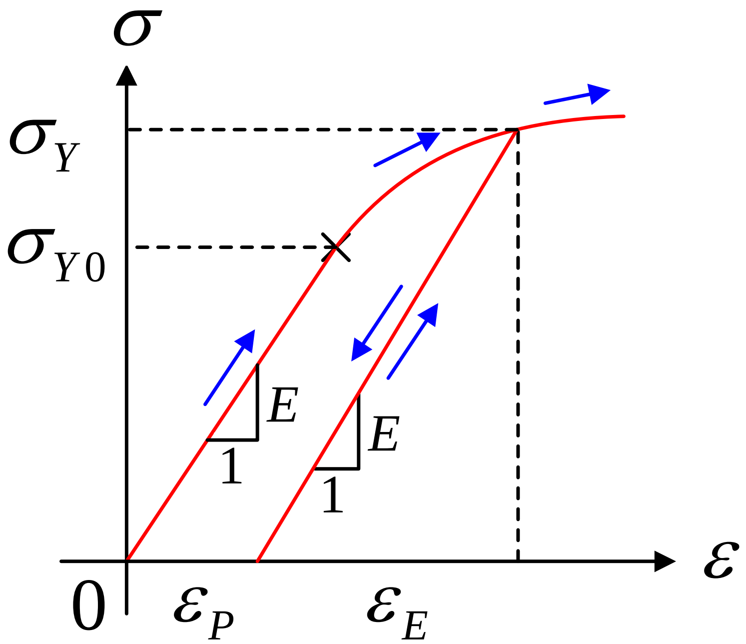
Plastic material behavior

Yield condition: $\sigma = \sigma_Y$

Plastic loading: $\sigma = \sigma_Y$, $\mathrm{d}\sigma > 0$



We will now denote the initial yield stress as σ_{Y0} and the current yield stress as

σ_Y; see above figure.

Decompose strain into elastic & plastic parts

$$\varepsilon = \varepsilon_E + \varepsilon_P$$

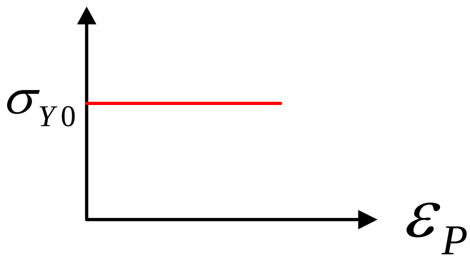
In incremental form:

$$\mathrm{d}\varepsilon_E = \frac{\mathrm{d}\sigma}{E}$$

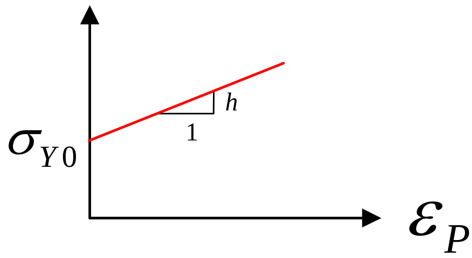
$$\mathrm{d}\varepsilon_P = \begin{cases} \dfrac{\mathrm{d}\sigma}{h}, & \text{if } \sigma = \sigma_Y, \mathrm{d}\sigma > 0 \\ 0, & \text{otherwise} \end{cases}$$

$h = \dfrac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon_P}$ is the tangent modulus of $\sigma - \varepsilon_P$ curve (which is obtained from experiments or fitting an assumed mathematical curve to experimental data).

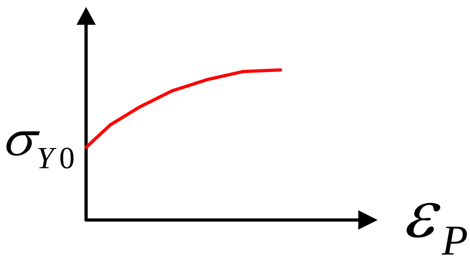
Perfectly plastic: $h=0$



Linear work hardening: $h=\text{const.}$



Power law hardening: $\sigma_Y = \sigma_{Y0}\left(1+\dfrac{E\varepsilon_P}{\sigma_{Y0}}\right)^N$



How to generalize this idea to 3D?

For elastic part,

$$\varepsilon_E = \frac{\sigma}{E} \qquad \Rightarrow \qquad \varepsilon_{ij}^E = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} = \frac{\sigma_{ij}^{'}}{2\mu} + \frac{\sigma_{kk}}{9K}\delta_{ij}$$

$$\mathrm{d}\varepsilon_{ij} = \mathrm{d}\varepsilon_{ij}^E + \mathrm{d}\varepsilon_{ij}^P$$

$$\mathrm{d}\varepsilon_{ij}^E = \frac{1}{2\mu}\mathrm{d}\sigma_{ij}^{'} + \frac{1}{9K}\delta_{ij}\mathrm{d}\sigma_{kk}$$

The plastic strain is typically modeled by the Levy-Mises theory,

$$\mathrm{d}\varepsilon_{ij}^P = \sigma_{ij}^{'}\mathrm{d}\lambda$$

The yield condition:

$$\sigma_e = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} = \sigma_Y \quad \text{(von Mises stress)}$$

shows that σ_e generalizes the 1D stress to an effective stress in 3D.

To utilize the measured plastic stress-strain behavior in 1D, we also need an effective measure of plastic strain in 3D. To see how this is generalized, consider strain energy,

$$\mathrm{d}w = \sigma_{ij}\mathrm{d}\varepsilon_{ij} = \sigma_{ij}\left(\mathrm{d}\varepsilon_{ij}^E + \mathrm{d}\varepsilon_{ij}^P\right) = \mathrm{d}w^E + \sigma'_{ij}\mathrm{d}\varepsilon_{ij}^P = \mathrm{d}w^E + \mathrm{d}w^P$$

$$\mathrm{d}w^P = \sigma'_{ij}\mathrm{d}\varepsilon_{ij}^P = \sigma_e \mathrm{d}\varepsilon_P = \sigma'_{ij}\sigma'_{ij}\mathrm{d}\lambda = \frac{2}{3}\sigma_e^2\mathrm{d}\lambda \quad \Rightarrow \quad \mathrm{d}\lambda = \frac{3}{2}\frac{\mathrm{d}\varepsilon_P}{\sigma_e}$$

$$\mathrm{d}\varepsilon_{ij}^P = \frac{3}{2}\sigma'_{ij}\frac{\mathrm{d}\varepsilon_\mathrm{P}}{\sigma_e}$$

Self consistency requires,

$$\mathrm{d}\varepsilon_{ij}^P\mathrm{d}\varepsilon_{ij}^P = \frac{3}{2}\sigma'_{ij}\frac{\mathrm{d}\varepsilon_\mathrm{P}}{\sigma_e}\cdot\frac{3}{2}\sigma'_{ij}\frac{\mathrm{d}\varepsilon_\mathrm{P}}{\sigma_e} = \frac{3}{2}\mathrm{d}\varepsilon_P^2$$

Therefore,

$$\varepsilon_\mathrm{P} = \sqrt{\frac{2}{3}\mathrm{d}\varepsilon_{ij}^P\mathrm{d}\varepsilon_{ij}^P}$$

Summary:

Generalization from 1D to 3D,

$$\sigma \to \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}}\,,\quad \mathrm{d}\varepsilon_\mathrm{P} \to \sqrt{\frac{2}{3}\mathrm{d}\varepsilon_{ij}^P\mathrm{d}\varepsilon_{ij}^P}$$

1D plastic stress-strain law

$$\mathrm{d}\varepsilon = \mathrm{d}\varepsilon_E + \mathrm{d}\varepsilon_P$$

$$\mathrm{d}\varepsilon_E = \frac{\mathrm{d}\sigma}{E}$$

$$\mathrm{d}\varepsilon_P = \begin{cases} \dfrac{\mathrm{d}\sigma}{h}, & \text{if } \sigma = \sigma_Y,\, \mathrm{d}\sigma > 0 \\ 0, & \text{otherwise} \end{cases}$$

is generalized to 3D plastic stress-strain law

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{E} + d\varepsilon_{ij}^{P}$$

$$d\varepsilon_{ij}^{E} = \frac{1}{2\mu} d\sigma_{ij}^{'} + \frac{1}{9K}\delta_{ij} d\sigma_{kk}$$

$$d\varepsilon_{ij}^{P} = \begin{cases} \frac{3}{2}\sigma_{ij}^{'} \frac{d\sigma_e}{h\sigma_e}, & \text{if } \sigma_e = \sigma_Y,\, d\sigma_e \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

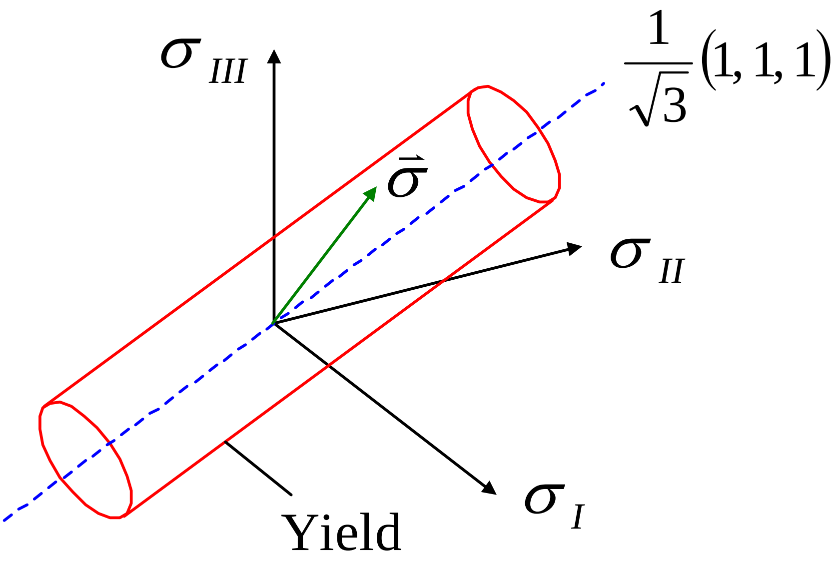
A few remarks:

1) Yield conditions:

Mises condition: $\sigma_e = \sqrt{\frac{3}{2}\sigma_{ij}^{'}\sigma_{ij}^{'}} = \sigma_Y$

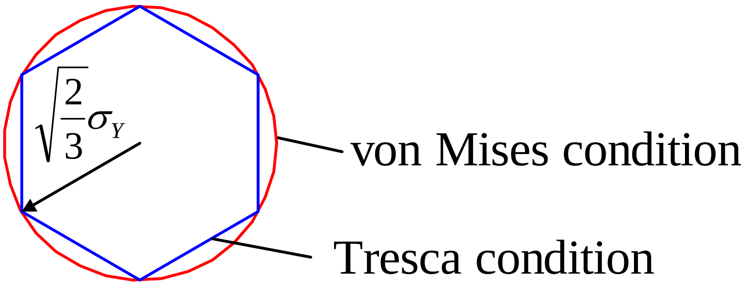
Tresca condition: $\sigma_I, \sigma_{II}, \sigma_{III}$, $\text{Max}\left(|\sigma_I - \sigma_{II}|, |\sigma_I - \sigma_{III}|, |\sigma_{II} - \sigma_{III}|\right) = \sigma_Y$

Representation of yield conditions in stress space:



This is called the yield surface. A perspective view along the $\frac{1}{\sqrt{3}}(1,1,1)$ direction would show

the projection of the von Mises yield surface as a circle



The Tresca yield condition corresponds to the inscribed hexagon inside the von Mises circle. The two yield conditions are actually quite close to each other.

2) J_2 - flow theory

The three invariants of stress σ_{ij} are sometimes denoted as

$$I_1 = \sigma_{kk}\,,\ \ I_2 = \frac{1}{2}\sigma_{ij}\sigma_{ij}\,,\ \ I_3 = \det(\sigma_{ij}) \ \ \text{(3 invariants of } \sigma_{ij}\text{)}$$

The three invariants of deviatoric stress σ'_{ij} are sometimes denoted as:

$$J_1 = \sigma'_{kk} = 0\,,\ \ J_2 = \frac{1}{2}\sigma'_{ij}\sigma'_{ij}\,,\ \ J_3 = \det(\sigma'_{ij}) \ \ \text{(3 invariants of } \sigma'_{ij}\text{)}$$

The Levy-Mises flow rule can be written as

$$\mathrm{d}\varepsilon^P_{ij} = \sigma'_{ij}\mathrm{d}\lambda = \frac{\partial J_2}{\partial \sigma'_{ij}}\mathrm{d}\lambda$$

where J_2 acts as a potential for plastic deformation. Therefore, this theory is also called the J_2-flow theory.