Summary of elementary strain concepts and their generalizations to 3D tensors:

1D	**3D**
$\varepsilon_{(1)} = \frac{\delta}{l_0} = \frac{l - l_0}{l_0} = \lambda - 1$	$(\underline{U} - \underline{I})$
$\varepsilon_{(2)} = \frac{\delta}{l} = \frac{l - l_0}{l} = 1 - \lambda^{-1}$	$(\underline{I} - \underline{V}^{-1})$
$\varepsilon_{(3)} = \frac{l^2 - l_0^2}{2l_0^2} = \frac{1}{2}(\lambda^2 - 1)$	$\frac{1}{2}(\underline{C} - \underline{I})$
$\varepsilon_{(4)} = \frac{l^2 - l_0^2}{2l^2} = \frac{1}{2}(1 - \lambda^{-2})$	$\frac{1}{2}(\underline{I} - \underline{B}^{-1})$
For small strain: $\varepsilon_{(1)} = \varepsilon_{(2)} = \varepsilon_{(3)} = \varepsilon_{(4)} = \varepsilon$	$\underline{\varepsilon} = \underline{U} - \underline{I} = \underline{I} - \underline{V}^{-1} = \frac{1}{2}(\underline{C} - \underline{I}) = \frac{1}{2}(\underline{I} - \underline{B}^{-1})$ $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ or $\underline{\varepsilon} = \frac{1}{2}(\nabla\vec{u} + \nabla^T\vec{u})$
1 equation: $\varepsilon = \frac{\partial u}{\partial x}$	6 equations: $\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$, $\varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$, $\varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$ $\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$, $\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$, $\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$

For small strain and small rotation, the rigid body rotation part is analyzed as follows:

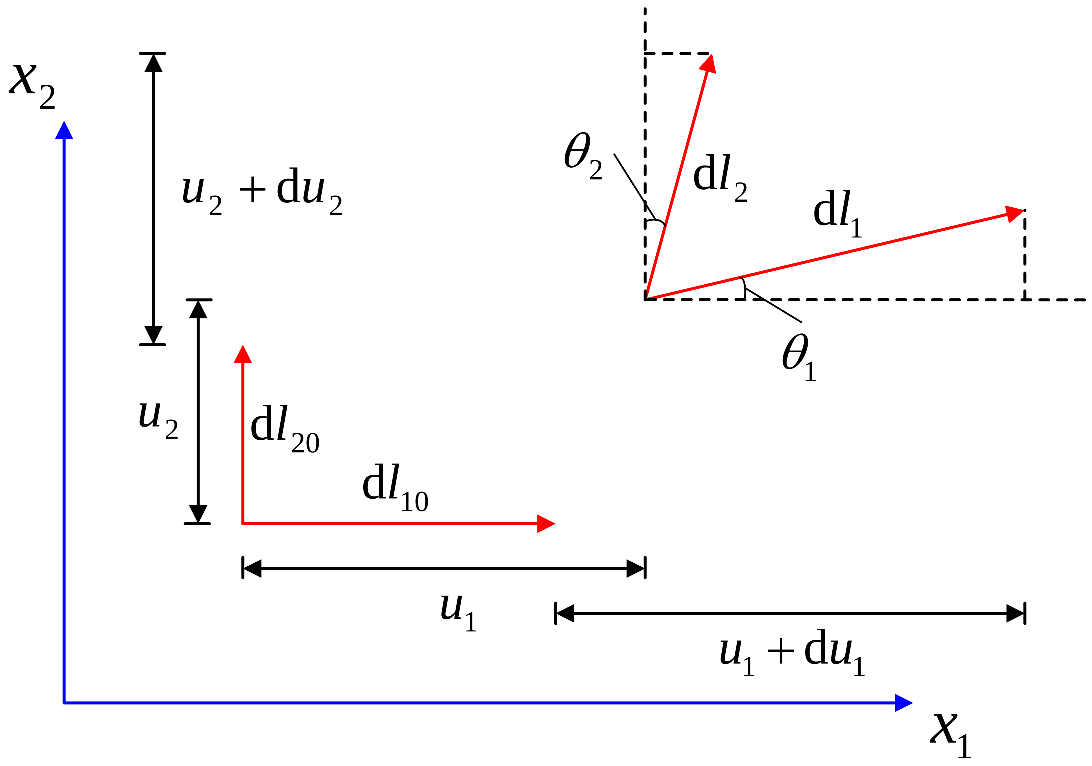
$\underline{R} = \underline{I} + \underline{\omega}$ where $\underline{\omega}$ is defined as the small rotation tensor.

$$\underline{R}\underline{U} = (\underline{I} + \underline{\omega})(\underline{I} + \underline{\varepsilon}) \cong \underline{I} + \underline{\varepsilon} + \underline{\omega} = \underline{F} = \underline{I} + \nabla\vec{u}$$

$$\underline{\omega} = \nabla\vec{u} - \underline{\varepsilon} = \nabla\vec{u} - \frac{1}{2}(\nabla\vec{u} + \nabla^T\vec{u}) = \frac{1}{2}(\nabla\vec{u} - \nabla^T\vec{u})$$

$\underline{\omega}^T = -\underline{\omega}$ (antisymmetric tensor)

Alternative way of deriving small strain tensor:



Normal strain:

$$\varepsilon_{11} = \frac{dl_1 - dl_{10}}{dl_{10}} = \frac{\sqrt{(dl_{10} + du_1)^2 + du_2^2} - dl_{10}}{dl_{10}} = \sqrt{\left(1 + \frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2} - 1 \cong \frac{\partial u_1}{\partial x_1}$$

$$\varepsilon_{22} = \frac{dl_2 - dl_{20}}{dl_{20}} = \frac{\sqrt{(dl_{20} + du_2)^2 + du_1^2} - dl_{20}}{dl_{20}} = \sqrt{\left(1 + \frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_1}{\partial x_2}\right)^2} - 1 \cong \frac{\partial u_2}{\partial x_2}$$

Shear strain:

$$\gamma_{12} = \theta_1 + \theta_2$$

$$\tan\theta_1 = \frac{du_2}{dl_{10} + du_1} = \frac{\partial u_2 / \partial x_1}{1 + \partial u_1 / \partial x_1} = \frac{\partial u_2}{\partial x_1} = \theta_1$$

$$\tan\theta_2 = \frac{du_1}{dl_{20} + du_2} = \frac{\partial u_1 / \partial x_2}{1 + \partial u_2 / \partial x_2} = \frac{\partial u_1}{\partial x_2} = \theta_2$$

$$\gamma_{12} = \theta_1 + \theta_2 = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 2\varepsilon_{12}$$

Volume change:

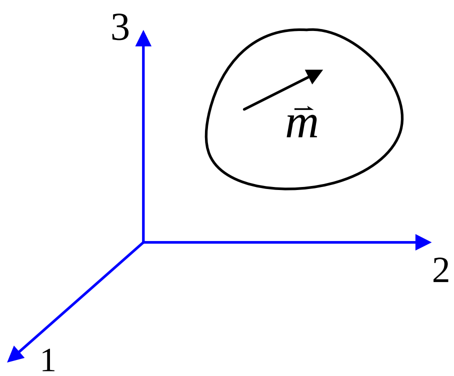
$$\frac{\mathrm{d}V}{\mathrm{d}V_0} = \det\left(\underline{F}\right) = \begin{vmatrix} 1+\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1+\frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1+\frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

If $\left|\frac{\partial u_i}{\partial x_j}\right| << 1$,

$$\frac{\mathrm{d}V}{\mathrm{d}V_0} = 1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} + o\left(\frac{\partial u_i}{\partial x_j}\right) \cong 1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 1 + \varepsilon_{kk}$$

Therefore:

$$\frac{\mathrm{d}V - \mathrm{d}V_0}{\mathrm{d}V_0} = \varepsilon_{kk} \quad \text{(bulk strain)}$$

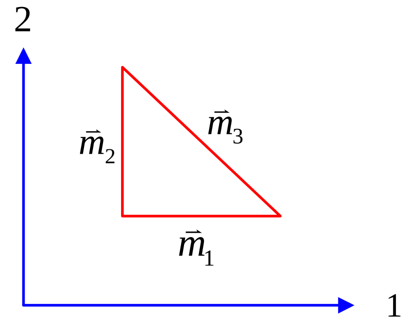


$$\vec{m}\cdot\underline{C}\vec{m} = \lambda^2 \Rightarrow \vec{m}\cdot\frac{1}{2}\left(\underline{C}-\underline{I}\right)\vec{m} = \frac{1}{2}\left(\lambda^2 - 1\right)$$

Note that $\underline{\varepsilon} = \frac{1}{2}\left(\underline{C}-\underline{I}\right)$,

$$\vec{m}\cdot\underline{\varepsilon}\vec{m} = \frac{1}{2}\left(\lambda^2 - 1\right) \cong \lambda - 1 \quad \text{(strain in } \vec{m} \text{ direction)}$$

Strain gages aligned along different directions at a solid surface can be used to measure strain in the plane of the surface. For example, assuming we can measure the normal strain



along the $\vec{m}_1, \vec{m}_2, \vec{m}_3$ directions as shown above, respectively, the components of surface strain tensor is simply

$$\vec{m}_1 \cdot \underline{\varepsilon} \vec{m}_1 = \varepsilon_{11}, \quad \vec{m}_2 \cdot \underline{\varepsilon} \vec{m}_2 = \varepsilon_{22}$$

The shear strain ε_{12} is to be determined from the reading in the $\vec{m}_3$ direction.

<u>Compatibility conditions of strains</u>

3 displacement components $\Rightarrow$ 6 strain components

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

Eliminating u from the above $\Rightarrow$ 3 equations among strain (compatibility conditions)

For example:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u_2}{\partial x_2 \partial x_1^2} = \frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}$$

Similarly:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} = 2\frac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_3}$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} = 2\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3}$$

<u>Chap. 5 Mechanical behavior of solids</u>

In a general boundary value problem in solid mechanics, the number of unknown variables is

u_i (3), ε_{ij} (6), σ_{ij} (6) → 15 unknown variables

Number of equations:

Equilibrium equations: $\sigma_{ij,j} + \rho_i = \ddot{u}_i$ (3 equations)

Strain-displacement equations: $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ (6 equations)

Therefore, we are still missing 6 equations to determine the 15 unknowns. The six additional equations to close the formulations can be generally expressed as:

$$\underline{\sigma} = f\left(\underline{\varepsilon}\right)$$

This leads to the constitutive model of different material behaviors which can be categorized into the following types:

Linear elastic material (Hooke's law)
Elastic-plastic material
Visco-elastic material
Visco-plastic material
…

Linear elastic material:

We have seen Hooke's law in 1D:

$$\sigma = E\varepsilon$$

In 3D, one might generalize this in tensor form as

$$\underline{\sigma} = \underline{C}\,\underline{\varepsilon} \quad \text{or} \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

wWhere C_{ijkl} is a 4th order tensor. For the most general case, a 4th order tensor has $3^4 = 81$ independent components. However, $\underline{\sigma}$ and $\underline{\varepsilon}$ are both symmetric tensors that only have six independent components. Therefore, $\underline{\sigma} = \underline{C}\,\underline{\varepsilon}$ can be rewritten in a matrix form as

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{pmatrix}$$

Clearly, the number of independent components of $\underline{C}$ is reduced to 36.

Furthermore, the 6×6 matrix of $\underline{C}$ is symmetric, which further reduces the number of independent components to 21 (the reason will be discussed in the next lecture).