School of Engineering
Brown University

# EN175: Advanced Mechanics of Solids

## Practice Final Examination

**NAME:** ________________________________________

## General Instructions

- No collaboration of any kind is permitted on this examination.
- You may use two double sided pages of reference notes but no other materials may be consulted
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

## Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

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**1. (5 points)** ______________

**2. (5 points)** ______________

**3. (5 points)** ______________

**4. (6 points)** ______________

**5. (3 points)** ______________

**TOTAL (25 points)**____________

**1.** Please answer the FEA related questions

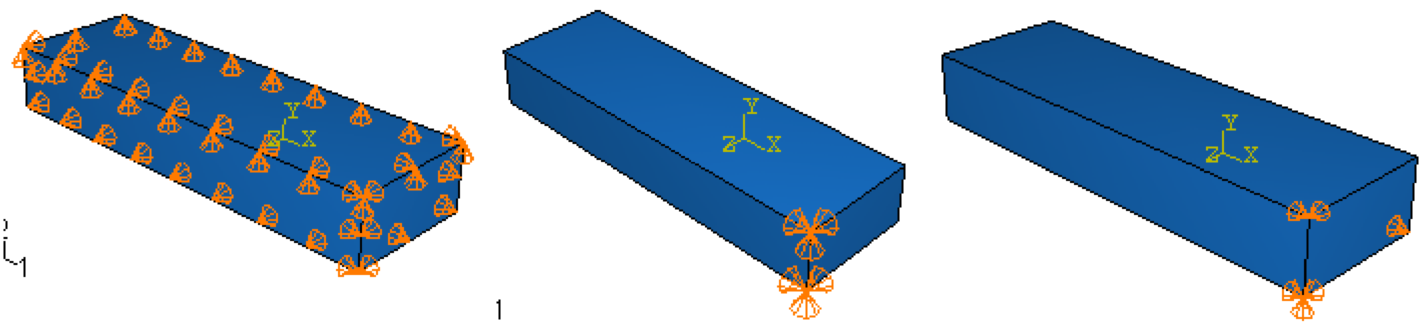
1. What is the difference between a truss element and a solid element (please limit your answer to a sentence!)

A truss element has only two nodes and assumes the line connecting them is a bar in uniaxial tension, while a solid element has at least 3 (in 2D or 4 (in 3D) nodes and interpolates a full 2D or 3D displacement field between the values defined at the nodes.

2. What is the difference between the displacement fields in 6 noded and 3 noded triangular elements, and which are generally more accurate?

6 noded elements assume a quadratic variation of displacements in the element; 3 noded elements have a linear variation. 6 noded elements are more accurate, partly because the quadratic variation provides a closer fit to the true displacement field, and partly because 6 noded elements are less prone to locking (for near incompressible materials)

3. Which of the boundary conditions shown below properly constrain the solid for a static analysis?




(a) (b) (c)

(a) And (c) are OK; (b) has an unconstrained rotational mode.

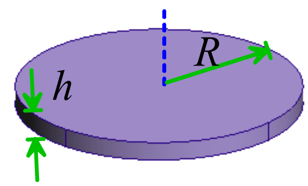
4. A linear elastic FE calculation predicts a maximum Mises stress of 100MPa in a component. The solid is loaded only by prescribing tractions and displacements on its boundary. If the applied loads and prescribed displacements are all doubled, what will be the magnitude of the maximum Mises stress?

The mises stress will double

5. An FE calculation is conducted on a part. The solid is idealized as an elastic-perfectly plastic solid, with Youngs modulus 210 GPa, Poisson ratio $\nu = 0.3$. Its plastic properties are idealized with Mises yield surface with yield stress 500MPa. The solid is loaded only by prescribing tractions and displacements on its boundary. The analysis predicts a maximum von-Mises stress of 400MPa in the component. If the applied loads and prescribed displacements are all doubled, what will be the magnitude of the maximum Mises stress?

The Mises stress will be 500 MPa since the material will be at yield, and there is no strain hardening.

**2.** A thin circular plate with radius $R$ has thickness $h$, Young's modulus $E$, Poisson's ratio $\nu$ and mass density $\rho_0$. The natural frequencies are calculated using FEA. Express the resulting relationship between the natural frequencies $\omega$, geometry, and mass density in dimensionless form.




We know that for a plate the governing equation is

$$\frac{Eh^3}{(1-\nu^2)}\nabla^4 u + \rho h \frac{\partial^2 u}{\partial t^2} = 0$$

Therefore

$$\omega = f(Eh^3/(1-\nu^2), \rho h, R)$$

(this is because the plate equations only contain the Youngs Modulus and Poissons ratio in the group $Eh^3/(1-\nu^2)$ and density in $\rho h$ ). The units are

$$\omega = 1/s \qquad Eh^3/(1-\nu^2) = kgm^2/s^2 \qquad \rho h = kg/m^2 \quad R = m$$

Therefore

$$\omega^2 R^4 \rho h(1-\nu^2)/Eh^3 = f(.) \qquad\qquad (1/s^2)m^4(kg/m^2)(s^2/kgm^2) = f(.)$$

and hence

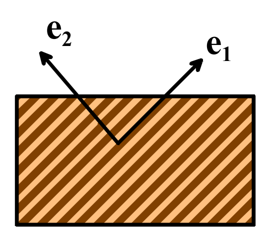
$$\omega^2 R^4 \rho(1-\nu^2)/Eh^2 = C$$

where $C$ is some constant.

You can also re-write the governing equation in dimensionless form and get the same answer.

**3.** The figure shows a fiber reinforced composite laminate.

(i) When loaded in uniaxial tension parallel to the fibers, it fails at a stress of 500MPa.
(ii) When loaded in uniaxial tension transverse to the fibers, it fails at a stress of 250 MPa.
(iii) When loaded at 45 degrees to the fibers, it fails at a stress of 223.6 MPa




The laminate is then loaded in uniaxial tension at 30 degrees to the fibers.
Calculate the expected failure stress under this loading, assuming that the material can be characterized using the Tsai-Hill failure criterion.

1. If the laminate is loaded in uniaxial tension parallel to the fibers, the material fails when $\sigma_{11} = \sigma_{TS1}$. It follows that $\sigma_{TS1} = 500MPa$

2. If the laminate is loaded in uniaxial tension perpendicular to the fibers. The material fails when $\sigma_{22} = \sigma_{TS2}$. It follows that $\sigma_{TS2} = 250MPa$
3. If the laminate in uniaxial tension with stress $\sigma_0$ at 45 degrees to the fibers (horizontally in the figure), we can use the basis change formulas to show that the stresses in the basis aligned parallel and perpendicular to the fibers $\sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_0 / 2$. (See problem 2 from HW4 2018 for details of this calculation). Substituting these into the failure criterion then shows that at failure

$$\left(\frac{\sigma_0/2}{\sigma_{TS1}}\right)^2 + \left(\frac{\sigma_0/2}{\sigma_{TS2}}\right)^2 - \frac{(\sigma_0/2)^2}{\sigma_{TS1}^2} + \frac{(\sigma_0/2)^2}{\sigma_{SS}^2} = 1$$

We can solve this for $\sigma_{SS}$

$$\sigma_{SS} = \sigma_{TS2}\sigma_0 / \sqrt{4\sigma_{TS2}^2 - \sigma_0^2} = \frac{250 \times 223.6}{\sqrt{4 \times 250^2 - 223.6^2}} = 125MPa$$

.

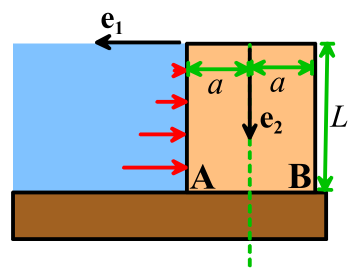
For the 30 degree loading case we have to use the basis change formulas to find the stress components in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis, then substitute the stress components into the failure criterion.

$$\begin{bmatrix} \sigma_{11}^{\mathbf{e}} & \sigma_{12}^{\mathbf{e}} \\ \sigma_{12}^{\mathbf{e}} & \sigma_{22}^{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \sigma_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} = \frac{\sigma}{4} \begin{bmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

$$\Rightarrow \frac{\sigma^2}{16}\left[\left(\frac{3}{500}\right)^2 + \left(\frac{1}{250}\right)^2 - \frac{3}{250^2} + \frac{3}{125^2}\right] = 1$$

$$\Rightarrow \sigma = 285.7MPa$$

**4.** A rectangular dam is subjected to pressure $p(x_2) = \rho_w x_2$ on one face, where $\rho_w$ is the weight density of water. The dam is made from concrete, with weight density $\rho_c$ (and is therefore subjected to a body force $\rho_c \mathbf{e_2}$ per unit volume). The goal is to calculate formulas for $a$ and $L$ to avoid failure.




4.1 Write down the boundary conditions on all four sides of the dam.

(i) On the top surface $\sigma_{22} = \sigma_{12} = 0$

(ii) On the right face $\sigma_{11} = \sigma_{12} = 0$

(iii) On the left face $\sigma_{12} = 0 \qquad \sigma_{11} = -\rho g x_2$

(iv) On the base $u_1 = u_2 = 0$

4.2 Consider the following approximate state of stress in the dam

$$\sigma_{22} = \frac{\rho_w x_2^3 x_1}{4a^3} + \frac{\rho_w x_2 x_1}{20a^3}\left(-10x_1^2 + 6a^2\right) - \rho_c x_2$$

$$\sigma_{11} = -\frac{\rho_w x_2}{2} + \frac{\rho_w x_2 x_1}{4a^3}\left(x_1^2 - 3a^2\right)$$

$$\sigma_{12} = \frac{3\rho_w x_2^2}{8a^3}(a^2 - x_1^2) - \frac{\rho_w}{8a^3}\left(a^4 - x_1^4\right) + \frac{3\rho_w}{20a}(a^2 - x_1^2)$$

Show that (i) The stress state satisfies the equilibrium equations (ii) the stress state exactly satisfies boundary conditions on the sides $x_1 = \pm a$, (iii) The stress does not satisfy the boundary condition on $x_2 = 0$ exactly.

(i) The equilibrium equation (in 2D) is

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + b_1 = 0$$

$$\frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{22}}{\partial x_2} + b_2 = 0$$

Substituting the given functions

$$\frac{\partial}{\partial x_1}\left(-\frac{\rho_w x_2}{2}+\frac{\rho_w x_2 x_1}{4a^3}\left(x_1^2-3a^2\right)\right)+\frac{\partial}{\partial x_2}\left(\frac{3\rho_w x_2^2}{8a^3}(a^2-x_1^2)-\frac{\rho_w}{8a^3}\left(a^4-x_1^4\right)+\frac{3\rho_w}{20a}(a^2-x_1^2)\right)$$

$$=\frac{\rho_w x_2}{4a^3}\left(3x_1^2-3a^2\right)+\frac{3\rho_w x_2}{4a^3}(a^2-x_1^2)=0$$

$$\frac{\partial}{\partial x_1}\left(\frac{3\rho_w x_2^2}{8a^3}(a^2-x_1^2)-\frac{\rho_w}{8a^3}\left(a^4-x_1^4\right)+\frac{3\rho_w}{20a}(a^2-x_1^2)\right)+\frac{\partial}{\partial x_2}\left(\frac{\rho_w x_2^3 x_1}{4a^3}+\frac{\rho_w x_2 x_1}{20a^3}\left(-10x_1^2+6a^2\right)-\rho_c x_2\right)+\rho_c$$

$$=\left(-\frac{3\rho_w x_2^2}{8a^3}x_1^1+\frac{4\rho_w}{8a^3}x_1^3-\frac{3\rho_w}{10a}x_1\right)+\left(\frac{3\rho_w x_2^2 x_1}{4a^3}+\frac{\rho_w x_1}{20a^3}\left(-10x_1^2+6a^2\right)-\rho_c\right)=0$$

The stresses on $x_1=a$ are

$$\sigma_{11}=-\frac{\rho_w x_2}{2}+\frac{\rho_w x_2 a}{4a^3}\left(a^2-3a^2\right)=-\rho_w x_2$$

$$\sigma_{12}=\frac{3\rho_w x_2^2}{8a^3}(a^2-a^2)-\frac{\rho_w}{8a^3}\left(a^4-a^4\right)+\frac{3\rho_w}{20a}(a^2-a^2)=0$$

The stresses on $x_1=-a$ are

$$\sigma_{11}=-\frac{\rho_w x_2}{2}-\frac{\rho_w x_2 a}{4a^3}\left(a^2-3a^2\right)=0$$

$$\sigma_{12}=\frac{3\rho_w x_2^2}{8a^3}(a^2-a^2)-\frac{\rho_w}{8a^3}\left(a^4-a^4\right)+\frac{3\rho_w}{20a}(a^2-a^2)=0$$

On $x_2=0$ we find that $\sigma_{22}=0$ (trivially) but

$$\sigma_{12}=\left\{-\frac{\rho_w}{8a^3}\left(a^4-x_1^4\right)+\frac{3\rho_w}{20a}(a^2-x_1^2)\right\}\neq 0$$

4.3 Show, however, that the resultant *force* acting on $x_2=0$ is zero, so by Saint Venant's principle the stress state will be accurate away from the top of the dam.

The resultant force is

$$F_1=-\int_{-a}^{a}\sigma_{12}\big|_{x_2=0}\,dx_1 \qquad F_2=-\int_{-a}^{a}\sigma_{22}\big|_{x_2=0}\,dx_1$$

$$\Rightarrow F_1=-\int_{-a}^{a}\left\{-\frac{\rho_w}{8a^3}\left(a^4-x_1^4\right)+\frac{3\rho_w}{20a}(a^2-x_1^2)\right\}dx_1$$

$$=\left\{-\frac{\rho_w}{8a^3}\left(2aa^4-\frac{2}{5}a^4\right)+\frac{3\rho_w}{20a}(2aa^2-\frac{2}{3}a^3)\right\}=0$$

$$F_2=-\int_{-a}^{a}0dx_1=0$$

4.4 The concrete cannot withstand any tension. Assuming that the greatest principal tensile stress is located at point A $(x_1 = a, x_2 = L)$, find an expression for the minimum width of the dam, in terms of $L$, $\rho_c, \rho_w$ .

At A:

$$\sigma_{22} = \frac{\rho_w L^3 a}{4a^3} + \frac{\rho_w La}{20a^3}\left(-10a^2 + 6a^2\right) - \rho_c L = L\left(\frac{\rho_w L^2}{4a^2} - \frac{4\rho_w}{20} - \rho_c\right)$$

$$\sigma_{11} = -\frac{\rho_w L}{2} + \frac{\rho_w La}{4a^3}\left(a^2 - 3a^2\right) = -\rho_w L$$

$$\sigma_{12} = 0$$

We require $\sigma_{22} < 0$ so

$$\left(\frac{\rho_w L^2}{4a^2} - \frac{4\rho_w}{20} - \rho_c\right) < 0 \Rightarrow \frac{\rho_w L^2}{4a^2} < \frac{\rho_w}{5} + \rho_c \Rightarrow \frac{a^2}{L^2} > \frac{4}{5} + 4\frac{\rho_c}{\rho_w} \Rightarrow a > L\sqrt{\frac{4}{5} + 4\frac{\rho_c}{\rho_w}}$$

4.5 The concrete fails by crushing when the minimum principal stress reaches $\sigma_{1\min} = -\sigma_c$. Assuming the greatest principal compressive stress is located at point B, $(x_1 = -a, x_2 = L)$ find an expression for the maximum height of the dam.

At B

$$\sigma_{22} = -\frac{\rho_w L^3 a}{4a^3} - \frac{\rho_w La}{20a^3}\left(-10a^2 + 6a^2\right) - \rho_c L = -L\left(\frac{\rho_w L^2}{4a^2} - \frac{4\rho_w}{20} + \rho_c\right)$$

$$\sigma_{11} = -\frac{\rho_w L}{2} - \frac{\rho_w La}{4a^3}\left(a^2 - 3a^2\right) = 0$$

$$\sigma_{12} = 0$$

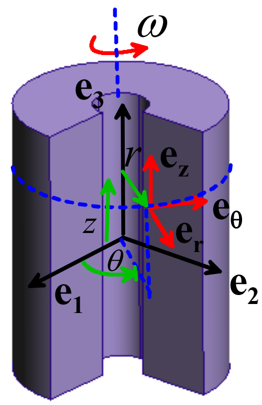
To maximize the dam's height we have to choose *a/L* to be the smallest allowable value, which requires

$$\frac{\rho_w a^2}{4L^2} = \frac{1}{5} + \rho_c$$

Therefore

$$L\left(\frac{\rho_w L^2}{4a^2} - \frac{4\rho_w}{20} + \rho_c\right) < \sigma_c \Rightarrow L < \sigma_c / 2\rho_c$$

**5.** The figure shows a long hollow cylindrical shaft with inner radius $a$ and outer radius $b$, which spins with angular speed $\omega$ about its axis. Assume that the disk is made from an elastic-perfectly plastic material with yield stress $Y$ and density $\rho$ The goal of this problem is to calculate the critical angular speed that will cause the cylinder to collapse.




Plane strain

5.1 Using the cylindrical-polar basis shown, list any stress or strain components that must be zero. Assume plane strain deformation.

Plane strain axisymmetry requires that

$$\varepsilon_{zz} = \varepsilon_{rz} = \varepsilon_{z\theta} = 0$$
$$\sigma_{r\theta} = \sigma_{z\theta} = \sigma_{rz} = 0$$

5.2 Write down the boundary conditions that the stress field must satisfy at $r=a$ and $r=b$

The inner and outer surfaces must be stress free, so that
$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0$

5.3 Write down the linear momentum balance equation in terms of the stress components, the angular velocity and the disk's density.

The EOM is $\nabla \cdot \boldsymbol{\sigma} = \rho \mathbf{a}$ and (circular motion) $\mathbf{a} = -\omega^2 r \mathbf{e}_r$

In cylindrical coordinates the divergence of a symmetric tensor is

$$\nabla \cdot \mathbf{S} \equiv \begin{bmatrix} \frac{\partial S_{rr}}{\partial r} + \frac{S_{rr}}{r} + \frac{1}{r}\frac{\partial S_{\theta r}}{\partial \theta} + \frac{\partial S_{zR}}{\partial z} - \frac{S_{\theta\theta}}{r} \\ \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial \theta} + \frac{\partial S_{r\theta}}{\partial r} + \frac{S_{r\theta}}{r} + \frac{S_{\theta r}}{r} + \frac{\partial S_{z\theta}}{\partial z} \\ \frac{\partial S_{zz}}{\partial z} + \frac{\partial S_{rz}}{\partial r} + \frac{S_{rz}}{r} + \frac{1}{r}\frac{\partial S_{\theta z}}{\partial \theta} \end{bmatrix}$$

If we assume cylindrical symmetry the only nontrivially satisfied equation is the first one. The linear momentum equation therefore reduces to

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = -\rho r \omega^2$$

5.4 Using the plastic flow rule, show that $\sigma_{zz} = (\sigma_{rr} + \sigma_{\theta\theta})/2$ if the cylinder deforms plastically under plane strain conditions

The plastic flow rule gives

$$\frac{d\varepsilon_{ij}^{p}}{dt} = \frac{d\varepsilon_e}{dt}\frac{S_{ij}}{\sigma_e}$$

Since the axial strain rate is zero $\frac{d\varepsilon_{zz}^{p}}{dt} = 0 \Rightarrow S_{zz} = 0$

Note that $S_{zz} = \sigma_{zz} - \frac{1}{3}(\sigma_{rr} + \sigma_{zz} + \sigma_{\theta\theta}) = 0 \Rightarrow \sigma_{zz} = \frac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta})$

5.5 Using Von-Mises yield criterion, show that the radial and hoop stress must satisfy $|\sigma_{\theta\theta} - \sigma_{rr}| = 2Y/\sqrt{3}$

Using the preceding result the Von Mises stress is

$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{rr} - \sigma_{zz})^2 + (\sigma_{\theta\theta} - \sigma_{zz})^2\right]}$$

$$= \sqrt{\frac{1}{2}\left[(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{rr} - \sigma_{\theta\theta})^2/4 + (\sigma_{\theta\theta} - \sigma_{rr})^2/4\right]} = \sqrt{3}|\sigma_{\theta\theta} - \sigma_{rr}|/2$$

Setting $\sigma_e = Y$ gives the answer stated.

5.6 Hence, show that the radial stress must satisfy the equation

$$\frac{d\sigma_{rr}}{dr} = -\rho r\omega^2 + \frac{2}{\sqrt{3}}\frac{Y}{r}$$

This is just a straight substitution of 5.5 in 5.3

5.7 Hence, calculate the critical angular speed that will cause plastic collapse.

Integrate the stresses (using the boundary conditions)

$$\frac{d\sigma_{rr}}{dr} = -\rho r\omega^2 + \frac{2}{\sqrt{3}}\frac{Y}{r} \Rightarrow \int_0^0 d\sigma_{rr} = \int_a^b \left( -\rho r\omega^2 + \frac{2}{\sqrt{3}}\frac{Y}{r} \right) dr$$

$$\Rightarrow -\rho\omega^2 (b^2 - a^2)/2 + 2Y \log(b/a)/\sqrt{3} = 0$$

$$\Rightarrow \omega = 2\sqrt{Y \log(b/a) / \rho(b^2 - a^2)\sqrt{3}}$$