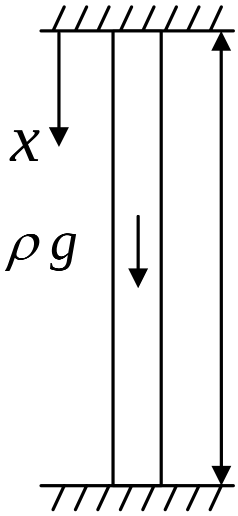
**Principle of minimum potential energy (continued)**

The potential energy of a system is

$$V = \int_V w \mathrm{d}V - \int_V f_i u_i \mathrm{d}V - \int_S t_i u_i \mathrm{d}S$$

Principle of minimum potential energy states that for all kinematically admissible $u_i$, the actual displacement field minimizes $V$.

Example 1:




We have shown in the beginning of the semester that the exact solution is: $u = \frac{\rho g}{2E} x(L-x)$. Now we discuss how to solve the same problem by using the principle of minimum potential energy.

Procedure:

1) Pick any displacement such that $u(0) = u(L) = 0$.
2) Minimize $V$ for the chosen parameters.

An obvious choice is $u(x) = x(L-x)f(x)$ since this satisfies the clamped displacement boundary conditions for any $f(x)$. We can assume $f(x)$ to be a polynomial function.

$$f(x) = C_0 + C_1 x + C_2 x^2 + \cdots + C_N x^N$$

To minimize the potential energy $V(C_0, C_1, C_2, \cdots, C_N)$, we take

$$\frac{\partial V}{\partial C_0} = 0, \quad \frac{\partial V}{\partial C_1} = 0, \ \ldots, \quad \frac{\partial V}{\partial C_N} = 0$$

This gives $N+1$ algebraic equations to determine $N+1$ parameters $C_0, C_1, C_2, \cdots, C_N$ .

For the present example, take the simplest form that $f(x) = C_0$ .

$$u(x) = C_0 x(L-x)$$

$$\varepsilon = \frac{\partial u}{\partial x} = C_0(L-2x)$$

$$\sigma = E\varepsilon = EC_0(L-2x)$$

$$w = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}EC_0^2(L-2x)^2$$

The potential energy of the system is

$$\begin{aligned} V &= \int_0^L w\mathrm{d}x \cdot A - \int_0^L \rho g C_0 x(L-x)\mathrm{d}x \cdot A \\ &= \int_0^L \frac{1}{2}EC_0^2(L-2x)^2\mathrm{d}x \cdot A - \int_0^L \rho g C_0 x(L-x)\mathrm{d}x \cdot A \\ &= \frac{1}{2}A\frac{L^3}{6}\left(EC_0^2 - \rho g C_0\right) \end{aligned}$$

To minimize $V$ ,

$$\frac{\partial V}{\partial C_0} = 0 \Rightarrow 2EC_0 - \rho g = 0 \Rightarrow C_0 = \frac{\rho g}{2E}$$

Therefore $u = \frac{\rho g}{2E}x(L-x)$, which is identical to the exact solution for this problem.

Suppose $f(x) = C_0 + C_1 x$ is assumed, we will get a function of potential energy in terms of $C_0$ and $C_1$ ,

$$V(C_0, C_1)$$

Minimizing the potential energy requires that

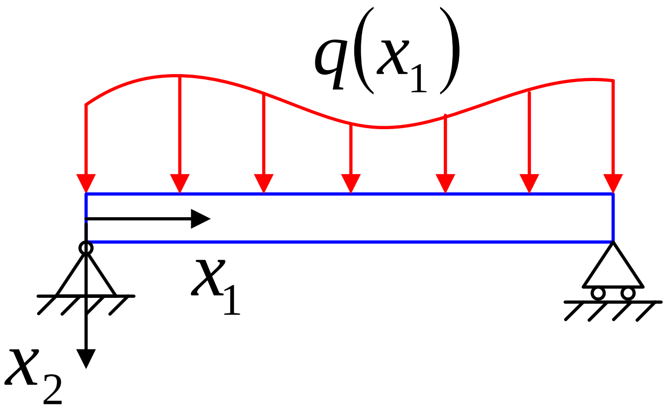
$$\frac{\partial V}{\partial C_0} = 0\,, \quad \frac{\partial V}{\partial C_1} = 0$$

The parameters turned out to be

$$C_0 = \frac{\rho g}{2E}\,, \quad C_1 = 0$$

Again we get the exact solution.

Re-derivation of the theory of beam bending




Kinematic assumptions (Kirchhoff):

$$u_2(x_1, x_2) = w(x_1)$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0$$

$$\frac{\partial \varepsilon_{12}}{\partial x_1} = 0 \Rightarrow \frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1}\right) + w''(x_1) = 0$$

$$\frac{\partial \varepsilon_{11}}{\partial x_2} = \frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1}\right) = -w''(x_1) = -\kappa\,, \quad \varepsilon_{11} = 0 \quad @ \quad x_2 = 0 \quad \text{(neutral plane)}$$

The axial strain is integrated to be

$$\varepsilon_{11} = -\kappa x_2 = -\kappa y$$

The axial stress is $\sigma_{11} = E\varepsilon_{11}$. All other stress components are zero. Therefore the strain energy density can be easily calculated

$$w = \frac{1}{2}\sigma_{11}\varepsilon_{11} = \frac{1}{2}E\varepsilon_{11}^2 = \frac{1}{2}Ey^2 w''^2$$

Then the potential energy of the system is

$$\begin{aligned} V &= \int_0^L \mathrm{d}x \int_A \frac{1}{2}Ey^2 w''^2 \mathrm{d}A - \int_0^L q(x)w(x)\mathrm{d}x \\ &= \int_0^L \frac{1}{2}EIw''^2 \mathrm{d}x - \int_0^L qw\mathrm{d}x \end{aligned}$$

where $\int_A y^2 \mathrm{d}A = I$ is the moment of inertia.

Recall that the governing equation of beam bending is $EIw'''' - q = 0$ plus boundary conditions.

Let's see if we get the same equation by using the principle of minimum potential energy.

Since $V$ is minimum,

$$\delta V = \int_0^L EIw'' \delta w'' \mathrm{d}x - \int_0^L q\,\delta w\,\mathrm{d}x = 0$$

The first term can be integrated by parts

$$\int_0^L EIw'' \delta w'' \mathrm{d}x = EIw'' \delta w' \Big|_0^L - \int_0^L EIw''' \delta w' \mathrm{d}x$$

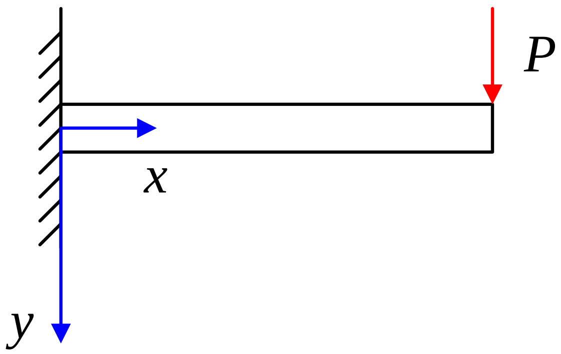
$$= EIw'' \delta w' \Big|_0^L - EIw''' \delta w \Big|_0^L + \int_0^L EIw'''' \delta w \mathrm{d}x$$

The second terms vanish due to the displacement boundary conditions at the ends, therefore

$$\delta V = EIw'' \delta w' \Big|_0^L + \int_0^L \left(EIw'''' - q\right)\delta w \mathrm{d}x = 0 \Rightarrow EIw'''' - q = 0 \text{ and } w'' = 0 \ @ \ x = 0, L$$

which is exactly the governing equation and moment-free boundary conditions for the simply supported beam.

Example 2:




The potential energy of the system is

$$V = \int_0^L \frac{1}{2} EIw''^2 \mathrm{d}x - Pw(L)$$

Pick $w(x)$ that satisfies $w(0) = 0$, $w'(0) = 0$, a simple guess is

$$w(x) = Cx^2$$

$$w'' = 2C$$

$$V = \frac{1}{2} EI \int_0^L (2C)^2 \mathrm{d}x - PCL^2 = 2C^2 EIL - PCL^2$$

Minimizing potential energy requires

$$\frac{\partial V}{\partial C} = 0 \Rightarrow 4CEIL - PL^2 = 0 \Rightarrow C = \frac{PL}{4EI}$$

Let us compare the resulted solution to exact one,

$$w(L) = \frac{PL^3}{4EI} \quad \text{(Exact solution: } w(L) = \frac{PL^3}{3EI}\text{)}$$

The relative error is: $\frac{1/3-1/4}{1/3}=25\%$

Te above solution can be improved by taking more terms.

Suppose $w(x)=C_1x^2+C_2x^3$. The system potential energy becomes

$$V=\frac{1}{2}EI\int_0^L(2C_1+6C_2x)^2\mathrm{d}x-P(C_1L^2+C_2L^3)$$

The same conditions that minimize potential energy

$$\frac{\partial V}{\partial C_1}=0,\ \frac{\partial V}{\partial C_2}=0$$

The parameters are determined as

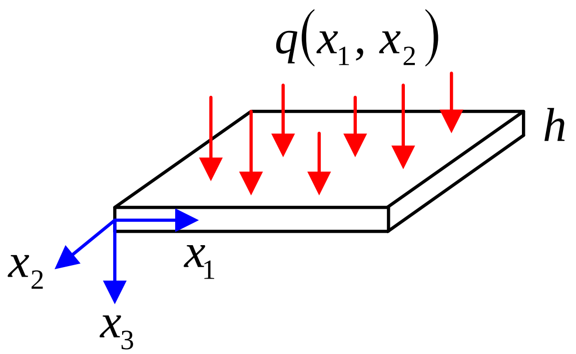
$$C_1=-\frac{P}{2EI}L,\ C_2=\frac{P}{6EI}$$

Therefore:

$$w(L)=C_1L^2+C_2L^3=\frac{PL^3}{3EI}$$

which is the exact solution.

Extend the analysis to plate bending




Kinematic assumption (Kirchhoff):

$u_3(x_1,x_2,x_3)=w(x_1,x_2)$

$\varepsilon_{13}=\varepsilon_{23}=0$

$$\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}=0 \qquad (1)$$

$$\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}=0 \qquad (2)$$

$$\frac{\partial(1)}{\partial x_1} \Rightarrow \frac{\partial \varepsilon_{11}}{\partial x_3} = -\frac{\partial^2 w}{\partial x_1^2}$$

$$\frac{\partial(2)}{\partial x_2} \Rightarrow \frac{\partial \varepsilon_{22}}{\partial x_3} = -\frac{\partial^2 w}{\partial x_2^2}$$

$$\frac{1}{2}\left(\frac{\partial(1)}{\partial x_2} + \frac{\partial(2)}{\partial x_1}\right) \Rightarrow \frac{\partial \varepsilon_{12}}{\partial x_3} = -\frac{\partial^2 w}{\partial x_1 \partial x_2}$$

Assume @ $x_3 = 0$ , $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{12} = 0$ (neutral plane), the strain components can be integrated out, respectively

$$\varepsilon_{11} = -\frac{\partial^2 w}{\partial x_1^2} x_3, \quad \varepsilon_{22} = -\frac{\partial^2 w}{\partial x_2^2} x_3, \quad \varepsilon_{12} = -\frac{\partial^2 w}{\partial x_1 \partial x_2} x_3$$

The strain energy density is

$$\begin{aligned} w &= \frac{1}{2}\sigma_{11}\varepsilon_{11} + \frac{1}{2}\sigma_{22}\varepsilon_{22} + \frac{1}{2}(\sigma_{12}\varepsilon_{12} + \sigma_{21}\varepsilon_{21}) \\ &= \frac{E}{2(1-\nu^2)}\left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\nu\varepsilon_{11}\varepsilon_{22} + 2(1-\nu)\varepsilon_{12}^2\right) \\ &= \frac{E}{2(1-\nu^2)}\left(w_{,11}^2 + w_{,22}^2 + 2\nu w_{,11} w_{,22} + 2(1-\nu) w_{,12}^2\right) x_3^2 \end{aligned}$$

The system potential energy is

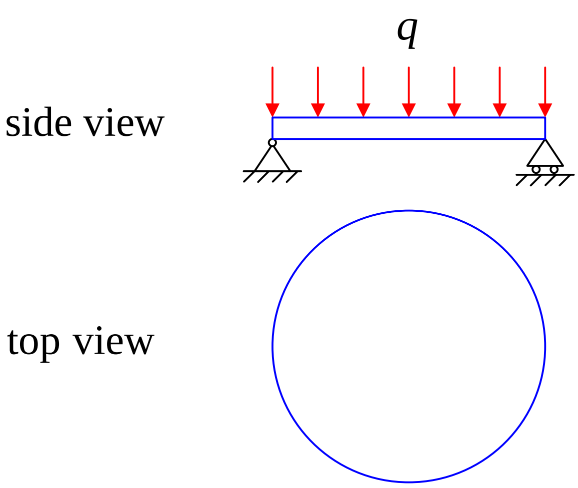
$$\begin{aligned} V &= \int_V w \mathrm{d}V - \int_A q w \mathrm{d}A \\ &= \frac{Eh^3}{24(1-\nu^2)} \int_A \left(w_{,11}^2 + w_{,22}^2 + 2\nu w_{,11} w_{,22} + 2(1-\nu) w_{,12}^2\right) \mathrm{d}A - \int_A q w \mathrm{d}A \end{aligned}$$

Define $D = \dfrac{Eh^3}{12(1-\nu^2)}$ and minimize $V$ . We have

$$\begin{aligned} \delta V &= \int_V w \mathrm{d}V - \int_A q w \mathrm{d}A \\ &= D\int_A \left(w_{,11}\delta w_{,11} + w_{,22}\delta w_{,22} + \nu w_{,11}\delta w_{,22} + \nu w_{,22}\delta w_{,11} + 2(1-\nu) w_{,12}\delta w_{,12}\right)\mathrm{d}A - \int_A q\delta w \mathrm{d}A \\ &= \int_A \left(D\nabla^2\nabla^2 w - q\right)\delta w \mathrm{d}A + \text{boundary terms} = 0 \\ &\qquad \Rightarrow D\nabla^2\nabla^2 w - q = 0 \end{aligned}$$

This is the classical governing equation for plate bending. In the following, we illustrate the application of energy theorem in solving plate bending problems.

Example 3: Simply supported circular plate under uniform pressure




Displacement boundary condition $w|_{r=a} = 0$.

To satisfy the boundary condition, pick the kinematically admissible displacement $w = C(r^2 - a^2) = C(x^2 + y^2 - a^2)$. With this assumption,

$$w_{,11} = w_{,22} = 2C, \quad w_{,12} = 0$$

The system potential energy can be easily integrated as

$$V = \frac{D}{2}\int_A \left((2C)^2 + (2C)^2 + 2\nu(2C)^2\right)dA - \int_A qC(r^2 - a^2)dA$$
$$= 4D(1+\nu)C^2 \cdot \pi a^2 + \frac{\pi}{2}a^4 qC$$

To minimize potential energy,

$$\frac{\partial V}{\partial C} = 0 \Rightarrow 8D(1+\nu)C \cdot \pi a^2 + \frac{\pi}{2}a^4 q \Rightarrow C = -\frac{qa^2}{16D(1+\nu)}$$

The deflection of the plate is

$$w = -\frac{qa^2}{16D(1+\nu)}(r^2 - a^2)$$

The maximum deflection occurs at the center,

$$w_{\max} = w|_{r=0} = \frac{qa^4}{16D(1+\nu)}$$

The exact solution to this problem shows: $w|_{r=0} = \dfrac{qa^4}{64D}\dfrac{5+\nu}{1+\nu}$. The relative error is

$$\frac{(5+\nu)/64 - 1/16}{(5+\nu)/64} = \frac{1+\nu}{5+\nu} \approx 25\%.$$

We see that a simple guess leads to a solution fairly close to exact.