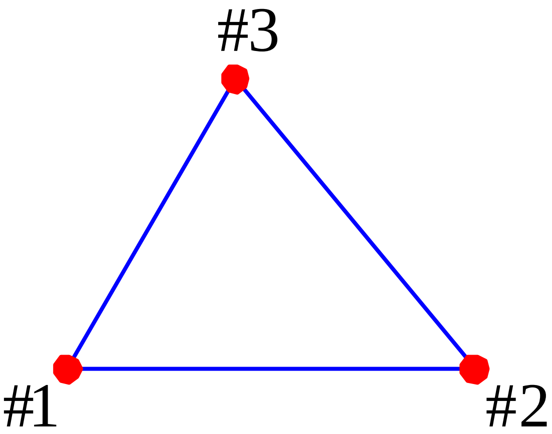
Finite element method (continued)

Summary of essential concepts:

FEM analysis begins with calculations on the element level.

Element



Element nodal displacement:

$$\underline{u}_{el} = \begin{Bmatrix} u_1^{\#1} \\ u_2^{\#1} \\ u_1^{\#2} \\ u_2^{\#2} \\ u_1^{\#3} \\ u_2^{\#3} \end{Bmatrix}$$

Element interpolation function:

$$N^1(x_1, x_2),\ N^2(x_1, x_2),\ N^3(x_1, x_2)$$

Element displacement field:

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} N^1 & 0 & N^2 & 0 & N^3 & 0 \\ 0 & N^1 & 0 & N^2 & 0 & N^3 \end{bmatrix} \underline{u}_{el}$$

Element strain field:

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial N^1}{\partial x_1} & 0 & \frac{\partial N^2}{\partial x_1} & 0 & \frac{\partial N^3}{\partial x_1} & 0 \\ 0 & \frac{\partial N^1}{\partial x_2} & 0 & \frac{\partial N^2}{\partial x_2} & 0 & \frac{\partial N^3}{\partial x_2} \\ \frac{\partial N^1}{\partial x_2} & \frac{\partial N^1}{\partial x_1} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^3}{\partial x_2} & \frac{\partial N^3}{\partial x_1} \end{bmatrix} \underline{u}_{el} = \underline{B}\underline{u}_{el}$$

Element stress field:

$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \underline{D}\underline{\varepsilon} = \underline{D}\underline{B}\underline{u}_{el}$$

Element strain energy:

$$U_{el} = \frac{1}{2}\int_{V_{el}} \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V = \frac{1}{2}\underline{u}_{el}^T \underline{K}_{el}\,\underline{u}_{el}$$

$$\underline{K}_{el} = A_{el}\,\underline{B}^T\,\underline{D}\underline{B}$$

Element nodal force:

$$\int_{V_{el}} f_i u_i \mathrm{d}V + \int_{A_{el}} t_i u_i \mathrm{d}S = \underline{F}_{el}\,\underline{u}_{el}$$

After the element quantities are calculated, the next step is to assemble the global stiffness matrix.

Global strain energy:

$$U = \sum_{elements} U_{el} = \sum_{elements} \frac{1}{2}\underline{u}_{el}^T \underline{K}_{el}\,\underline{u}_{el} = \frac{1}{2}\underline{u}^T\,\underline{K}\,\underline{u}$$

$$\underline{u}_{el} = \begin{Bmatrix} u_1^{\#1} \\ u_2^{\#1} \\ u_1^{\#2} \\ u_2^{\#2} \\ u_1^{\#3} \\ u_2^{\#3} \end{Bmatrix}_{6\times 1} \rightarrow \quad \underline{u} = \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ \vdots \end{Bmatrix}_{2n\times 1} \qquad \left(\underline{K}_{el}\right)_{6\times 6} \rightarrow \underline{K}_{2n\times 2n}$$

Element connectivity

$$(\#1, \#2, \#3) \Rightarrow (a, b, c)$$

$$\alpha(1, 2, 3, 4, 5, 6) \Rightarrow z_\alpha(2a-1, 2a, 2b-1, 2b, 2c-1, 2c)$$

$$K_{z_\alpha z_\beta} = K_{z_\alpha z_\beta} + K^{el}_{\alpha\beta}$$ (assembly of global stiffness matrix over all elements)

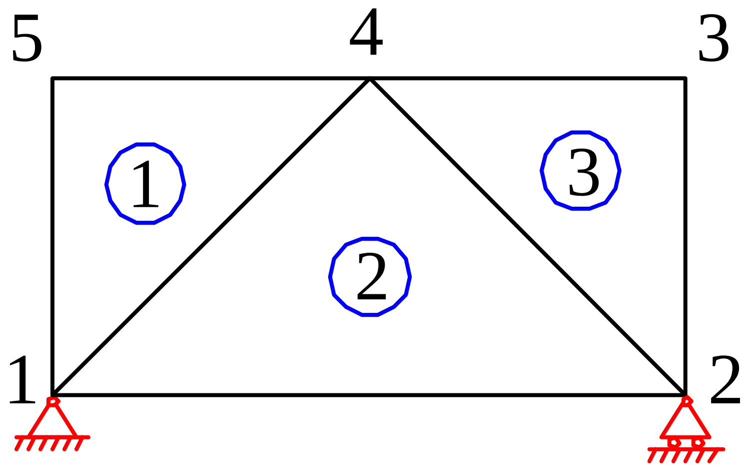
Nodal force

$$F_{z_\alpha} = F_{z_\alpha} + F^{el}_{\alpha}$$

FEM equation:

$$\underline{K}_{2n\times 2n}\underline{u}_{2n\times 1} = \underline{F}_{2n\times 1}$$

For n nodes, there will be $2n$ unknown displacement components (2 degrees of freedom each node) to be solved. For displacement boundary value problems, there are further restrictions on the degrees of freedom.



At node 1, the displacements at both directions are fixed; at node 2, the displacement in 2-direction is fixed. The global displacement vector becomes

$$\underline{u}_{2n\times 1} = \begin{bmatrix} 0 \\ 0 \\ \times \\ 0 \\ \times \\ \vdots \end{bmatrix}$$

Example: suppose $u_2^{(1)} = \Delta$

For this prescribed displacement, we can simply replace the equation for the appropriate degree of freedom with the displacement constraint, but the resulting stiffness matrix is no longer symmetric. This can be remedied by placing the prescribed displacement to the right side of equation. The resulting FEM equation becomes

$$\begin{bmatrix} K_{11} & 0 & K_{13} & \cdots & \cdots & \cdots & K_{1,2n} \\ 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ K_{31} & 0 & K_{33} & \cdots & \cdots & \cdots & K_{3,2n} \\ \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{2n,1} & 0 & K_{2n,3} & \cdots & \cdots & \cdots & K_{2n,2n} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ \vdots \\ u_1^{(n)} \\ u_2^{(n)} \end{bmatrix} = \begin{bmatrix} F_1^{(1)} - K_{12}\Delta \\ \Delta \\ F_1^{(2)} - K_{32}\Delta \\ F_2^{(2)} - K_{42}\Delta \\ \vdots \\ F_1^{(n)} - K_{2n-1,2}\Delta \\ F_2^{(n)} - K_{2n,2}\Delta \end{bmatrix}_{3n\times 1}$$

The modified $\underline{K}$ then remains symmetric, as shown above.

Post-processing

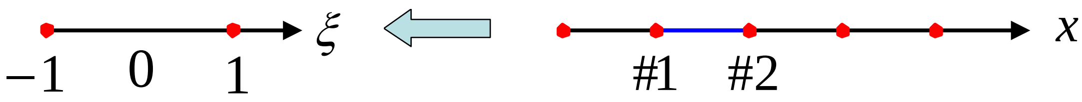
Extract nodal displacement using element connectivity: $\underline{u} \Rightarrow \underline{u}_{el}$

$$\underline{\varepsilon} = \underline{B}\underline{u}_{el}$$

$$\underline{\sigma} = \underline{D}\underline{B}\underline{u}_{el}$$

Isoparametric elements:

The actual nodal coordinates can be mapped to a normalized domain $\xi \subset [-1,1]$, which is particularly convenient because displacements and positions for different elements can be interpolated using the same shape functions.



$$u(x) = u^{\#1} N^1(\xi) + u^{\#2} N^2(\xi)$$

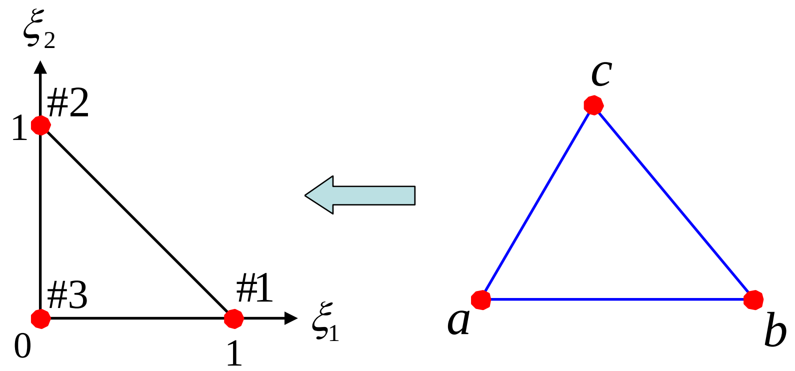
$$x = x^{\#1} N^1(\xi) + x^{\#2} N^2(\xi)$$

$$N^1(\xi) = \begin{cases} 1, & \xi = -1 \\ 0, & \xi = 1 \end{cases}, \qquad N^2(\xi) = \begin{cases} 0, & \xi = -1 \\ 1, & \xi = 1 \end{cases}$$

Interpolation functions:

$$N^1(\xi) = \frac{1-\xi}{2}, \qquad N^2(\xi) = \frac{1+\xi}{2}$$

2D linear triangular element:



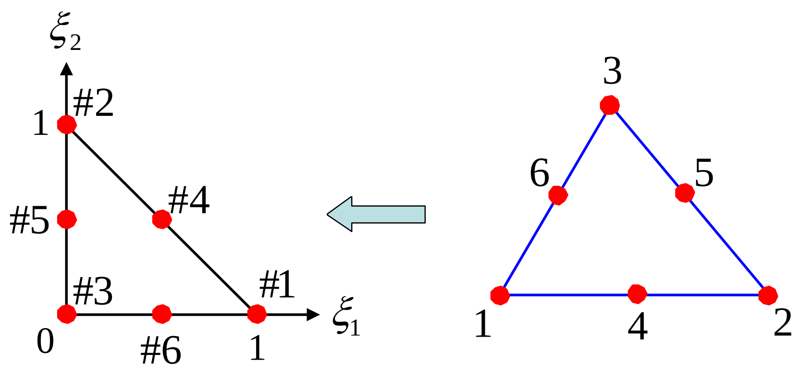
$$\vec{u} = \vec{u}^a N^1(\xi_1, \xi_2) + \vec{u}^b N^2(\xi_1, \xi_2) + \vec{u}^c N^3(\xi_1, \xi_2)$$

$$\vec{x} = \vec{x}^a N^1(\xi_1, \xi_2) + \vec{x}^b N^2(\xi_1, \xi_2) + \vec{x}^c N^3(\xi_1, \xi_2)$$

Interpolation functions:

$$N^1(\xi) = \xi_1, \qquad N^2(\xi) = \xi_2, \qquad N^3(\xi) = 1 - \xi_1 - \xi_2$$

2D quadratic triangular element:



Interpolation functions:

$$N^1 = (2\xi_1 - 1)\xi_1$$

$$N^2 = (2\xi_2 - 1)\xi_2$$

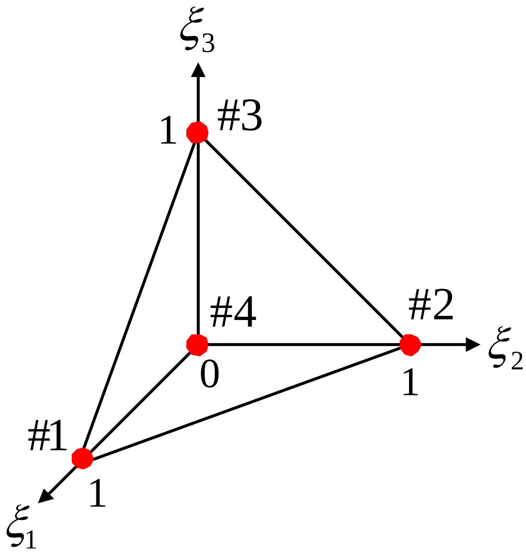
$$N^3 = (2(1 - \xi_1 - \xi_2) - 1)(1 - \xi_1 - \xi_2)$$

$$N^4 = 4\xi_1\xi_2$$

$$N^5 = 4\xi_2(1 - \xi_1 - \xi_2)$$

$$N^6 = 4\xi_1(1-\xi_1-\xi_2)$$

3D linear tetrahedral element:



Interpolation functions:

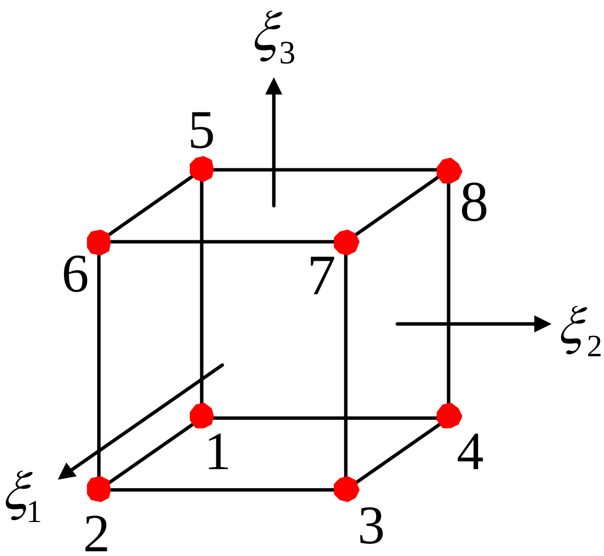
$$N^1 = \xi_1$$

$$N^2 = \xi_2$$

$$N^3 = \xi_3$$

$$N^4 = 1-\xi_1-\xi_2-\xi_3$$

3D brick element:



$$-1 \le \xi_1, \xi_2, \xi_3 \le 1$$

Interpolation functions:

$$N^1 = \frac{1}{8}(1-\xi_1)(1-\xi_2)(1-\xi_3), \quad N^2 = \frac{1}{8}(1+\xi_1)(1-\xi_2)(1-\xi_3)$$

$$N^3 = \frac{1}{8}(1+\xi_1)(1+\xi_2)(1-\xi_3), \quad N^4 = \frac{1}{8}(1-\xi_1)(1+\xi_2)(1-\xi_3)$$

$$N^5 = \frac{1}{8}(1-\xi_1)(1-\xi_2)(1+\xi_3), \quad N^6 = \frac{1}{8}(1+\xi_1)(1-\xi_2)(1+\xi_3)$$

$$N^7 = \frac{1}{8}(1+\xi_1)(1+\xi_2)(1+\xi_3), \quad N^8 = \frac{1}{8}(1-\xi_1)(1+\xi_2)(1+\xi_3)$$

Element stiffness matrix:

$$K^{el}_{aibk} = \int_{V_{el}} C_{ijkl} \frac{\partial N^a}{\partial x_j} \frac{\partial N^b}{\partial x_l} \mathrm{d}V = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} C_{ijkl} \frac{\partial N^a(\xi_1, \xi_2, \xi_3)}{\partial x_j} \frac{\partial N^b(\xi_1, \xi_2, \xi_3)}{\partial x_l} J \,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$$

$J = \det\left(\frac{\partial x_i}{\partial \xi_j}\right)$ is the Jacobian associated with the mapping, which can be computed by

$$\frac{\partial x_i}{\partial \xi_j} = \frac{\partial}{\partial \xi_j}\left(N^a(\xi) x_i^a\right) = \frac{\partial N^a}{\partial \xi_j} x_i^a$$

$$\frac{\partial N^a}{\partial x_j} = \frac{\partial N^a}{\partial \xi_l}\left(\frac{\partial \xi_l}{\partial x_j}\right) = \frac{\partial N^a}{\partial \xi_l}\left(\frac{\partial x_j}{\partial \xi_l}\right)^{-1}$$

Calculate: $\frac{\partial x_j}{\partial \xi_l} = A_{jl}$

Then: $\frac{\partial \xi_l}{\partial x_j} = A^{-1}_{jl}$

<u>Integration scheme for isoparametric elements:</u>

Gaussian integration:

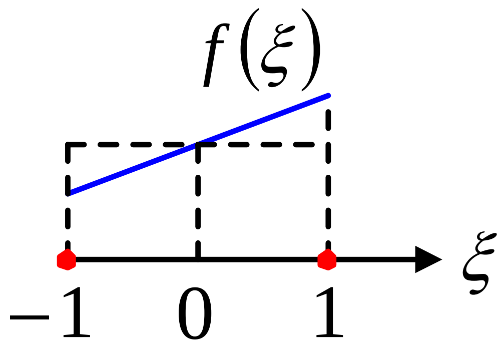
$$\int_{-1}^{1} f(\xi)\mathrm{d}\xi = \sum_{I=1}^{N_I} W_I f(\xi_I)$$

ξ_I : Gaussian integration points

W_I : weighting coefficients

Gaussian integration ensures that polynomials of order of $(2N_I - 1)$ are exactly integrated.

For $N_I = 1$, $\xi_1 = 0$, $W_1 = 2$.



$$\int_{-1}^{1} f(\xi)\mathrm{d}\xi = 2f(0)$$

Check:

$$f(\xi) = C_0 + C_1\xi$$

$$\int_{-1}^{1}(C_0 + C_1\xi)\mathrm{d}\xi = 2C_0 = 2f(0)$$

For $N_I = 2$, $\xi_1 = -\frac{1}{\sqrt{3}}$, $\xi_2 = \frac{1}{\sqrt{3}}$, $W_1 = W_2 = 1$.

$$\int_{-1}^{1} f(\xi)\mathrm{d}\xi = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Check:

$$f(\xi) = C_0 + C_1\xi + C_2\xi^2 + C_3\xi^3$$

$$\int_{-1}^{1} f(\xi)\mathrm{d}\xi = \int_{-1}^{1}\left(C_0 + C_2\xi^2\right)\mathrm{d}\xi = 2C_0 + \frac{2}{3}C_2 = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

For $N_I = 3$, $\xi_1 = -0.775$, $\xi_2 = 0$, $\xi_3 = 0.775$, $W_1 = W_3 = 0.556$, $W_2 = 0.889$.

$$\int_{-1}^{1} f(\xi)\mathrm{d}\xi = 0.556f(-0.775) + 0.889f(0) + 0.556f(0.775)$$

h-method: refine mesh, i.e. reduce the element size to achieve sufficient accuracy
p-method: increase order of polynomial interpolation function

Summary about integration schemes for 2D and 3D elements

Linear triangular element: 1 integration point
Quadratic triangular element: 3 integration points

Linear quadrilateral element: 4 integration points ($N_I = 2$)

Quadratic quadrilateral element: 4 integration points ($N_I = 2$)

Linear brick element: 8 integration points ($N_I = 2$)

Quadratic brick element: 27 integration points ($N_I = 3$)