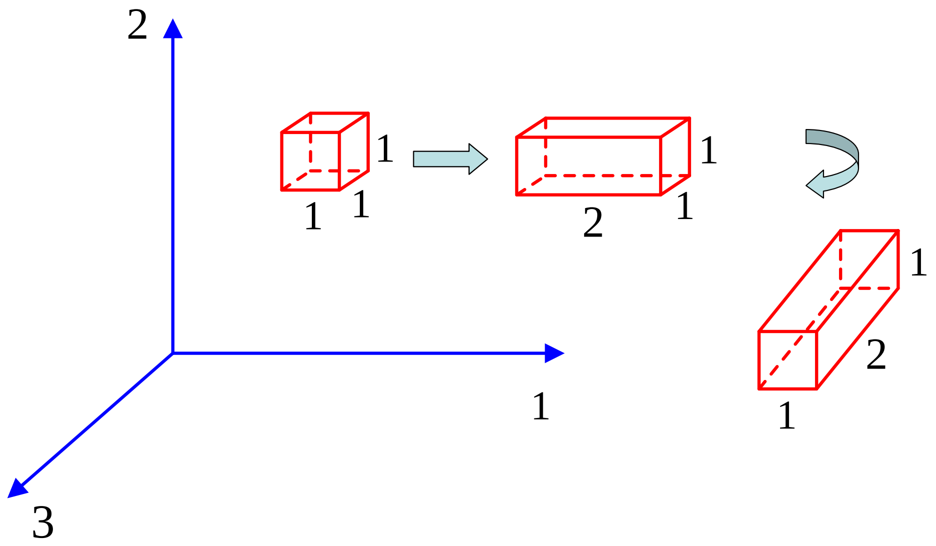
$$\underline{B}=\underline{F}\underline{F}^{T}=\lambda^{2}\underline{I}$$

$$\underline{V}=\lambda\underline{I}$$

$$\underline{E}=\frac{1}{2}\left(\lambda^{2}-1\right)\underline{I}=\underline{E}^{*}$$

$$\frac{\mathrm{d}V}{\mathrm{d}V_{0}}=J=\det\left(\underline{F}\right)=\lambda^{3}$$

Example 2: Uniaxial stretch (stretch along 1-direction first, then rotate 90° about 2-axis)



$$\lambda_I = \lambda, \ \ \lambda_{II} = \lambda_{III} = 1$$

$$\underline{U} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{R} = \vec{e}_i \cdot \underline{R}\,\vec{e}_j = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\underline{F} = \underline{R}\underline{U} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

$$\underline{V}\,\underline{R} = \underline{F} \Rightarrow \underline{V} = \underline{F}\,\underline{R}^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} == \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\underline{C} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \ \underline{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\underline{E} = \frac{1}{2}\left(\underline{C} - \underline{I}\right) = \frac{1}{2}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{E}^* = \frac{1}{2}\left(\underline{I} - \underline{B}^{-1}\right) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\frac{\mathrm{d}V}{\mathrm{d}V_0} = J = 2$$

If rotated 45° about 2-axis,

$$\underline{U} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

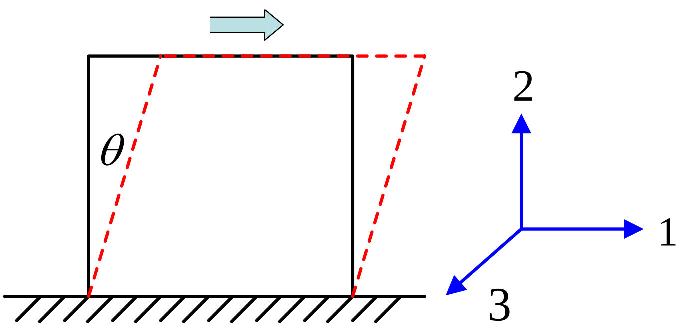
$$\underline{R} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The other tensors can be worked out by

$$\underline{F} = \underline{R}\underline{U}, \quad \underline{V}\,\underline{R} = \underline{R}\underline{U}$$

$$\underline{V} = \underline{F}\,\underline{R}^T, \quad \underline{B} = \underline{F}\,\underline{F}^T$$

Example 3: Simple shear



$$y_1 = x_1 + x_2 \tan\theta$$

$$y_2 = x_2$$

$$y_3 = x_3$$

$$\underline{F} = \frac{\partial y_i}{\partial x_j} = \begin{bmatrix} 1 & \tan\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{C} = \underline{F}^T \underline{F} = \begin{bmatrix} 1 & 0 & 0 \\ \tan\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \tan\theta & 0 \\ \tan\theta & 1+\tan^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Small strain & small rotation theory

Assumption: displacements in solids are usually small compared to relevant structure sizes, i.e.

$$\frac{\partial u_i}{\partial x_j} \propto \left(\frac{u}{L} \right) << 1$$

In this case, it proves to be useful to Linearize all equations about $\frac{\partial u_i}{\partial x_j}$,

$$\underline{F} = \frac{\partial y_i}{\partial x_j} = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j} \right) = \delta_{ij} + u_{i,j} \quad \text{(because } y_i = x_i + u_i \text{)}$$

$$\underline{C} = F_{ki} F_{kj} = (\delta_{ki} + u_{k,i})(\delta_{kj} + u_{k,j}) = \delta_{ij} + u_{i,j} + u_{j,i} + u_{k,i} u_{k,j} \cong \delta_{ij} + u_{i,j} + u_{j,i}$$

$$\underline{U} = \sqrt{\underline{C}} = \delta_{ij} + \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\underline{E} = \frac{1}{2}(\underline{C} - \underline{I}) = \frac{1}{2}(u_{i,j} + u_{j,i}) = \varepsilon_{ij} \quad \text{(small strain tensor)}$$

$$\underline{B} = F_{ik} F_{jk} = (\delta_{ik} + u_{i,k})(\delta_{jk} + u_{j,k}) \cong \delta_{ij} + u_{i,j} + u_{j,i}$$

$$\underline{V} = \delta_{ij} + \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\underline{E}^* = \frac{1}{2}(\underline{I} - \underline{B}^{-1}) = \frac{1}{2}(u_{i,j} + u_{j,i}) = \varepsilon_{ij} = \underline{E}$$

For small strain & small rotation, there is no longer a need to distinguish between Lagrangian and Eulerian strain tensors. Basically, all strain tensors become reduced to one

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

This greatly simplifies the mathematical problem.