



School of Engineering
Brown University

EN175: Advanced Mechanics of Solids

Midterm Examination

Nov 1 2018

NAME: _____

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may use two pages of reference notes
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

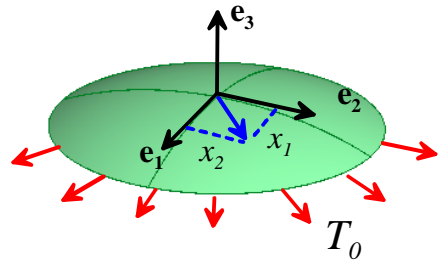
Please initial the statement below to show that you have read it

By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

- _____
1. (3 points) _____
 2. (5 points) _____
 3. (5 points) _____
 4. (5 points) _____
 5. (12 points) _____

TOTAL (30 points) _____

1. A thin circular membrane with radius R and mass per unit area m is stretched by a uniform force per unit length of perimeter T_0 .



1.1 Re-write the equation $\omega = f(T_0, m, R)$ for the natural frequency of vibration of the membrane ω in dimensionless form.

[2 POINTS]

1.2 A membrane with mass per unit area 100 grams and radius 10cm is stretched by a force per unit length of 400N. It is observed to have a vibration mode with natural frequency 7.7 Hz. If the radius is doubled, but other parameters are unchanged, what is the new natural frequency (in Hz)?

[1 POINT]

2. Express the following equations in index notation

(a) $\eta = \mathbf{a} \cdot \mathbf{b}$

(b) $\mathbf{c} = \mathbf{a} \times \mathbf{b}$

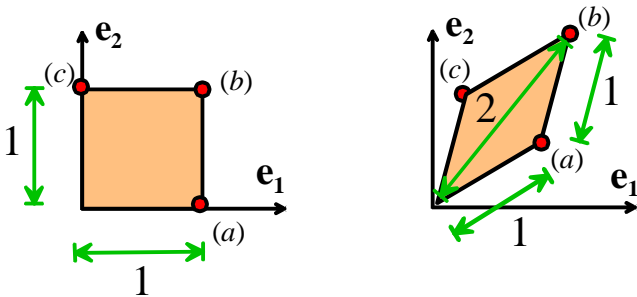
[2 POINTS]

Hence, use index notation to show that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$. Recall that $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$ and

$\epsilon_{ijk} \epsilon_{ipq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}$. Please show your work clearly

[3 POINTS]

3. The figure shows a small element of material in a deformable solid before and after deformation. Determine the components of the 2D Lagrange strain tensor (eg as a 2x2 matrix).



[5 POINTS]

4. For the stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} -400 & 300 & 0 \\ 300 & 400 & 0 \\ 0 & 0 & 600 \end{bmatrix}$$

please calculate

4.1 The hydrostatic stress

[1 POINT]

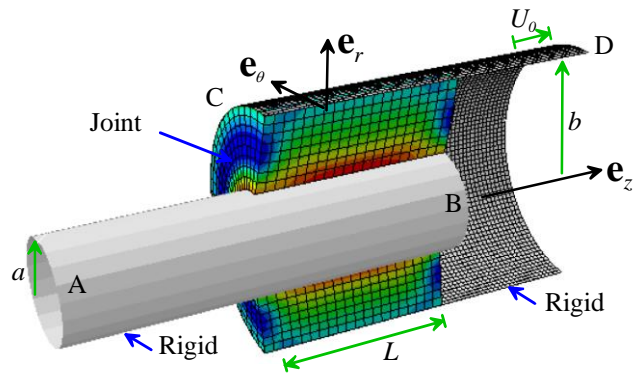
4.2 The principal stresses and their directions

[3 POINTS]

4.3 The tractions acting on a plane with normal $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) / \sqrt{3}$

[1 POINT]

5. The figure shows a cross-section through a joint connecting two hollow cylindrical shafts. The joint is a hollow cylinder with external radius b and internal radius a . It is bonded to the two rigid shafts AB and CD. Shaft AB is fixed (no translation or rotation), and an axial displacement $\mathbf{u} = U_0 \mathbf{e}_z$ is applied to the hollow cylinder CD.



The goal of this problem is to estimate the axial force necessary to produce displacement of the shaft CD, and hence determine the stiffness of the joint.

5.1 Assume that the displacement field in the joint can be approximated by $\mathbf{u} = u(r)\mathbf{e}_z$, where $u(r)$ is a function to be determined. Calculate the infinitesimal strain tensor in the joint in terms of $u(r)$ and its derivatives (you can assume that the gradient of a vector in cylindrical-polar coordinates is

$$\nabla(v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z) \equiv \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r} \frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

[2 POINTS]

5.2 Assume that the joint can be idealized as an isotropic, linear elastic material with Young's modulus E and Poisson's ratio ν . Find a formula for the stress in the joint $a < r < b$, in terms of derivatives of $u(r)$ and the material properties.

[1 POINT]

5.3 Use the equation of static equilibrium $\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}$ to show that $u(r)$ must satisfy

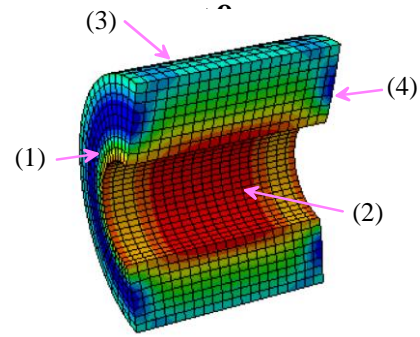
$$\left\{ \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} \right\} = \frac{1}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) = 0$$

You can use the formula for the divergence of a tensor \mathbf{S} in cylindrical-polar coordinates

$$\nabla \cdot \mathbf{S} \equiv \begin{bmatrix} \frac{\partial S_{rr}}{\partial r} + \frac{S_{rr}}{r} + \frac{1}{r} \frac{\partial S_{\theta r}}{\partial \theta} + \frac{\partial S_{zr}}{\partial z} - \frac{S_{\theta\theta}}{r} \\ \frac{1}{r} \frac{\partial S_{\theta\theta}}{\partial \theta} + \frac{\partial S_{r\theta}}{\partial r} + \frac{S_{r\theta}}{r} + \frac{S_{\theta r}}{r} + \frac{\partial S_{z\theta}}{\partial z} \\ \frac{\partial S_{zz}}{\partial z} + \frac{\partial S_{rz}}{\partial r} + \frac{S_{rz}}{r} + \frac{1}{r} \frac{\partial S_{\theta z}}{\partial \theta} \end{bmatrix}$$

[2 POINTS]

5.4 Write down the boundary conditions for displacements and/or stresses on the four external surfaces of the joint (i.e. give any known values for displacement components u_r, u_θ, u_z , or stress $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{r\theta}, \sigma_{rz}, \sigma_{z\theta}$)



[3 POINTS]

5.5 Find a solution for $u(r)$ that satisfies 5.3 and boundary conditions on $r=a$ and $r=b$. Does the solution satisfy all the boundary conditions in 5.4? If not, which boundary conditions are not satisfied?

[2 POINTS]

5.6 Find a formula for the axial force F_z that must be applied to shaft CD to cause the necessary axial displacement U_0 . Hence, find a formula for the stiffness of the joint.

[2 POINTS]