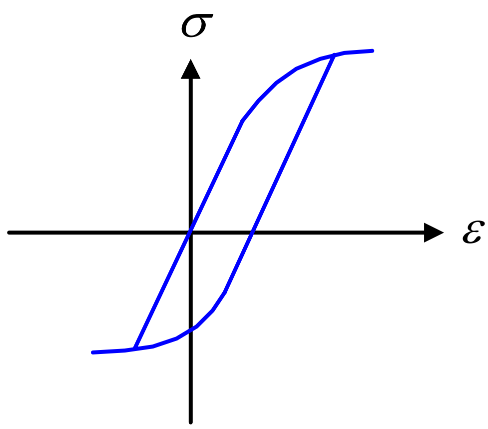
9. Boundary value problems in plasticity

Slip line theory

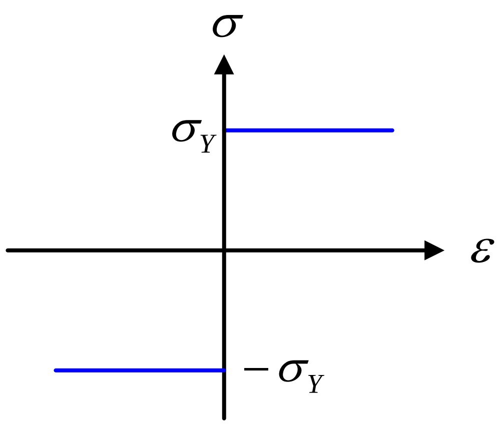
An important theory in the plane problems of plasticity is the slip line theory. This theory simplifies the governing equations for plastic deformation by making several assumptions:

1) rigid-plastic material response (see explanations below).
2) plane strain deformation;
3) quasi-static loading;
4) no temperature change and no body force;
5) isotropic material
6) no Baushinger effect
7) No work hardening

Typical experimental stress-strain relation on plastic material behavior has the form:



As an idealized model, rigid-plastic model simplifies the above elastic-plastic stress-strain relation to a much simpler form described in the figure below:

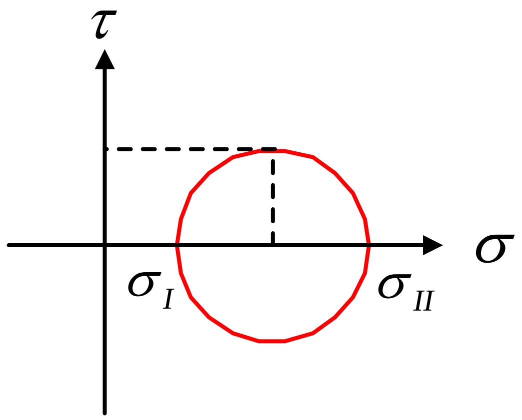


This indicates that the von Mises stress shall be kept constant at the yield stress

$$\sigma_e = \sigma_Y \text{, where } \sigma_e = \sqrt{\frac{3}{2}\sigma_{ij}^{'}\sigma_{ij}^{'}}$$

once plastic deformation occurs.

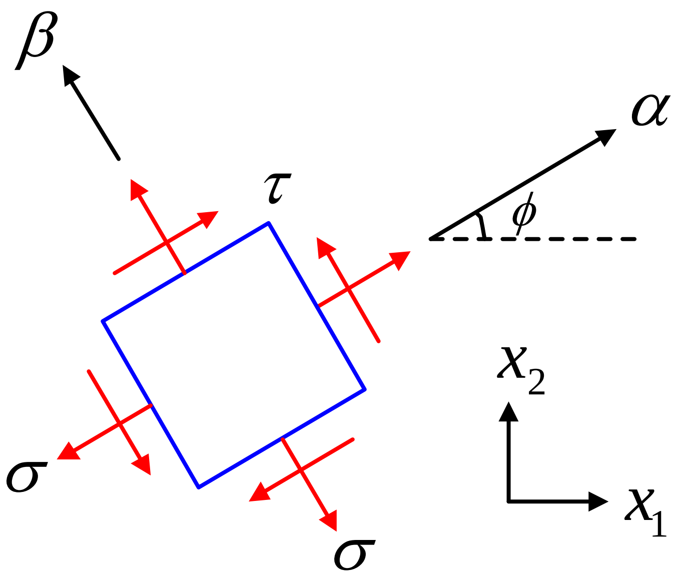
Slip lines are defined as trajectories of the directions of maximum shear stress. For a 2D stress state, the orientation of the maximum shear can be found by the techniques of stress transformation or Mohr's circle.



The above Mohr circle suggests that the 2D stress state in the orientations of the maximum shear stress can be expressed as:

$$\sigma_{\alpha\alpha} = \sigma_{\beta\beta} = \sigma \text{, } \sigma_{\alpha\beta} = \tau$$

$$\sigma_{zz} = \sigma$$



α, β : slip directions corresponding to the directions of maximum shear stress.

Since the hydrostatic part of stress causes no plastic deformation, consider the deviatoric stress

$$\underline{\sigma}' = \tau\left(\vec{e}_\alpha \otimes \vec{e}_\beta + \vec{e}_\beta \otimes \vec{e}_\alpha\right)$$

Within the plastic zone,

$$\sigma_e = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} = \sqrt{\frac{3}{2}\times 2\tau^2} = \sqrt{3}\,\tau = \sigma_Y$$

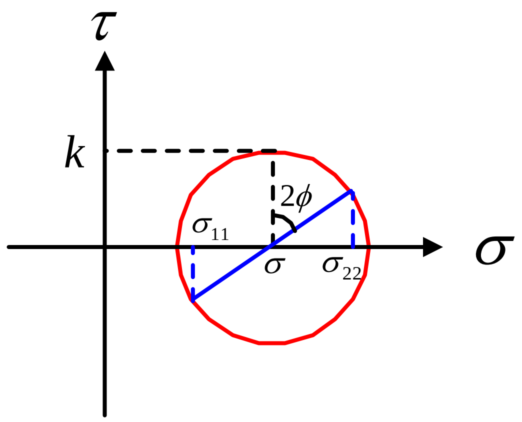
or

$$\tau = \frac{\sigma_Y}{\sqrt{3}} = \text{const.} = k$$

Equilibrium equations (in-plane):

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$



We now transform the stresses from the orientation of maximum shear stress to the x-y coordinates:

$$\sigma_{11} = \vec{e}_1 \cdot \underline{\sigma}\,\vec{e}_1 = \sigma - k\sin 2\phi$$

$$\sigma_{22} = \vec{e}_2 \cdot \underline{\sigma}\,\vec{e}_2 = \sigma + k\sin 2\phi$$

$$\sigma_{12} = \vec{e}_1 \cdot \underline{\sigma}\,\vec{e}_2 = k\cos 2\phi$$

Plug these stress components into the equilibrium equations, we get

$$\frac{\partial \sigma}{\partial x_1} - 2k\left(\cos 2\phi\frac{\partial \phi}{\partial x_1} + \sin 2\phi\frac{\partial \phi}{\partial x_2}\right) = 0$$

$$\frac{\partial \sigma}{\partial x_2} - 2k\left(\sin 2\phi\frac{\partial \phi}{\partial x_1} - \cos 2\phi\frac{\partial \phi}{\partial x_2}\right) = 0$$

In the α, β directions, the above equations have the form

$$\frac{\partial}{\partial s_\alpha}(\sigma - 2k\phi) = 0$$

$$\frac{\partial}{\partial s_\beta}(\sigma + 2k\phi) = 0$$

One can verify that these equations are equivalent to the original equilibrium equations using

$$\frac{\partial}{\partial s_\alpha} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial s_\alpha} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial s_\alpha} = \cos\phi\frac{\partial}{\partial x_1} + \sin\phi\frac{\partial}{\partial x_2}$$

$$\frac{\partial}{\partial s_\beta} = -\sin\phi\frac{\partial}{\partial x_1} + \cos\phi\frac{\partial}{\partial x_2}$$

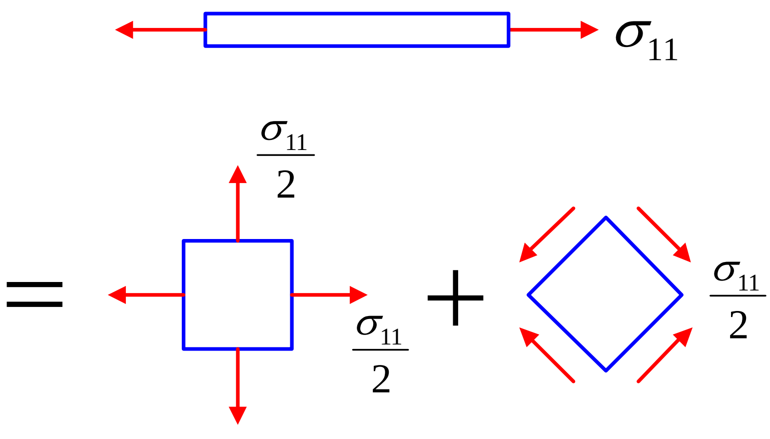
Hencky's equations:

$\sigma - 2k\phi = \text{const.}$ along α -slip line

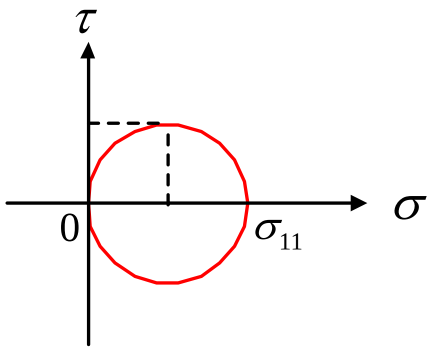
$\sigma + 2k\phi = \text{const.}$ along β -slip line

Example 1:

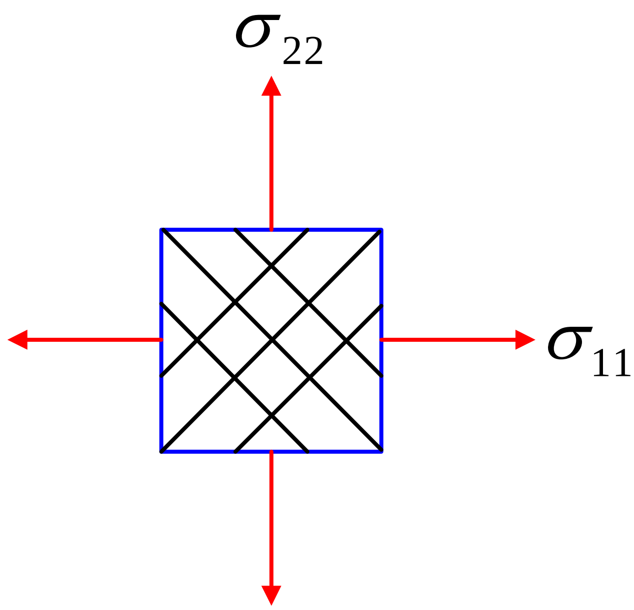
Consider uniaxial tension,



We can easily see that the slip lines are along 45° and -45° directions, which is straightforward by using Mohr's circle.

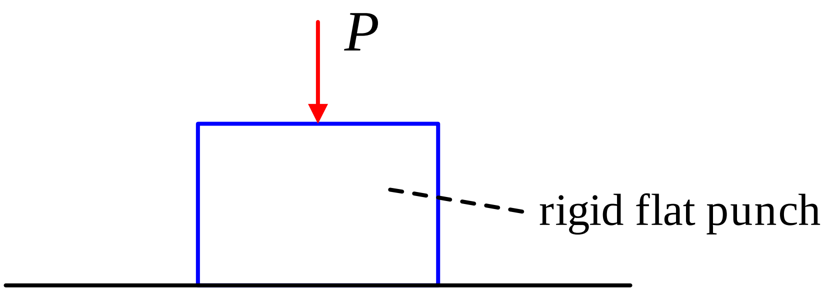


Consider tension in two 1 and 2 directions,

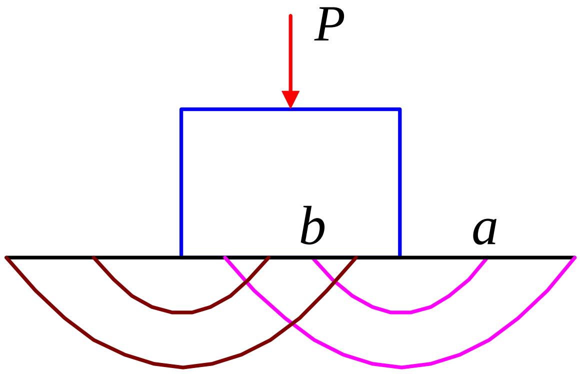


Also, slip lines are along 45° and -45° directions because the present configuration is just the superposition of two uni-axial tensions.

Example 2: Hill's problem of a rigid punch indenting a plastic solid



What's the value of P that allows the punch to indent into the plastic solid?



The slip lines consist of a curvilinear mesh of two families of lines, which always cross each other at right angles. Follow one slip line ending with two point a, b at boundaries,

At point a, $\phi_a = 45^\circ$

$$\sigma_{22} = \sigma_{12} = 0 \quad \text{(free boundary)}$$

At point b, $\phi_b = -45^\circ$

$$\sigma_{22} \neq 0, \ \sigma_{11} \neq 0, \ \sigma_{12} = 0 \quad \text{(frictionless contact)}$$

@ Point a:

$$\sigma_{12} = k\cos 2\phi_a = 0 \quad \rightarrow \quad \phi_a = 45^\circ$$

$$\sigma_{22} = \sigma_a + k\sin 2\phi_a = \sigma_a + k = 0 \Rightarrow \sigma_a = -k$$

$$\sigma_{11} = \sigma_a - k\sin 2\phi_a = -2k$$

@ Point b:

$$\sigma_{12} = k\cos 2\phi_b = 0 \quad \rightarrow \quad \phi_b = 45^\circ$$

Using Hencky's equation: $\sigma_b - 2k\phi_b = \sigma_a - 2k\phi_a$, we obtain

$$\sigma_b = \sigma_a - 2k\phi_a + 2k\phi_b = -k - 2k\cdot\frac{\pi}{4} + 2k\cdot\left(-\frac{\pi}{4}\right) = -(1+\pi)k$$

Therefore, $\sigma_{22} = \sigma_b + k\sin 2\phi_b = -(1+\pi)k + k\sin\left(-\frac{\pi}{2}\right) = -(2+\pi)k$

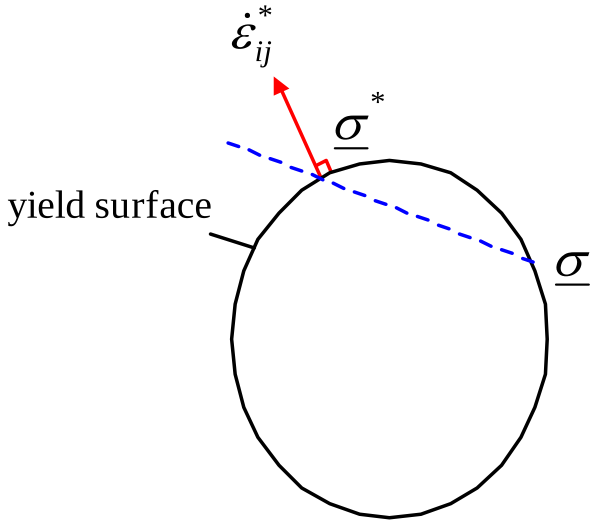
The pressure under the punch, σ_{22}, turns out to be uniform, so integral of σ_{22} over the area should balance the applied force P,

$$P=\sigma_{22}^{b}\cdot w=-(2+\pi)kw=-(2+\pi)\frac{\sigma_Y}{\sqrt{3}}w\approx-3\sigma_Y w$$

w is the width of the punch.

(Note: The pressure to indent a material is often called the hardness of a material. The above relation shows that the hardness is roughly 3 times the yield stress. This relation between hardness and yield stress, sometimes called the Taber relation, is widely used in materials science.)

Energy theorems and bounds for limit load in plasticity



Since the yield surface is always convex, the principle of maximum plastic resistance indicates an inequality:

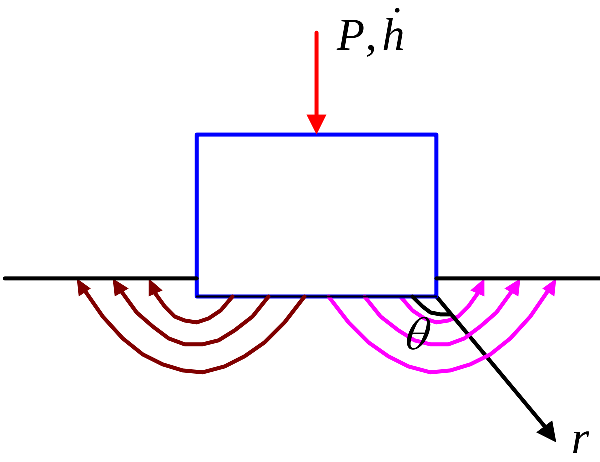
$$\left(\sigma_{ij}^{*}-\sigma_{ij}\right)\dot{\varepsilon}_{ij}^{*}\geq 0$$

$$\sigma_{ij}^{*}\,\dot{\varepsilon}_{ij}^{*}\geq\sigma_{ij}\,\dot{\varepsilon}_{ij}^{*}$$

$$\int_V\sigma_{ij}^{*}\,\dot{\varepsilon}_{ij}^{*}\,\mathrm{d}V\geq\int_V\sigma_{ij}\,\dot{\varepsilon}_{ij}^{*}\,\mathrm{d}V \qquad (1)$$

This relation can be used to determine an upper and lower bound for the plastic limit load. First, suppose we take kinematically admissible $\dot{\varepsilon}_{ij}^{*}$ from some velocity field v_i^{*}, and σ_{ij}^{*} is the stress that drives $\dot{\varepsilon}_{ij}^{*}$. In this case, (1) suggests a way to estimate the upper bound of the limit load. This is best illustrate by an example as follows.

Consider Hill's problem by energy theorem. We attempt to find an upper bound for the force required to push a rigid flat punch into a rigid plastic solid.



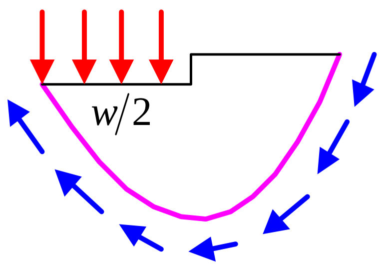
Assume a set of kinematically admissible velocity field in local polar coordinates as shown above:

$$v_r^* = 0\,,\;\; v_\theta^* = \dot{h}\,,$$

(i.e., This field satisfies all displacement boundary conditions in the problem.)
The associated strain components are

$$\dot{\varepsilon}_{rr}^* = \dot{\varepsilon}_{\theta\theta}^* = 0\,,\;\; \dot{\varepsilon}_{r\theta}^* = -\frac{1}{2}\frac{v_\theta^*}{r} = -\frac{\dot{h}}{2r} = \dot{\varepsilon}_{\theta r}^*$$

$$\int_V \sigma_{ij}^* \,\dot{\varepsilon}_{ij}^* \,\mathrm{d}V = 2\int_0^{w/2}\int_0^{\pi} \frac{\sigma_Y}{\sqrt{3}}\frac{\dot{h}}{2r}\, r\,\mathrm{d}r\,\mathrm{d}\theta = \pi\,\frac{w}{2}\frac{\sigma_Y}{\sqrt{3}}\,\dot{h}$$



$$\int_V \sigma_{ij}\,\dot{\varepsilon}_{ij}^*\,\mathrm{d}V = \int_V \sigma_{ij}\,\dot{u}_{i,j}^*\,\mathrm{d}V = \int_V \sigma_{ij}\,v_{i,j}^*\,\mathrm{d}V = \int_V \left(\sigma_{ij}\,v_i^*\right)_{,j}\mathrm{d}V = \int_S t_i\,v_i^*\,\mathrm{d}A$$

$$= \frac{P}{w}\dot{h}\frac{w}{2} - \frac{\sigma_Y}{\sqrt{3}}\dot{h}\cdot\pi\frac{w}{2}$$

$$\int_V \sigma_{ij}^* \, \dot{\varepsilon}_{ij}^* \, \mathrm{d}V \ge \int_V \sigma_{ij} \, \dot{\varepsilon}_{ij}^* \, \mathrm{d}V \;\Rightarrow\; \frac{\pi}{2} w\dot{h} \frac{\sigma_Y}{\sqrt{3}} \ge \frac{P}{w} \dot{h} \frac{w}{2} - \frac{\sigma_Y}{\sqrt{3}} \dot{h} \cdot \pi \frac{w}{2}$$

$$\Rightarrow P \le 2\pi \frac{\sigma_Y}{\sqrt{3}} w$$

Compare to the exact solution shown previously,

$$P_{exact} = (2+\pi) \frac{\sigma_Y}{\sqrt{3}} w$$

The relative error is

$$\frac{2\pi - (2+\pi)}{2+\pi} \approx 20\%$$

Added note:

To estimate lower bound, we switch the starred and the non-stared field in eq. (1) as

$$\int_V \sigma_{ij} \, \dot{\varepsilon}_{ij} \, \mathrm{d}V \ge \int_V \sigma_{ij}^* \, \dot{\varepsilon}_{ij} \, \mathrm{d}V$$

where σ_{ij}^* corresponds to a statically admissible field (satisfying equilibrium and not violating the yield condition). This equation then provides an estimate for the lower bound.

Consider the same example as above. A statically admissible stress field not violating the yield is uniform stress equal to yield underneath the indenter. Therefore,

$$P \ge \sigma_Y w$$