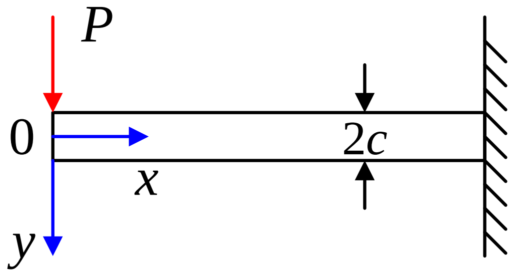
Continue on the Airy stress function method in elasticity:

$$\nabla^2\nabla^2\phi = 0 \quad (\phi: \text{Airy stress function})$$

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}, \quad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}, \quad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}$$

Example 4:




thickness : 1

Consider bending of a beam (height: $2c$; thickness: 1) caused by a concentrated force $P$ at the end. The Airy stress function

$$\nabla^2\nabla^2\phi = \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0$$

for this problem has the form: $\phi = Axy^3$.

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = 6Axy, \quad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -3Ay^2$$

To take care of the traction free boundary on top & bottom surfaces, we can superpose a constant shear stress term to modify $\sigma_{xy}$ as

$$\sigma_{xy} = 3A\left(c^2 - y^2\right) \qquad \text{such that } \sigma_{xy}\big|_{y=\pm c} = 0.$$

The Airy stress function should be modified accordingly,

$$\phi = Axy^3 - 3Ac^2xy$$

The constant $A$ is determined by the boundary condition at the end, i.e. the integral of shear stress across the section should balance the applied force $P$, i.e.

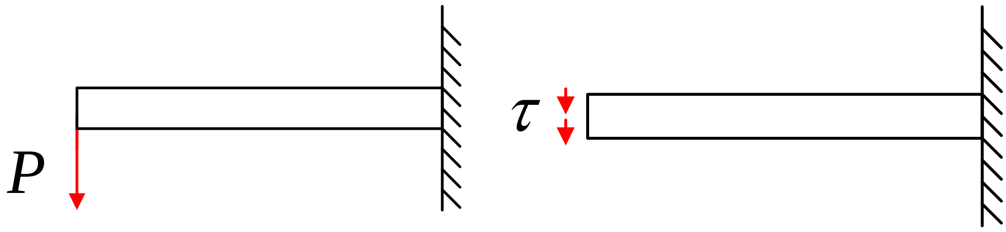
$$\int_{-c}^{c}\sigma_{xy}\mathrm{d}y = -P \Rightarrow 3Ac^2y - Ay^3\Big|_{-c}^{c} = -P \Rightarrow A = -\frac{P}{4c^3}$$

Therefore:

$$\phi = -\frac{P}{4c^3}\left(y^2 - 3c^2\right)xy$$

$$\sigma_{xx} = -\frac{3P}{2c^3}xy, \quad \sigma_{xy} = -\frac{3P}{4c^3}\left(c^2 - y^2\right), \quad \sigma_{yy} = 0$$

Remark 1:




St. Venent principle:

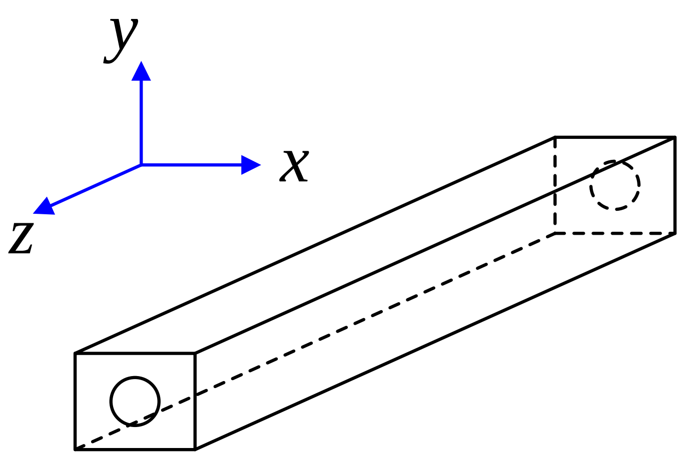
Boundary condition at the end only affects a zone of dimension on the order of beam thickness, as long as the resultant force is the same.

Remark 2:

$$\sigma_{xx} = -\frac{3P}{2c^3}xy = \frac{M}{I}y \quad \Rightarrow \quad \begin{cases} M = -Px \\ I = \frac{1}{12}(2c)^3 = \frac{2}{3}c^3 \end{cases}$$

The result from Airy stress function is identical to that from elementary beam bending theory even though the Kirchhoff assumption of a cross section plane remaining a plane during bending is no longer valid for this problem.

**Plane strain problems ( $\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$ ):**




$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}\right) \qquad (1)$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz}\right) \quad (2)$$

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy}\right) = 0 \Rightarrow \sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right) \quad (3)$$

$$\varepsilon_{xz} = \frac{1+\nu}{E}\sigma_{xz} = 0 \quad (4)$$

$$\varepsilon_{yz} = \frac{1+\nu}{E}\sigma_{yz} = 0 \quad (5)$$

$$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy} \quad (6)$$

Insert (3) into (1):

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy} - \nu^2\left(\sigma_{xx} + \sigma_{yy}\right)\right) = \frac{1-\nu^2}{E}\left(\sigma_{xx} - \frac{\nu}{1-\nu}\sigma_{yy}\right)$$

Similarly, insert (3) into (2):

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx} - \nu^2\left(\sigma_{xx} + \sigma_{yy}\right)\right) = \frac{1-\nu^2}{E}\left(\sigma_{yy} - \frac{\nu}{1-\nu}\sigma_{xx}\right)$$

Suppose we define $E^{'} = \frac{E}{1-\nu^2}$, $\nu^{'} = \frac{\nu}{1-\nu}$, the equations of plane strain condition become

$$\varepsilon_{xx} = \frac{1}{E^{'}}\left(\sigma_{xx} - \nu^{'}\sigma_{yy}\right)$$

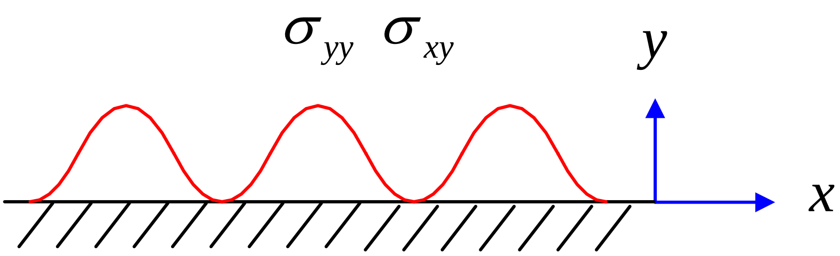
$$\varepsilon_{yy} = \frac{1}{E^{'}}\left(\sigma_{yy} - \nu^{'}\sigma_{xx}\right)$$

$$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy} = \frac{1+\nu^{'}}{E^{'}}\sigma_{xy}$$

which are identical to plane stress condition.

Check: $\frac{1+\nu^{'}}{E^{'}} = \frac{1+\frac{\nu}{1-\nu}}{\frac{E}{1-\nu^2}} = \frac{1+\nu}{E}$.

Example 5: Periodic stresses applied on the surface of an elastic body




$$\sigma_{yy} = \sigma_0 \sin \omega x\,,\quad \sigma_{xy} = \tau_0 \cos \omega x$$

The applied stresses suggest the Airy stress function should have the form of $\phi = f(y)\sin \omega x$.

Insert $\phi = f(y)\sin \omega x$ into $\nabla^2\nabla^2\phi = 0$,

$$\begin{aligned}
& f''''(y)\sin \omega x + 2 f''(y)\left(-\omega^2 \sin \omega x\right) + f(y)\left(\omega^4 \sin \omega x\right) = 0 \\
& \Rightarrow f''''(y) - 2\omega^2 f''(y) + \omega^4 f(y) = 0 \\
& \Rightarrow f = e^{\lambda y} \Rightarrow \lambda^4 - 2\lambda^2\omega^2 + \omega^4 = 0 \\
& \Rightarrow \lambda = \omega, \omega, -\omega, -\omega
\end{aligned}$$

For stress to be finite, $\lambda > 0$

$$f = \left( e^{\lambda y}\Big|_{\lambda=\omega}, \frac{\partial}{\partial \lambda} e^{\lambda y}\Big|_{\lambda=\omega} \right) = C_1 e^{\omega y} + C_2 y e^{\omega y}$$

Therefore,

$$\phi = \left(C_1 e^{\omega y} + C_2 y e^{\omega y}\right)\sin \omega x$$

At the surface of the body ( $y = 0$ ):

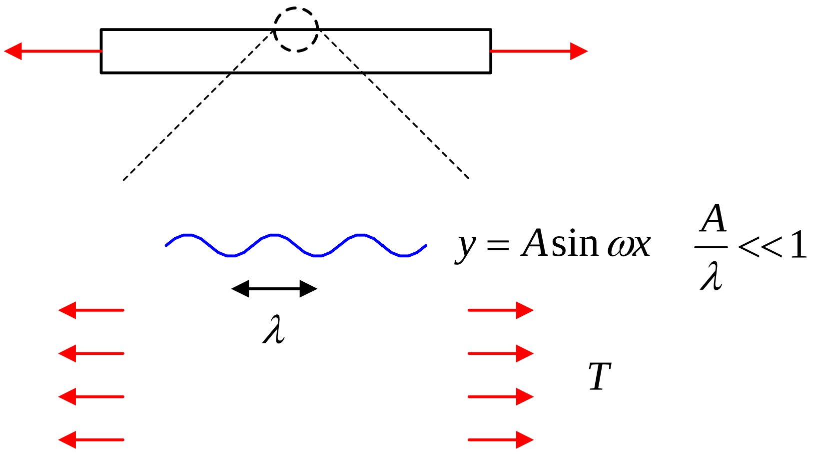
$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}\bigg|_{y=0} = -C_1\omega^2 \sin \omega x = \sigma_0 \sin \omega x$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}\bigg|_{y=0} = -\left(C_1\omega + C_2\right)\omega \cos \omega x = \tau_0 \cos \omega x$$

The constants $C_1$ and $C_2$ can be determined as

$$C_1 = -\frac{\sigma_0}{\omega^2}\,,\quad C_2 = -\frac{\tau_0}{\omega} - C_1\omega = \frac{\sigma_0 - \tau_0}{\omega}$$

Example 6: Stress concentration at a slightly wavy rough surface.

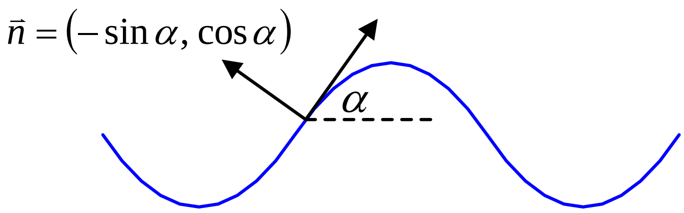



Consider a solid surface with a wavy roughness profile: $y = A\sin\omega x$, $\frac{A}{\lambda} << 1$. The slope of tangential vector along the profile is

$$\tan\alpha = y' = A\omega\cos\omega x \approx \alpha$$

The normal vector is

$$\vec{n} = (-\sin\alpha, \cos\alpha)$$




Traction free boundary condition: $\underline{\sigma}\,\vec{n} = 0$

$$\underline{\sigma} = T\vec{e}_1 \otimes \vec{e}_1 + \hat{\underline{\sigma}}$$

$$\vec{n} = \vec{n}_0 + \hat{\vec{n}}$$

$$\vec{n} = (-\sin\alpha, \cos\alpha)$$

$$\vec{n}_0 = (0,1)$$

$$\hat{\vec{n}} = \vec{n} - \vec{n}_0 = (-\sin\alpha, \cos\alpha) - (0,1) = (-\sin\alpha, \cos\alpha - 1) \approx (-\alpha, 0)$$

Keeping terms of the first order in $\frac{A}{\lambda} << 1$ only,

$$\left(T\vec{e}_1 \otimes \vec{e}_1 + \hat{\underline{\sigma}}\right)\left(\vec{e}_2 + \hat{\vec{n}}\right) = 0 \quad \text{at wavy surface} \qquad \Rightarrow \qquad T\vec{e}_1(-\alpha) + \hat{\underline{\sigma}}\,\vec{e}_2 = 0$$

$$\Rightarrow \quad \hat{\underline{\sigma}}\,\vec{e}_2 = \alpha T\,\vec{e}_1 = AT\omega\cos\omega x\,\vec{e}_1 \qquad \text{at the reference flat surface}$$

Therefore, the rough surface problem can be converted into the problem discussed in Example 5, i.e. a flat surface subjected to sinusoidal stress given by

$$\hat{\underline{\sigma}}_{21} = AT\omega\cos\omega x\,, \quad \hat{\underline{\sigma}}_{22} = 0 \qquad \text{at} \quad y = 0$$

Use the solution in Example 5 ($\sigma_0 = 0$, $\tau_0 = AT\omega$),

$$\hat{\phi} = -\frac{\tau_0}{\omega}\, y e^{\omega y} \sin \omega x = -ATy e^{\omega y} \sin \omega x$$

$$\hat{\sigma}_{xx} = \left.\frac{\partial^2 \hat{\phi}}{\partial y^2}\right|_{y=0} = -2AT\omega \sin \omega x$$

We see that the tangential stress along the surface is

$$\sigma_{xx} = \hat{\sigma}_{xx} + T = T(1 - 2A\omega \sin \omega x) = T\left(1 - \frac{4\pi A}{\lambda}\sin \omega x\right)$$

Maximum stress occurs at $\sin \omega x = -1$, corresponding to surface valleys/troughs.

At $\sin \omega x = -1$,

$$\sigma_{xx} = T\left(1 + \frac{4\pi A}{\lambda}\right)$$

Assuming $\frac{A}{\lambda} = 0.1$, $\sigma_{xx} = 2.25T$, which means a stress concentration factor more than 2 can be induced by a slightly wavy surface.

The concept of stress concentration plays an important role in Engineering. For the configurations given below, the marked points (with negative surface curvature) are where stress concentration and material failure is likely to occur.