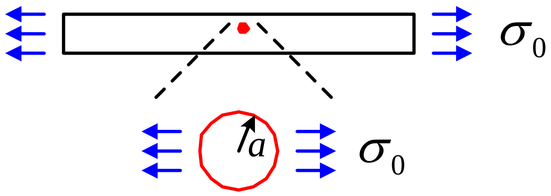
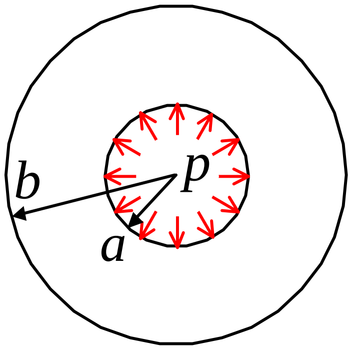
Linear elasticity solution in polar coordinates

Typical problems: Stress around a circular hole in an elastic solid.



Boundary conditions:

Traction free @ $r = a$: $\underline{\sigma}\,\vec{e}_r = \sigma_{rr}\vec{e}_r + \sigma_{r\theta}\vec{e}_\theta = 0$, i.e. $\sigma_{rr} = 0$, $\sigma_{r\theta} = 0$



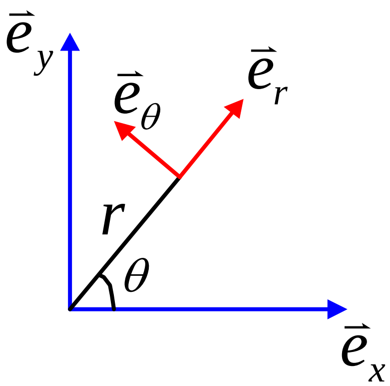
Boundary conditions:

$\sigma_{rr} = -p$, $\sigma_{r\theta} = 0$ @ $r = a$

$\sigma_{rr} = 0$, $\sigma_{r\theta} = 0$ @ $r = b$

Governing equation: $\nabla^2\nabla^2\phi = 0$

In Cartesian coordinates: $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)^2 \phi = 0$



Proposition: use (r,θ) instead of (x,y), $\phi=\phi(r,\theta)$

$$\nabla=\vec{e}_x\frac{\partial}{\partial x}+\vec{e}_y\frac{\partial}{\partial y}=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\theta\frac{\partial}{r\partial\theta}$$

$$\nabla^2=\nabla\cdot\nabla=\left(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)$$

$$\vec{e}_r=\cos\theta\,\vec{e}_x+\sin\theta\,\vec{e}_y$$

$$\vec{e}_\theta=-\sin\theta\,\vec{e}_x+\cos\theta\,\vec{e}_y$$

$$\frac{\partial\vec{e}_r}{\partial\theta}==-\sin\theta\,\vec{e}_x+\cos\theta\,\vec{e}_y=\vec{e}_\theta$$
$$\frac{\partial\vec{e}_\theta}{\partial\theta}==-\cos\theta\,\vec{e}_x-\sin\theta\,\vec{e}_y=-\vec{e}_r$$

It follows from above that $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$

Governing equation in polar coordinates: $\phi=\phi(r,\theta)$

$$\nabla^2\nabla^2\phi=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\phi=0$$

Stress components in Cartesian coordinates:

$$\sigma_{xx}=\frac{\partial^2\phi}{\partial y^2},\quad \sigma_{xy}=-\frac{\partial^2\phi}{\partial x\partial y},\quad \sigma_{yy}=\frac{\partial^2\phi}{\partial x^2}$$

$$\sigma_{xx}+\sigma_{yy}=\nabla^2\phi$$

Stress components in polar coordinates:

$$\sigma_{\theta\theta}=\frac{\partial^2\phi}{\partial r^2},\quad \sigma_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right),\quad \sigma_{rr}=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

Equilibrium equations in polar coordinates:

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+f_r=0$$
$$\frac{\partial\sigma_{\theta\theta}}{\partial r}-\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{2\sigma_{r\theta}}{r}+f_\theta=0$$

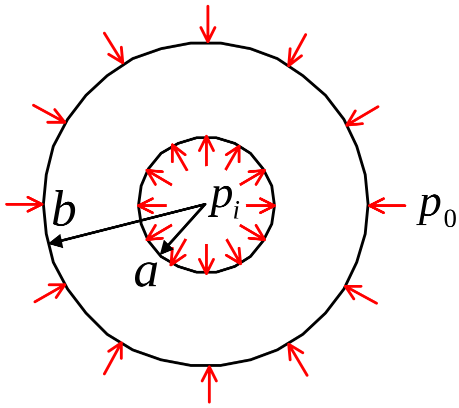
Hooke's law in polar coordinates:

$$\varepsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu\sigma_{\theta\theta}\right)$$
$$\varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right)$$
$$\varepsilon_{r\theta} = \frac{1+\nu}{E}\sigma_{r\theta}$$

Strain-displacement relations in polar coordinates:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r},\quad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r},\quad 2\varepsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

Example 1: Thick-walled pressure vessel



Since the problem is axisymmetric,

$$\phi = \phi(r)$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)$$

Boundary conditions:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} = -p_i,\quad \sigma_{r\theta} = 0 \quad @ \quad r = a$$
$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} = -p_0,\quad \sigma_{r\theta} = 0 \quad @ \quad r = b$$

In mathematical description, the problem becomes an ordinary differential equation

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r}\phi = 0$$

with boundary conditions

$$\phi'(a) = -ap_i$$

$$\phi'(b) = -bp_0$$

The above differential equation can be directly integrated and has the solution

$$\phi = A\ln r + Br^2 \ln r + Cr^2 + D$$

The constant term D is nothing but a rigid body motion and can be neglected in stress analysis, i.e. $D = 0$.

The tangental displacement associated with the term $Br^2 \ln r$ comes out to be $u_\theta = \frac{4B}{E} r\theta$ plus a rigid body motion, which is not a single-valued function. Actually, the term $Br^2 \ln r$ represents a so-called disclination (think of gluing a cut-opened ring back into a circle). For the present problem, take $B = 0$. Therefore, the solution to thickwalled cylinder is

$$\phi = A\ln r + Cr^2$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} = \frac{A}{r^2} + 2C$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = -\frac{A}{r^2} + 2C$$

The constants A, C are determined from the boundary conditions:

$$\sigma_{rr}\big|_{r=a} = \frac{A}{a^2} + 2C = -p_i$$

$$\sigma_{rr}\big|_{r=b} = \frac{A}{b^2} + 2C = -p_0$$

The results are

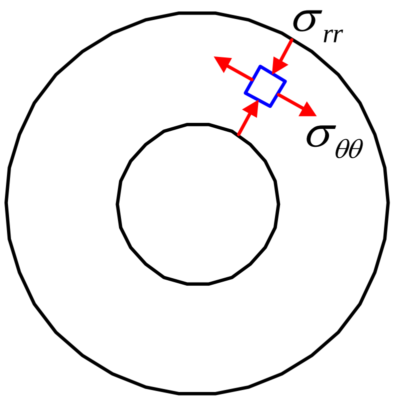
$$A = \frac{a^2 b^2 (p_0 - p_i)}{b^2 - a^2}$$

$$2C = \frac{p_i a^2 - p_0 b^2}{b^2 - a^2}$$

In the special case of $p_0 = 0$:

$$\sigma_{rr} = \frac{a^2 p_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \qquad (<0\text{, compressive})$$

$$\sigma_{\theta\theta} = \frac{a^2 p_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \qquad (>0\text{, tensile})$$



Consider the tensile hoop stress,

$$@\ \ r = b\,,\ \ \sigma_{\theta\theta} = \frac{2a^2 p_i}{b^2 - a^2}$$

$$@\ \ r = a\,,\ \ \sigma_{\theta\theta} = \frac{a^2 + b^2}{b^2 - a^2}\, p_i = SCF \cdot p_i$$

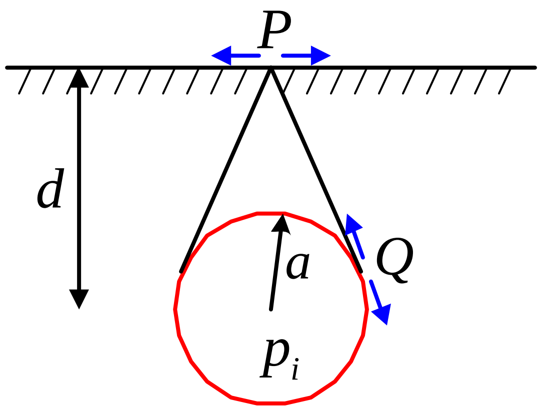
Maximum stress occurs @ $r = a$.

$SCF = \dfrac{a^2 + b^2}{b^2 - a^2}$ is called stress concentration factor.

In the case of a pressurized circular hole in an infinite medium, i.e. $p_0 = 0$ and $b = \infty$:

$$\sigma_{rr} = -p_i \frac{a^2}{r^2}, \quad \sigma_{\theta\theta} = p_i \frac{a^2}{r^2}$$

Example 2: Pressurized underground tunnel



The solution is discussed in Timoshenko's book (Timoshenko and Goodier, 1987). The interesting features are that the maximum stress occurs at two potential sites

@ point P: $\sigma_{xx} = \dfrac{4a^2}{d^2 - a^2} p_i$

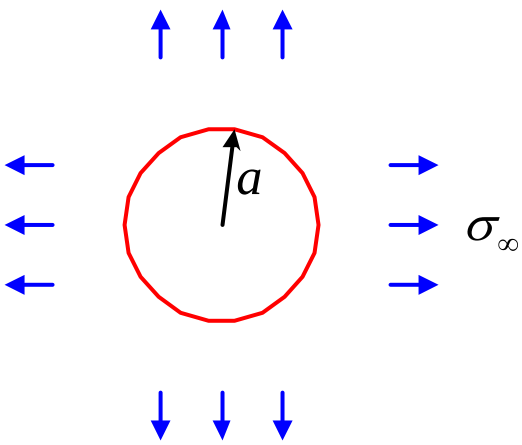
@ point Q: $\sigma_{\theta\theta} = \dfrac{d^2 + a^2}{d^2 - a^2} p_i$

For $d = \sqrt{3}a$, $\sigma_{xx}^P = \sigma_{\theta\theta}^Q$

If $d < \sqrt{3}a$, maximum stress occurs at ground point P.

If $d > \sqrt{3}a$, maximum stress occurs at the hole boundary point Q.

Example 3:



This is a special case of example 1. Take $p_i = 0$, $p_0 = -\sigma_\infty$, $b \to \infty$. We find

$A = -a^2 \sigma_\infty$

$2C = \sigma_\infty$

The stress fields are:

$$\sigma_{rr} = \sigma_\infty \left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_{\theta\theta} = \sigma_\infty \left(1 + \frac{a^2}{r^2}\right)$$

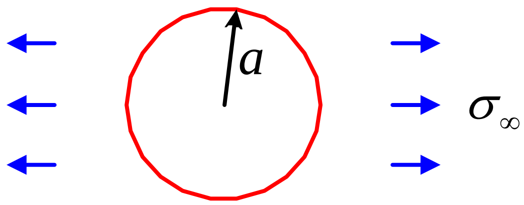
The maximum stress occurs at $r = a$, $\sigma_{\theta\theta}\big|_{r=a} = 2\sigma_\infty$ with a stress concentration factor of 2.

The general solution of $\nabla^2\nabla^2\phi = 0$ in polar coordinates (i.e. for any 2D elasticity problem) can be expressed as:

$$\phi = \left(a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r\right) + \left(d_0 r^2\theta + a_0^{'}\theta\right) + \left(\frac{a_1}{2} r\theta \sin\theta - \frac{c_1}{2} r\theta\cos\theta\right)$$
$$+ \left(b_1 r^3 + a_1^{'} r^{-1} + b_1^{'} r \ln r\right)\cos\theta + \left(d_1 r^3 + c_1^{'} r^{-1} + d_1^{'} r\ln r\right)\sin\theta$$
$$+ \sum_{n=2}^{\infty}\left(a_n r^n + b_n r^{n+2} + a_n^{'} r^{-n} + b_n^{'} r^{-n+2}\right)\cos n\theta + \sum_{n=2}^{\infty}\left(c_n r^n + d_n r^{n+2} + c_n^{'} r^{-n} + d_n^{'} r^{-n+2}\right)\sin n\theta$$

The general solution can be conveniently used to solve boundary value problems.

Example 4: Circular hole under uniaxial tension (remote)



Governing equation: $\nabla^2\nabla^2\phi = 0$

Boundary conditions: $\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} = -p_i$, $\sigma_{r\theta} = 0$ @ $r = a$

$$\sigma_{xx} = \sigma_\infty, \ \sigma_{yy} = 0, \ \sigma_{xy} = 0 \ @ \ r = \infty$$

First, let us transform the remote stresses into polar coordinates

$$\underline{\sigma} = \sigma_\infty \vec{e}_x \otimes \vec{e}_x$$

$$\sigma_{rr} = \vec{e}_r \cdot \underline{\sigma}\vec{e}_r = \sigma_\infty \cos^2\theta = \frac{\sigma_\infty}{2}(1+\cos 2\theta)$$

$$\sigma_{\theta\theta} = \vec{e}_\theta \cdot \underline{\sigma}\vec{e}_\theta = \sigma_\infty \sin^2\theta = \frac{\sigma_\infty}{2}(1-\cos 2\theta)$$

$$\sigma_{r\theta} = \vec{e}_r \cdot \underline{\sigma}\vec{e}_\theta = -\frac{\sigma_\infty}{2}\sin 2\theta$$

The above expressions suggest that the Airy stress function should have the form of $\phi = C_1 \ln r + C_2 r^2 + f(r)\cos 2\theta$. Pick the corresponding expression in the general solution associated with $\cos 2\theta$, we see $f(r) = a_2 r^2 + a_2' r^{-2} + b_2'$ (the r^4 term is discarded since it generates infinite stress at large r). Applying the boundary conditions allow all the parameters to be determined. The solution is

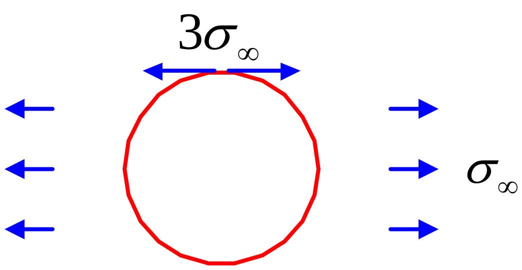
$$\phi = -\frac{\sigma_\infty a^2}{2}\ln r + \frac{\sigma_\infty}{4}r^2 + \left(-\frac{r^2}{4} - \frac{a^4}{4r^2} + \frac{a^2}{2}\right)\sigma_\infty \cos 2\theta$$

The associated stress fields are

$$\sigma_{rr} = \frac{\sigma_\infty}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{\sigma_\infty}{2}\left(1-\frac{a^2}{r^2}\right)\left(1-\frac{3a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{\sigma_\infty}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{\sigma_\infty}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\sigma_{r\theta} = -\frac{\sigma_\infty}{2}\left(1-\frac{a^2}{r^2}\right)\left(1+\frac{3a^2}{r^2}\right)\sin 2\theta$$



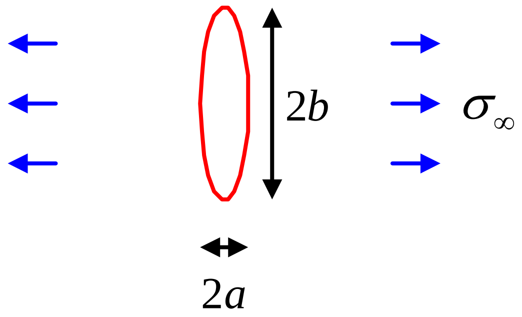
The maximum tensile stress occurs at $r = a$, $\theta = \dfrac{\pi}{2}$

$$\sigma_{\theta\theta}^{\max} = 3\sigma_\infty$$

Therefore, the stress concentration factor is 3.

The above problem can also be directly treated without knowing the general solution (next lecture).

For an elliptic hole under uniaxial tension (remote),



$$\sigma_{\theta\theta}^{\max} = \sigma_\infty\left(1+\frac{b}{a}\right)$$

The stress concentration factor is $\left(1+2\frac{b}{a}\right)$, which depends on the aspect ratio of the elliptic hole.