Chap. 6 Boundary value problems in linear elasticity

Equilibrium equations: $\sigma_{ij,j} + f_i = \rho \ddot{u}_i$ (1)

Kinematic equations: $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ (2)

Hooke's law: $\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$ or $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$ (3)

For the most general problems of linear elasticity, you have to solve a system of 15 independent equations with 15 unknown variables.

There are two kinds of solution techniques:

1. Displacement based solution methods

2. Stress based solution methods

1. Displacement based method (Navier displacement equation)

Eliminate strain by inserting (2) into (3):

$$\sigma_{ij} = \mu\left(u_{i,j} + u_{j,i}\right) + \lambda u_{k,k}\delta_{ij} \quad (4)$$

Eliminate stress by inserting (4) into (1):

$$\begin{aligned} &\mu\left(u_{i,jj} + u_{j,ij}\right) + \lambda u_{k,kj}\delta_{ij} + f_i = \rho\ddot{u}_i \\ &\Rightarrow \mu u_{i,jj} + (\mu+\lambda)u_{k,ki} + f_i = \rho\ddot{u}_i \end{aligned} \quad (5)$$

Equation (5) is called Navier displacement equation. The vector form of this equation is

$$\mu\nabla^2\vec{u} + (\mu+\lambda)\nabla\nabla\cdot\vec{u} + \vec{f} = \rho\ddot{\vec{u}}$$

Similar to the concept of splitting stress/strain into a volumetric part and a deviatoric part, the displacement field can also be decomposed into a dilatational part plus a distortional part,

$$\vec{u} = \nabla\varphi + \nabla\times\vec{\psi}$$

where $\nabla\varphi$ is the dilatational term and $\nabla\times\vec{\psi}$ is the distortional term.

The volume change $\frac{\Delta V}{V} = \varepsilon_{kk} = u_{k,k} = \nabla\cdot\vec{u}$. If $\vec{u} = \nabla\times\vec{\psi}$, $\nabla\cdot\vec{u} = \nabla\cdot\nabla\times\psi = 0$, which means the distortional part causes no volume change.

Longitudinal/dilatational elastic wave:

If $\vec{u} = \nabla \varphi$, $\nabla \cdot \vec{u} = \nabla^2 \varphi$, $\nabla\nabla \cdot \vec{u} = \nabla\nabla^2 \varphi = \nabla^2 \nabla \varphi = \nabla^2 \vec{u}$. In this case,

The Navier equation becomes

$$(\lambda + 2\mu)\nabla^2 \vec{u} = \rho \ddot{\vec{u}}$$

where body force is neglected.

Comparing this to the standard form of wave equation $\nabla^2 \vec{u} = \frac{1}{C_l^2}\ddot{\vec{u}}$ indicates that the dilatational part of elastic deformation travels with velocity

$$C_l = \sqrt{\frac{\lambda + 2\mu}{\rho}} \text{ : longitudinal wave speed}$$

Shear/distortional elastic wave:

If $\vec{u} = \nabla \times \vec{\psi}$, $\nabla \cdot \vec{u} = 0$, the Navier equation becomes

$$\mu \nabla^2 \vec{u} = \rho \ddot{\vec{u}}$$

Comparing this to the standard form of wave equation $\nabla^2 \vec{u} = \frac{1}{C_s^2}\ddot{\vec{u}}$ indicates that the distortional part of elastic deformation travels with velocity

$$C_s = \sqrt{\frac{\mu}{\rho}} \text{ : shear wave speed}$$

For a rough estimate of these speeds

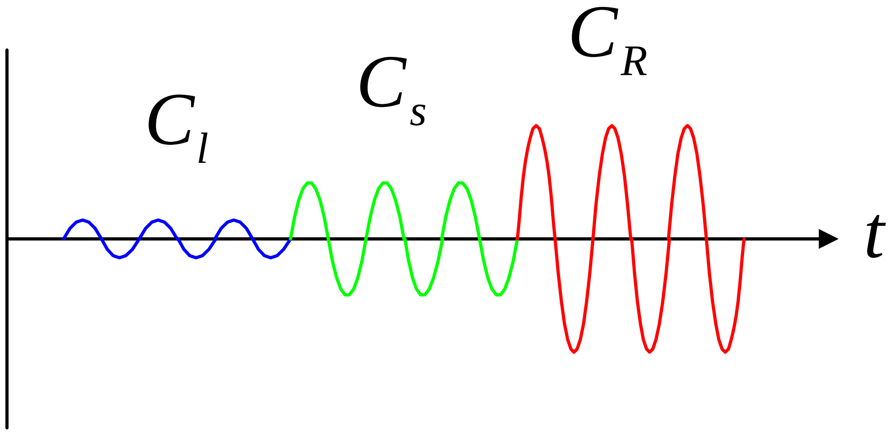
$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = \mu \frac{2\nu}{(1-2\nu)} \cong \mu \quad \text{if} \quad \nu \cong \frac{1}{4}$$

$$\mu = \frac{E}{2(1+\nu)}$$

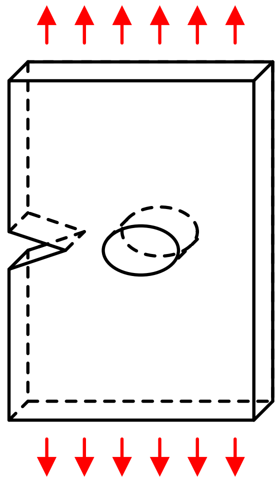
$$C_l = \sqrt{3} C_s \approx 1.73 C_s$$

Generally, the elastic wave speeds in solids are on the order of a few km/s (~5 km/s in steel). The longitudinal wave propagates about 70% faster than the shear wave.

Besides longitudinal and shear waves, there are also surface waves, also called the Rayleigh waves, that propagate at the speed roughly equal to $C_R \approx 0.9 C_s$. Surface wave is the slowest one among the three kinds of elastic wave but usually causes most damages in earthquake. The response signal of an earthquake detector may look like the figure below.

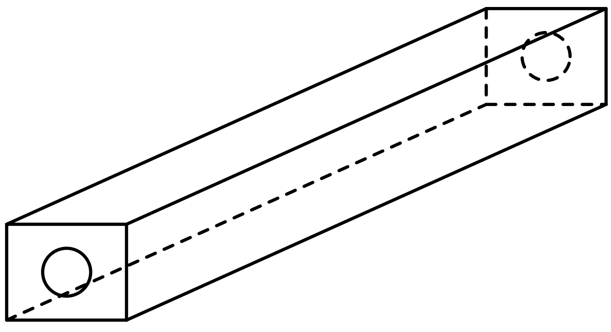


2. Stress based method (2D static equilibrium problems)



Plane stress (thin plates):

$$\sigma_{3i} = 0 \qquad \Rightarrow \qquad \sigma_{31} = \sigma_{32} = \sigma_{33} = 0$$



Plane strain (infinitely long in 3-direction):

$$\varepsilon_{3i} = 0 \Rightarrow \varepsilon_{31} = \varepsilon_{32} = \varepsilon_{33} = 0$$

If we reduce 3D elasticity problems to 2D, the number of unknown variables will decrease from 15 to 8. The equations of 2D plane stress problems are listed as follows:

Equilibrium: $\dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} = 0$ (1); $\dfrac{\partial \sigma_{12}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} = 0$ (2)

Kinematic: $\varepsilon_{11} = \dfrac{\partial u_1}{\partial x_1}$ (3); $\varepsilon_{22} = \dfrac{\partial u_2}{\partial x_2}$ (4); $\varepsilon_{12} = \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right)$ (5)

Hooke's law: $\varepsilon_{11} = \dfrac{1}{E}\left(\sigma_{11} - \nu\sigma_{22}\right)$ (6); $\varepsilon_{22} = \dfrac{1}{E}\left(\sigma_{22} - \nu\sigma_{11}\right)$ (7); $\varepsilon_{12} = \dfrac{1+\nu}{E}\sigma_{12}$ (8)

The objective of stress based method is to eliminate $\vec{u}$ and $\underline{\varepsilon}$ from the above equations.

Eliminate $\vec{u}$ from (3, 4, 5),

$$2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} \quad \text{(compatibility condition)} \qquad (9)$$

Eliminate $\underline{\varepsilon}$ by inserting (6, 7, 8) into (9):

$$2\frac{1+\nu}{E}\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = \frac{1}{E}\left(\frac{\partial^2 \sigma_{11}}{\partial x_2^2} - \nu\frac{\partial^2 \sigma_{22}}{\partial x_2^2}\right) + \frac{1}{E}\left(\frac{\partial^2 \sigma_{22}}{\partial x_1^2} - \nu\frac{\partial^2 \sigma_{11}}{\partial x_1^2}\right) \qquad (10)$$

Differentiating equation (1, 2) with respect to x_1, x_2 gives:

$$\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = -\frac{\partial^2 \sigma_{11}}{\partial x_1^2}, \quad \frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = -\frac{\partial^2 \sigma_{22}}{\partial x_2^2}$$

Inserting the above two relations into (10) leads to

$$\nabla^2\left(\sigma_{11} + \sigma_{22}\right) = 0$$

Therefore, the plane stress equations are reduced to a system of 3 equations (to solve for three unknowns σ_{11}, σ_{22}, σ_{12}):

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} = 0 \\ \dfrac{\partial \sigma_{12}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} = 0 \\ \nabla^2 (\sigma_{11} + \sigma_{22}) = 0 \end{cases}$$

<u>Airy stress function method:</u>

Suppose we represent σ_{ij} by a scalar function ϕ as

$$\sigma_{11} = \phi_{,22}, \quad \sigma_{12} = -\phi_{,12}, \quad \sigma_{22} = \phi_{,11}$$

We see the first two equation of the equation system are always satisfied for any ϕ. The third equation becomes

$$\sigma_{11} + \sigma_{22} = \phi_{,11} + \phi_{,22} = \nabla^2 \phi \qquad \Rightarrow \qquad \nabla^2 \nabla^2 \phi = 0$$

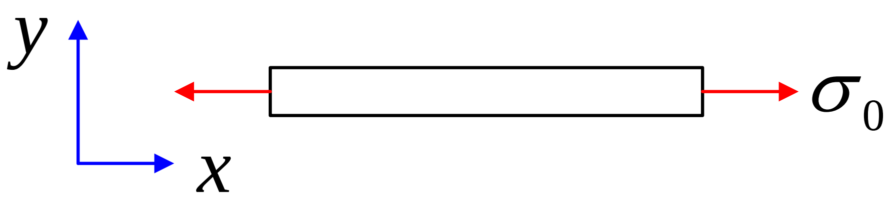
The problem now becomes solving a biharmonic equation. The Airy stress function ϕ is a called biharmonic function.

Example 1: $\phi = a_1 + b_1 x + c_1 y$

$\sigma_{11} = \sigma_{12} = \sigma_{22} = 0$ (rigid body motion)

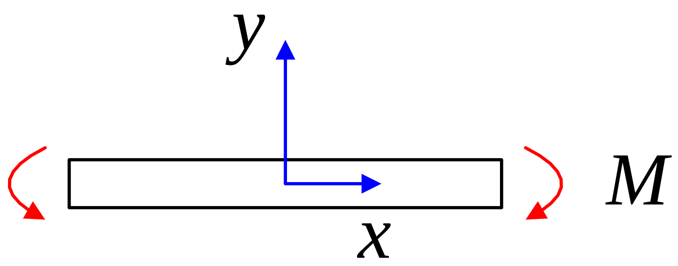
Example 2: $\phi = a_2 x^2 + b_2 xy + c_2 y^2$

$\sigma_{11} = 2c_2$, $\sigma_{12} = -b_2$, $\sigma_{22} = 2a_2$ (states of constant stress)



For uniaxial tension in the x direction, $\phi = \dfrac{\sigma_0}{2} y^2$.

Example 3: Pure bending



$\sigma_{xx} = \frac{My}{I}$, $\sigma_{xy} = 0$, $\sigma_{yy} = 0$

These stress expressions correspond to the Airy stress function

$$\phi = \frac{M}{6I} y^3$$

Therefore, the beam bending problem can also be treated as a 2D elasticity problem in an elongated rectangular domain.