School of Engineering
Brown University

EN175: Advanced Mechanics of Solids

Midterm Examination

NAME: ____

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may use two pages of reference notes
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

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1. (3 points) _____________

2. (5 points) _____________

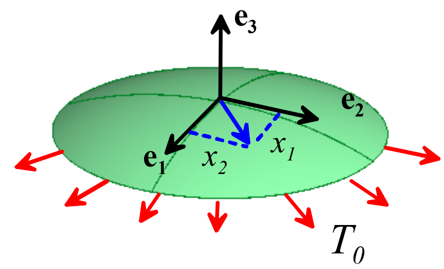
3. (5 points) _____________

4. (5 points) _____________

5. (12 points) _____________

TOTAL (35 points)___________

1. A thin circular membrane with radius R and mass per unit area m is stretched by a uniform force per unit length of perimeter T_0 .



1.1 Re-write the equation $\omega = f(T_0, m, R)$ for the natural frequency of vibration of the membrane ω in dimensionless form

The units are $\omega = 1/\mathrm{s}$ $\quad m = \mathrm{kg/m^2}$ $\quad T_0 = \mathrm{kg.m.s^{-2}/m}$ $\qquad R = \mathrm{m}$

We need to make the LHS and the function arguments dimensionless. There is no way to combne the function arguments into a dimensionless group, but they can be combined to make something with dimensions of seconds. Hence the dimensionless formula is

$$\omega\sqrt{\frac{mR^2}{T_0}} = f(.)$$

[2 POINTS]

1.2 A membrane with mass per unit area 100 grams and radius 10cm is stretched by a force per unit length of 400N. It is observed to have a vibration mode with natural frequency 7.7 Hz. If the radius is doubled, but other parameters are unchanged, what is the new natural frequency (in Hz)?

We have

$$\omega\sqrt{\frac{mR^2}{T_0}} = C$$

If the radius is doubled, the frequency will halve, i.e. $\omega = 3.85\,\mathrm{Hz}$

[1 POINT]

2. Express the following equations in index notation

(a) $\eta = \mathbf{a}\cdot\mathbf{b}$

(b) $\mathbf{c} = \mathbf{a}\times\mathbf{b}$

(a) $\eta = a_i b_i$

(b) $c_i = \in_{ijk} a_j b_k$

[2 POINTS]

Hence, use index notation to show that $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2$. Recall that $|\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{a}$ and $\in_{ijk}\in_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$. Please show your work clearly

From (a), (b) $|\mathbf{a}\times\mathbf{b}|^2 \equiv \left(\in_{ijk} a_j b_k\right)\left(\in_{ipq} a_p b_q\right)$

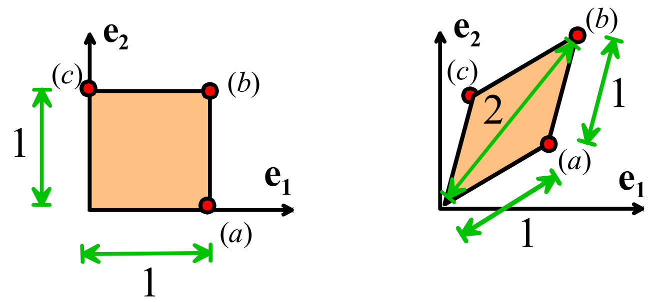
Recall the identities $\in_{ijk}\in_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$ and $a_j = \delta_{ji} a_i$. Hence

$$|\mathbf{a}\times\mathbf{b}|^2 \equiv \left(\in_{ijk} a_j b_k\right)\left(\in_{ipq} a_p b_q\right) = \left(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}\right) a_j b_k a_p b_q$$

$$= a_j b_k a_j b_k - a_j b_k a_k b_j = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2$$

[3 POINTS]

3. The figure shows a small element of material in a deformable solid before and after deformation. Determine the components of the 2D Lagrange strain tensor (eg as a 2x2 matrix).



We apply the identity $\dfrac{l^2 - l_0^2}{2l_0^2} = E_{ij}m_i m_j$, taking $\mathbf{m}$ to be the two sides of the square and the diagonal. This gives

$$\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}E_{11} & E_{12}\\ E_{12} & E_{22}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = E_{11} = 0$$

$$\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}E_{11} & E_{12}\\ E_{12} & E_{22}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = E_{22} = 0$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}E_{11} & E_{12}\\ E_{12} & E_{22}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}(E_{11} + E_{22} + 2E_{12}) = \frac{4-2}{4}$$

$$\Rightarrow E_{11} = E_{22} = 0 \qquad E_{12} = 1/2$$

(Strictly speaking this is not a physically admissible deformation since the volume of the deformed solid goes to zero, but the numbers were chosen to make the arithmetic easy!)

[5 POINTS]

4. For the stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} -400 & 300 & 0 \\ 300 & 400 & 0 \\ 0 & 0 & 600 \end{bmatrix}$$

please calculate

4.1 The hydrostatic stress

$$\sigma_h = \text{trace}(\boldsymbol{\sigma}) / 3 = 600 / 3 = 200$$

[1 POINT]

4.2 The principal stresses and their directions

$$\det(\boldsymbol{\sigma} - \lambda \mathbf{I}) = 0$$
$$\Rightarrow (600 - \lambda)\left(-(400 - \lambda)(400 + \lambda) - 300^2\right) = 0$$
$$\Rightarrow \lambda^2 - 250000 = 0 \Rightarrow \lambda = \pm 500$$

So $[\sigma_1, \sigma_2, \sigma_3] = [600, 500, -500]$

The principal stress directions are the null vectors of

$$\begin{bmatrix} -400 - \lambda & 300 & 0 \\ 300 & 400 - \lambda & 0 \\ 0 & 0 & 600 - \lambda \end{bmatrix}$$

$$\mathbf{m}^1 = [0, 0, 1] \qquad \mathbf{m}^2 = [3, 9, 0] / \sqrt{90} \qquad \mathbf{m}^3 = [-9, 3, 0] / \sqrt{90}$$

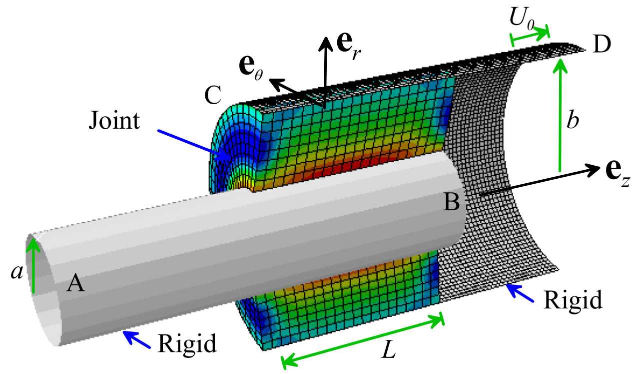
[3 POINTS]

4.3 The tractions acting on a plane with normal $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) / \sqrt{3}$

$$\mathbf{n} \cdot \boldsymbol{\sigma} = \frac{1}{\sqrt{3}} [1, 1, 1] \begin{bmatrix} -400 & 300 & 0 \\ 300 & 400 & 0 \\ 0 & 0 & 600 \end{bmatrix} = [-100, 700, 600] / \sqrt{3}$$

[1 POINT]

5. The figure shows a cross-section through a joint connecting two hollow cylindrical shafts. The joint is a hollow cylinder with external radius b and internal radius a. It is bonded to the two rigid shafts AB and CD. Shaft AB is fixed (no translation or rotation), and an axial displacement $\mathbf{u} = U_0 \mathbf{e}_z$ is applied to the hollow cylinder CD.



The goal of this problem is to estimate the axial force necessary to produce displace the shaft CD, and hence determine the stiffness of the joint.

5.1 Assume that the displacement field in the joint can be approximated by $\mathbf{u} = u(r)\mathbf{e}_z$, where $u(r)$ is a function to be determined. Calculate the infinitesimal strain tensor in the joint in terms of $u(r)$ and its derivatives (you can assume that the gradient of a vector in cylindrical-polar coordinates is

$$\nabla(v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z) \equiv \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

The strain is $\boldsymbol{\varepsilon} = \frac{1}{2}\left\{\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right\}$

Substituting the given displacement gives

$$\boldsymbol{\varepsilon} \equiv \frac{1}{2}\begin{bmatrix} 0 & 0 & \frac{du}{dr} \\ 0 & 0 & 0 \\ \frac{du}{dr} & 0 & 0 \end{bmatrix}$$

[1 POINT]

5.2 Assume that the joint can be idealized as an isotropic, linear elastic material with Youngs modulus E and Poisson's ratio ν . Find a formula for the stress in the joint $a<r<b$, in terms of derivatives of $u(r)$ and the material properties.

The only nonzero stress component is $\sigma_{zr} = \frac{E\varepsilon_{zr}}{(1+\nu)} = \frac{E}{2(1+\nu)}\frac{du}{dr}$

[2 POINTS]

5.3 Use the equation of static equilibrium $\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}$ to show that $u(r)$ must satisfy

$$\left\{ \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} \right\} = \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr} \right) = 0$$

You can use the formula for the divergence of a tensor **S** in cylindrical-polar coordinates

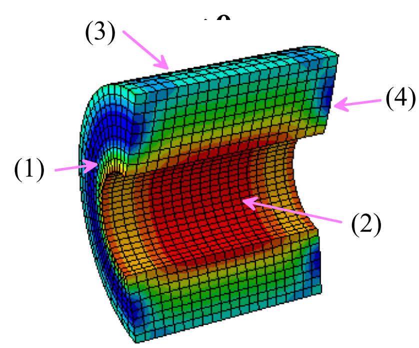
$$\nabla \cdot \mathbf{S} \equiv \begin{bmatrix} \frac{\partial S_{rr}}{\partial r} + \frac{S_{rr}}{r} + \frac{1}{r}\frac{\partial S_{\theta r}}{\partial \theta} + \frac{\partial S_{zR}}{\partial z} - \frac{S_{\theta\theta}}{r} \\ \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial \theta} + \frac{\partial S_{r\theta}}{\partial r} + \frac{S_{r\theta}}{r} + \frac{S_{\theta r}}{r} + \frac{\partial S_{z\theta}}{\partial z} \\ \frac{\partial S_{zz}}{\partial z} + \frac{\partial S_{rz}}{\partial r} + \frac{S_{rz}}{r} + \frac{1}{r}\frac{\partial S_{\theta z}}{\partial \theta} \end{bmatrix}$$

Substituting the stress from the previous problem into the third equation (the first two are zero) gives

$$\frac{\partial \sigma_{zr}}{\partial r} + \frac{\sigma_{zr}}{r} = \frac{E}{2(1+\nu)}\left\{ \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} \right\} = 0$$

[2 POINTS]

5.4 Write down the boundary conditions for displacements and/or stresses on the four external surfaces of the joint (i.e. give any known values for displacement components u_r, u_θ, u_z , or stress $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{r\theta}, \sigma_{rz}, \sigma_{z\theta}$)



On (1) and (4) $\sigma_{zz} = \sigma_{z\theta} = \sigma_{rz} = 0$

On (2) $u_r = u_\theta = u_z = 0$

On (3) $u_r = u_\theta = 0 \qquad u_z = U_0$

[3 POINTS]

5.5 Find a solution for $u(r)$ that satisfies 5.3 and boundary conditions on $r=a$ and $r=b$. Does the solution satisfy all the boundary conditions in 5.4? If not, which boundary conditions are not satisfied?

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)=0 \Rightarrow \left(r\frac{du}{dr}\right)=C \Rightarrow \frac{du}{dr}=\frac{C}{r} \Rightarrow u=A+C\log(r)$$

The boundary condition at $r=a$ gives $A=-C\log(a) \Rightarrow u=C\log(r/a)$

The boundary condition at $r=b$ gives $U_0=C\log(b/a) \Rightarrow C=U_0/\log(b/a)$

Hence

$$u=\frac{U_0}{\log(b/a)}\log(r/a)$$

All the boundary conditions are satisfied except that on (1) and (4) $\sigma_{rz}\neq 0$

[2 POINTS]

5.6 Find a formula for the axial force F_z that must be applied to shaft CD to cause the necessary axial displacement U_0. Hence, find a formula for the stiffness of the joint.

The force can be found by integrating the traction exerted by the joint on the rigid outer cylinder. The force magnitude is

$$F_z=\int_A \sigma_{zr}dA=2\pi bL\frac{E}{2(1+\nu)}\left.\frac{du}{dr}\right|_{r=b}=2\pi bL\frac{E}{2(1+\nu)}\frac{U_0}{\log(b/a)}\frac{1}{b}=\frac{\pi LEU_0}{(1+\nu)\log(b/a)}$$

The stiffness is therefore

$$k=\frac{\pi LE}{(1+\nu)\log(b/a)}$$

[2 POINTS]