School of Engineering
Brown University

EN175: Advanced Mechanics of Solids

Final Examination
Monday Dec 17 2018

NAME: ____

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may use two pages of reference notes
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

____

1-5. (11 points) _____________

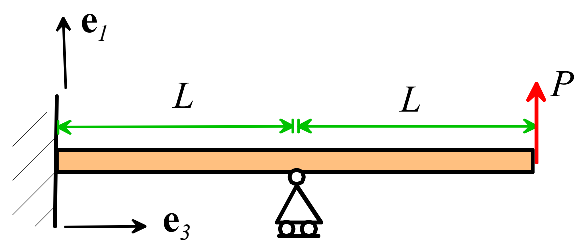
6. (16 points) _____________

7. (6 points) _____________

8. (12 points) _____________

TOTAL (45 points) ____________

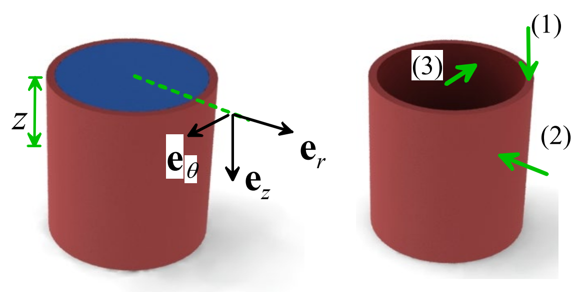
1. The figure shows a beam subjected to constraints and forces. Indicate whether the following functions are kinematically admissible displacement fields for calculating the beam's potential energy



	Admissible?	
(a) $u_1 = x_3(L - x_3)$	Y	**[N]**
(b) $u_1 = x_3^3(L - x_3)^2$	**[Y]**	N
(c) $u_1 = x_3^2(L - x_3)(2L - x_3)$	**[Y]**	N
(d) $u_1 = x_3^2(2L - x_3)$	Y	**[N]**

[2 POINTS]

2. The figure shows a tumbler with mass density ρ_T filled with fluid with mass density ρ . Using the cylindrical-polar basis shown, write down the boundary conditions for stress components on the three surfaces marked (1), (2), (3) in the figure.



(1) (Top surface, $z = 0, \quad a < r < a + t$)

$\sigma_{zz} = 0, \ \sigma_{rz} = 0 \ \ \sigma_{z\theta} = 0$

(2) (Outer surface $r = a + t$)

$\sigma_{rr} = 0, \ \sigma_{rz} = 0 \ \ \sigma_{r\theta} = 0$

(3) (Inner surface $r = a$)

$\sigma_{rr} = -\rho g z, \ \sigma_{rz} = 0 \ \ \sigma_{r\theta} = 0$

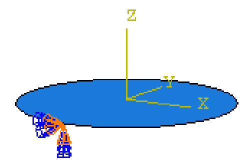
[3 POINTS]

3. The circular plate shown in the figure is meshed with plate elements for a static analysis. Indicate whether the boundary conditions shown properly constrain the solid.

Properly constrained?

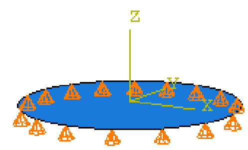
Y N

U1=U2=U3=0
UR1=UR2=UR3=0



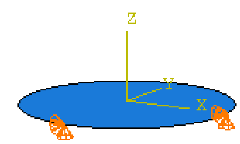
Y **N**

U3=0



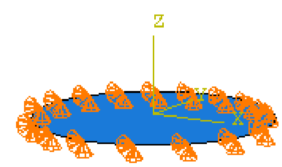
Y **N**

U1=U2=U3=0



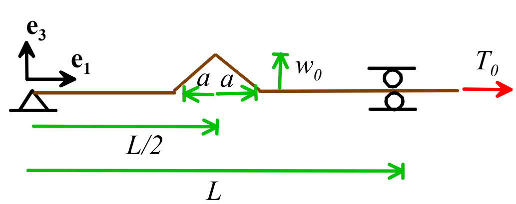
Y N

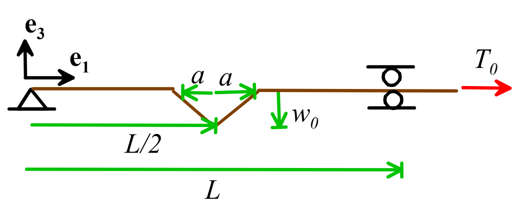
U1=U2=U3=0



[2 POINTS]

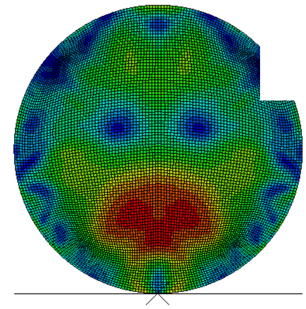
4. The string shown in the figure has mass per unit length m and is stretched by an axial tension T_0 . It is prevented from moving in a vertical direction at its ends $x_3 = 0, x_3 = L$. At time *t=0* it is released from rest with the displacement distribution shown in the figure. Use the space below to draw the shape of the string at time $t = L\sqrt{m/T_0}$





[2 POINTS]

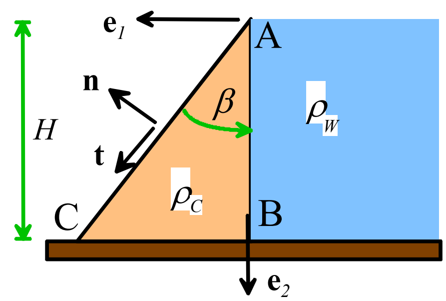
5. The figure shows a finite element simulation of an elastic cylinder with radius *R*, mass density ρ, Young's modulus *E* and Poisson's ratio ν colliding with a frictionless rigid surface. The goal of the simulation is to calculate the contact time *T* as a function of material properties and geometry. Re-write the relationship $T = f(E,\rho,\nu,R)$ in dimensionless form.



$$\frac{T}{R}\sqrt{\frac{E}{\rho}} = f(\nu)$$

[2 POINTS]

6. The figure shows a simple design for a dam. The wedge-shaped dam is made from a material with weight density ρ_C . It is loaded on its vertical face by fluid pressure $p = -\rho_W x_2$, where ρ_W is the weight density of the fluid.



6.1 Write down formulas for unit vectors **t, n** tangent and normal to the back face of the dam, in terms of β (you will need this information for part 7.2 and 7.3)

$$\mathbf{t} = \sin\beta\mathbf{e}_1 + \cos\beta\mathbf{e}_2$$
$$\mathbf{n} = -\sin\beta\mathbf{e}_2 + \cos\beta\mathbf{e}_1$$

[2 POINTS]

6.2 Write down the boundary conditions on the two faces AB, AC of the dam, in terms of the 2D stress components $\sigma_{11}, \sigma_{22}, \sigma_{12}$.

On face AB, the condition $\mathbf{n}\cdot\boldsymbol{\sigma} = \rho_W x_2 \mathbf{e}_1$ gives $[-1,0]\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix} = \begin{bmatrix}-\sigma_{11}\\ -\sigma_{12}\end{bmatrix} = \begin{bmatrix}\rho_w x_2\\ 0\end{bmatrix}$

On face AC, the condition $\mathbf{n}\cdot\boldsymbol{\sigma} = \mathbf{0}$ gives $[\cos\beta, -\sin\beta]\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix} = \begin{bmatrix}\sigma_{11}\cos\beta - \sigma_{12}\sin\beta\\ \sigma_{12}\cos\beta - \sigma_{22}\sin\beta\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

[2 POINTS]

6.2 Consider the stress state

$$\sigma_{11} = -\rho_W x_2$$
$$\sigma_{22} = \rho_C\left(x_1 \cot(\beta) - x_2\right) - \rho_W \cot^2\beta(2x_1\cot(\beta) - x_2)$$
$$\sigma_{12} = -\rho_W x_1 \cot^2\beta$$

Show that the stress state satisfies equilibrium in the dam (inside the wedge) and also satisfies the boundary conditions on the faces AB and AC. Note that on face AC $x_1 = x_2 \tan\beta$.

The equilibrium equations are

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \rho_C = -\rho_W \cot^2\beta - \rho_C + \rho_W \cot^2\beta + \rho_C = 0$$

[2 POINTS]

The boundary condition on face AB is $\sigma_{11} = -\rho_W x_2 \qquad \sigma_{12} = 0 \qquad x_1 = 0$. This is clearly satisfied.

[2 POINTS]

The boundary condition on face AC is

$$t_1 = \sigma_{11}\cos\beta - \sigma_{12}\sin\beta = 0$$
$$t_2 = \sigma_{12}\cos\beta - \sigma_{22}\sin\beta = 0$$

on $x_1 = s\sin\beta, \qquad x_2 = s\cos\beta$ Substituting the given formulas

$$t_1 = -\rho_W s\cos^2\beta + \rho_W s\sin^2\beta\cot^2\beta = 0$$
$$t_2 = -\rho_W s\sin\beta\cos\beta\cot^2\beta - \rho_C(s\sin\beta\cot\beta - s\cos\beta) + \rho_W\cot^2\beta\sin\beta(2s\sin\beta\cot\beta - s\cos\beta) = 0$$

[2 POINTS]

6.3 Suppose that the dam is made from concrete, which cannot safely withstand tensile stress. Assuming that the maximum principal tensile stress occurs at B, find a formula for the minimum allowable value for the angle β , in terms of ρ_c, ρ_W

The condition

$$\sigma_{22} < 0 \quad x_1 = 0$$
$$\Rightarrow \rho_C\left(-x_2\right) - \rho_W \cot^2\beta(-x_2) < 0 \Rightarrow \cot^2\beta < \rho_C / \rho_W$$

[2 POINTS]

6.4 Assume that the concrete fails by crushing if the minimum principal stress reaches $\sigma_3 = -\sigma_c$. Assuming that the minimum principal stress occurs at point C, and that β has the minimum value calculated in 7.3, find an expression for the maximum height of the dam, in terms of ρ_C, ρ_W . You may find the trig identity $1+\tan^2\beta = 1/\cos^2\beta$ helpful.

Since the rear face of the dam is free of traction, the direction of the nonzero principal stress must be parallel to **t**. Therefore

$$\sigma_3 = \mathbf{t}\cdot\boldsymbol{\sigma}\mathbf{t} = \begin{bmatrix}\sin\beta & \cos\beta\end{bmatrix}\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix}\begin{bmatrix}\sin\beta\\ \cos\beta\end{bmatrix} = \sigma_{11}\sin^2\beta + \sigma_{22}\cos^2\beta + 2\sigma_{12}\sin\beta\cos\beta$$

Note that at C $x_1 = H\tan\beta \quad x_2 = H$ so

$$\sigma_{11} = -\rho_W H \qquad \sigma_{22} = -\rho_W H\cot^2\beta \qquad \sigma_{12} = -\rho_W \cot\beta$$

This gives

$$\sigma_3 = -\rho_W H\left(\sin^2\beta + \cos^2\beta\cot^2\beta + 2\sin\beta\cos\beta\cot\beta\right)$$

$$= -\rho_W H\left(1+\cos^2\beta + \cos^2\beta\cot^2\beta\right) = -\frac{\rho_W H}{\sin^2\beta}\left(\sin\beta^2 + \cos^2\beta\sin^2\beta + \cos^4\beta\right) = -\frac{\rho_W H}{\sin^2\beta}$$

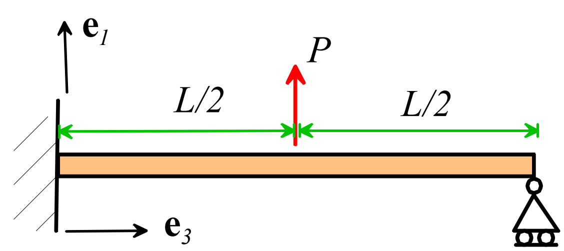
From 7.3 we know that

$$\tan^2\beta = \rho_W/\rho_C \Rightarrow \frac{1}{\cos^2\beta} = 1+\rho_W/\rho_C \Rightarrow \sin^2\beta = 1-\frac{1}{1+\rho_W/\rho_C} = \frac{\rho_W}{\rho_W+\rho_C}$$

So $\sigma_3 = -H(\rho_W+\rho_C)$, and therefore $H < \sigma_c/(\rho_W+\rho_C)$

[4 POINTS]

7. The figure shows a beam that is clamped at $x_3 = 0$ and pinned at $x_3 = L$. It is subjected to a point force P at mid-span $x_3 = L/2$.



7.1 Show that $\hat{v} = Cx_1^2(L-x_1)$ is a kinematically admissible deflection for the beam

The boundary conditions are

$$v = dv/dx_3 = 0 \qquad x_3 = 0$$
$$v = 0 \qquad x_3 = L$$

Note $\frac{dv}{dx_3} = 2x_1 L - 3x_1^2$

The displacement field given therefore satisfies all three of these conditions.

[2 POINTS]

7.2 Hence, find a formula for the potential energy of the beam, in terms of E,I,C,P,L . You can assume that the potential energy of a beam is

$$\Pi = \int_0^L \frac{1}{2} EI \left(\frac{d^2 v}{dx^2} \right)^2 dx - \int_0^L q(x) v(x) dx$$

Assume a displacement field $\hat{v} = C x_1^2 (L - x_1)$

We have that $\frac{d^2 v}{dx_3^2} = 2L - 6x_1$

The potential energy is therefore

$$\begin{aligned}
\Pi &= \int_0^L 2EIC^2 \left(L - 3x_1 \right)^2 dx - CP \left(\frac{L}{2} \right)^2 \frac{L}{2} \\
&= 2EIC^2 \left[-\frac{1}{9} (L - 3x_1)^3 \right]_0^L - CP \frac{L^3}{8} \\
&= 2EIC^2 \frac{1}{9} (8L^3 + L^3) - CP \frac{L^3}{8} \\
&= 2EIC^2 L^3 - CP \frac{L^3}{8}
\end{aligned}$$

[2 POINTS]

7.3 Hence, use the Rayleigh-Ritz method to estimate the deflection of the beam at $x_3 = L/2$.

We need to minimize the PE

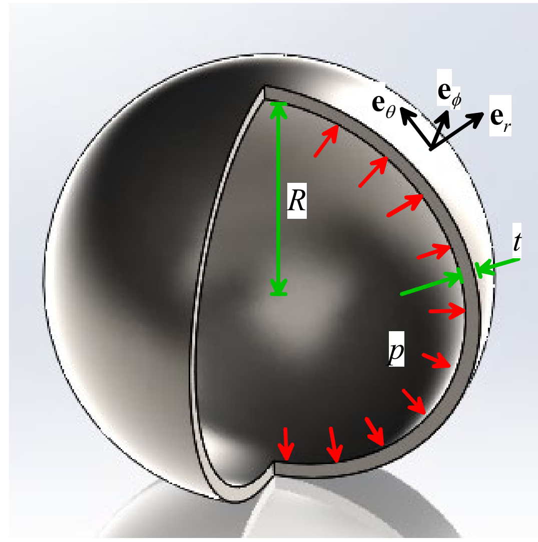
$$\frac{d\Pi}{dC} = 4EICL^3 - P\frac{L^3}{8} = 0 \Rightarrow C = \frac{P}{32EI}$$

The deflection at midspan is therefore

$$\hat{v} = \frac{P}{32EI} \left(\frac{L}{2} \right)^3 = \frac{PL^3}{256EI}$$

[2 POINTS]

8. A thin-walled sphere with radius R and wall thickness t is made from an elastic-plastic material with Youngs modulus E , Poissons ratio ν and a linear hardening relation $Y = Y_0 + h\varepsilon_e$. The sphere is subjected to monotonically increasing internal pressure p (with $dp/dt > 0$) , which generates a stress state (in spherical-polar coordinates) $\sigma_{rr} \approx 0,\ \sigma_{\phi\phi} = \sigma_{\theta\theta} = pR/(2t)$ (note that these are principal stresses)



9.1 Find a formula for the Von-Mises stress in the sphere wall, in terms of p,R and t. Hence, calculate the pressure that will first cause yield in the sphere wall.

$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right]} = pR/2t$$

At yield $\sigma_e = Y \Rightarrow p_y = 2tY/R$

[2 POINTS]

8.2 Find the hydrostatic and deviatoric stresses in the sphere wall.

$$\sigma_h = \sigma_{kk}/3 = pR/3t$$

$$S_{rr} = -pR/3t \qquad S_{\theta\theta} = S_{\phi\phi} = pR/6t$$

[2 POINTS]

8.3 Hence, find a formula for the Von Mises plastic strain rate $d\varepsilon_e/dt$ in the sphere wall, in terms of $dp/dt, h, R, t$

From notes, the plastic strain rate is zero below yield while above yield, the formula is

$$\frac{d\varepsilon_e}{dt} = \frac{3}{2}\frac{1}{h\sigma_e}\left\langle S_{ij}\frac{d\sigma_{ij}}{dt}\right\rangle = \frac{3}{2}\frac{1}{h(pR/2t)}\left\langle 2\frac{pR}{6t}\frac{dp}{dt}\frac{R}{2t}\right\rangle = \frac{1}{h}\frac{R}{2t}\frac{dp}{dt}$$

[2 POINTS]

8.4 Hence, find a formula for the total strain rates $d\varepsilon_{rr} / dt, d\varepsilon_{\theta\theta} / dt$ (include both elastic and plastic strain rates, and give solutions for pressure both below and above yield) in the shell.

From notes

$$\frac{d}{dt}\begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \end{bmatrix} = \begin{cases} \dfrac{R}{2Et}\begin{bmatrix} -2\nu \\ 1-\nu \end{bmatrix}\dfrac{dp}{dt} & pR/(2t) < Y_0 \\ \dfrac{R}{2Et}\begin{bmatrix} -2\nu \\ 1-\nu \end{bmatrix}\dfrac{dp}{dt} + \dfrac{1}{h}\dfrac{R}{2t}\dfrac{dp}{dt}\begin{bmatrix} -1 \\ 1/2 \end{bmatrix} & pR/(2t) < Y_0 \end{cases}$$

[2 POINTS]

8.5 Find the total hoop strains $\varepsilon_{\theta\theta}, \varepsilon_{\phi\phi}$ when the pressure reaches a value $p = 4tY_0 / R$

Integrating

$$\begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \end{bmatrix} = \frac{Y_0}{E}\begin{bmatrix} -2\nu \\ 1-\nu \end{bmatrix} + \frac{R}{2t}\begin{bmatrix} -2\nu / E - 1/h \\ (1-\nu)/E + 1/(2h) \end{bmatrix}\left(p - 2Y_0 t / R\right)$$
$$= Y_0\begin{bmatrix} -4\nu / E - 1/h \\ 2(1-\nu)/E + 1/(2h) \end{bmatrix}$$

[2 POINTS]

8.6 Find a formula for the change in radius of the sphere when the pressure reaches a value $p = 4tY_0 / R$

The strains are related to the radial displacements by $\varepsilon_{\theta\theta} = u / R$. Therefore

$$u = Y_0 R\left(\frac{2(1-\nu)}{E} + \frac{1}{2h}\right)$$

[2 POINTS]