School of Engineering
Brown University

# EN175: Advanced Mechanics of Solids

## Practice Midterm Examination

**NAME:** ___________________________________________

## General Instructions

- No collaboration of any kind is permitted on this examination.
- Two double sided pages of reference notes are permitted
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

## Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

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**1. (8 points)** ______________

**2. (5 points)** ______________

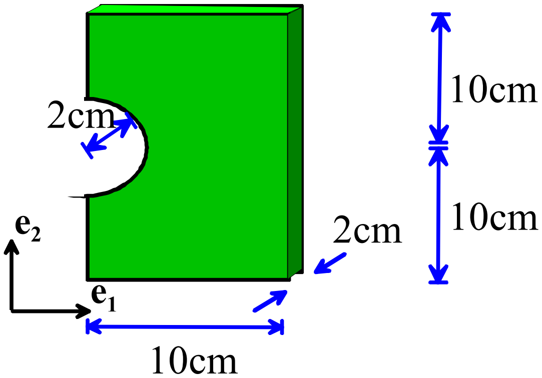
**3. (6 points)** ______________

**4. (3 points)** ______________

**5. (4 points)** ______________

**TOTAL (25 points)** ____________

**1.** You are asked to conduct a finite element analysis of an edge notched compact tension specimen, illustrated in the picture below. The specimen is made from steel, and will be loaded in tension by clamping the top and bottom edges of the specimen in the grips of a tensile testing machine. You are to determine the load that will initiate plastic deformation in the solid.



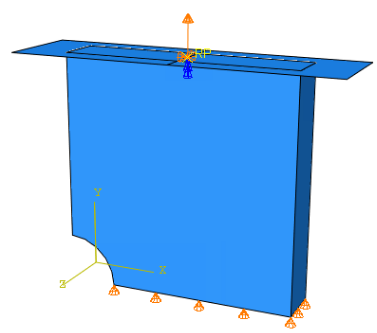

1.1 Write out the field equations that determine the solution (e.g. the equations of stress equilibrium, and other governing equations)

The equations are (1) Strain displacement relation $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$

(2) Stress-strain relation; for steel we would use isotropic linear elasticity so $\sigma_{ij} = \frac{E}{1+\nu}\left\{\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right\}$

(3) Equilibrium $\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}$ (there are no body forces; we wouldn't consider gravity….)

1.2 Draw a sketch showing the region of the specimen that you will model (if you plan to model the whole thing, sketch the entire specimen). Using your sketch, state the boundary conditions that you will apply.



We would use symmetry; the sides of the specimen are stress free; vertical motion of the symmetry plane should be constrained, and a load should be applied to the top of the specimen. Some constraint should be present to eliminate rigid body modes. This could be done in various ways; it's hard to know exactly what the grips are going to do to the top of the specimen. For analysis purposes it would be convenient to bond the top to a rigid surface, so that the forces applied to the grips can be calculated, or a prescribed force could be applied to the top. In the example shown a rigid surface is tied to the top face; the reference point has U1=U3=0 and a prescribed value for U2 (a load could be applied in the 2 direction as well); and UR2 =0 to remove the rotational mode about the y axis.

Applying a vertical traction to the top face is fine too; in that case you'd have to fix u1=u3=0 at a corner somewhere, and prescribe either u1 or u3 at the other end of a horizontal edge passing through this corner to constrain rigid body motion.

1.3 Select an appropriate element type for your computation (you don't need to specify the ABAQUS name for the element, simply state, e.g. that you will use axisymmetric constant strain triangles)

We would use 3D quadratic hex elements for the most accurate solution. The elements could be reduced integration or fully integrated elements; the difference between the two should be small. We might use a plane stress approximation for a quicker simulation (2D plane stress quadratic hex elements would be best in this case)

1.4 Explain briefly how you would determine the load required to initiate plastic deformation from your finite element solution.

(1) Run a simulation with an arbitrary load $P$ applied to the top of the specimen
(2) Find the max Mises stress $\sigma_e$
(3) We know that $\sigma_e$ is proportional to $P$ (since solutions to small strain elasticity problems are linear) so if the yield stress is $Y$, the load required to cause yield is

$$P_y = \frac{P}{\sigma_e} Y$$

**3.** A solid with volume $V$ is subjected to a displacement field (which corresponds to a homogeneous deformation) given by

$$u_1 = \lambda_1 x_1 + a_1 \qquad u_2 = \lambda_2 x_2 + a_2 \qquad u_3 = \lambda_3 x_3$$

Calculate formulas for

(3.1) The components of the deformation gradient tensor

$$\mathbf{F} = \nabla \mathbf{u} + \mathbf{I} = \begin{bmatrix} \lambda_1 + 1 & 0 & 0 \\ 0 & \lambda_2 + 1 & 0 \\ 0 & 0 & \lambda_3 + 1 \end{bmatrix}$$

(3.2) The volume of the solid after deformation

$$V \det(\mathbf{F}) = (1+\lambda_1)(1+\lambda_2)(1+\lambda_3)V$$

(3.3) The components of the Lagrange strain tensor

$$\mathbf{E} = (\mathbf{F}^T \mathbf{F} - \mathbf{I})/2 = \frac{1}{2} \begin{bmatrix} (\lambda_1 + 1)^2 - 1 & 0 & 0 \\ 0 & (\lambda_2 + 1)^2 - 1 & 0 \\ 0 & 0 & (\lambda_3 + 1)^2 - 1 \end{bmatrix}$$

(3.4) The components of the infinitesimal strain tensor

$$\boldsymbol{\varepsilon} = [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]/2 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

(3.5) Using the Lagrange strain tensor, calculate the length after deformation of a line element that has unit length and direction $\mathbf{m} = (3\mathbf{e}_1 + 4\mathbf{e}_3)/5$ in the undeformed solid

Use the formula

$$\frac{l^2 - l_0^2}{2l_0^2} = \mathbf{m} \cdot \mathbf{E} \cdot \mathbf{m} = \frac{1}{50}(9\left[(\lambda_1 + 1)^2 - 1\right] + 16\left[(\lambda_3 + 1)^2 - 1\right]$$

$$\Rightarrow l = \sqrt{1 + \frac{1}{25}(9\left[(\lambda_1 + 1)^2 - 1\right] + 16\left[(\lambda_3 + 1)^2 - 1\right]}$$

**4.** The matrix below gives components of a stress tensor (in MPa) in a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

$$\begin{bmatrix} 300 & -100 & 0 \\ -100 & 300 & 0 \\ 0 & 0 & 300 \end{bmatrix}$$

Calculate

(4.1) The hydrostatic stress

$$\sigma_h = \text{trace}(\boldsymbol{\sigma})/3 = 300$$

(4.2) The deviatoric stress tensor

$$\mathbf{S} = \boldsymbol{\sigma} - \text{trace}(\boldsymbol{\sigma})/3 = \begin{bmatrix} 0 & -100 & 0 \\ -100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(4.3) The principal stresses and the components of the principal stress directions in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis

The matrix happens to have nice eigenvalues – set det( $\det(\boldsymbol{\sigma} - \lambda \mathbf{I}) = 0$ which gives

$$(300-\lambda)\left[(300-\lambda)^2-100^2\right]=0$$
$$\Rightarrow (300-\lambda)\left[\lambda^2-600\lambda+80000\right]=0$$
$$\Rightarrow (300-\lambda)\left[\lambda^2-600\lambda+800^2\right]=0$$
$$\Rightarrow (300-\lambda)(\lambda-200)(\lambda-400)=0$$

So principal stresses are $\sigma_1=400, \sigma_2=300, \sigma_3=200$

The principal stress directions can be found by looking for the null vectors of

$$\begin{bmatrix} 300-\lambda & -100 & 0 \\ -100 & 300-\lambda & 0 \\ 0 & 0 & 300-\lambda \end{bmatrix}$$

Which gives $\mathbf{m}^{(1)}=[1,-1,0]/\sqrt{2},\ \mathbf{m}^{(2)}=[0,0,1],\ \mathbf{m}^{(3)}=[1,1,0]/\sqrt{2}$

(4.4) The Von-Mises stress

For this stress it's easiest to use the formula in terms of deviatoric stress

$$\sigma_e=\sqrt{\frac{3}{2}\mathbf{S}:\mathbf{S}}=\sqrt{\frac{3}{2}\left(2\times 100^2\right)}=100\sqrt{3}$$

(4.5) The internal traction vector acting on an internal surface with normal $\mathbf{n}=\left(\mathbf{e}_1+\mathbf{e}_2\right)/\sqrt{2}$

$$\mathbf{t}=\mathbf{n}\cdot\boldsymbol{\sigma}=\frac{1}{\sqrt{2}}[1,1,0]\begin{bmatrix} 300 & -100 & 0 \\ -100 & 300 & 0 \\ 0 & 0 & 300 \end{bmatrix}=200[1,1,0]/\sqrt{2}$$

(4.6) The normal and tangential components of traction acting on an internal surface with normal $\mathbf{n}=\left(\mathbf{e}_1+\mathbf{e}_2\right)/\sqrt{2}$

$$t_n=\mathbf{t}\cdot\mathbf{n}=200$$
$$\mathbf{t}_t=\mathbf{t}-t_n\mathbf{n}=\mathbf{0}$$

You can get 4.5 and 4.6 directly from the principal stresses as well, of course, because $\mathbf{n}=\left(\mathbf{e}_1+\mathbf{e}_2\right)/\sqrt{2}$ is one of the principal stress directions.

**5.** State whether the stress field given below satisfies the equations of static equilibrium

$$\sigma_{11} = A\left(x_1^2 x_2 - 2x_2^3/3\right) \qquad \sigma_{22} = A(x_2^3/3 - c^2 x_2 + 2c^3/3) \qquad \sigma_{12} = \sigma_{21} = A(c^2 - x_2^2)x_1$$

(with all other components zero).

Equilibrium is

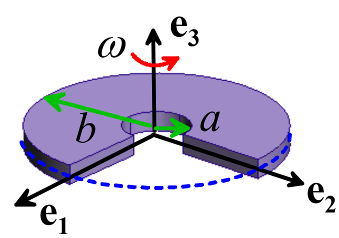
$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} = 2Ax_1x_2 - 2Ax_1x_2 = 0$$
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} = A(c^2 - x_2^2) + A(x_2^2 - c^2) = 0$$

So equilibrium is satisfied (assuming no body forces, of course)

**6.** The stress field in a *solid* disk with outer radius *b*, mass density $\rho_0$ and Poisson's ratio $\nu$ that spins with angular speed $\omega$ is given by

$$\sigma_{rr} = (3+\nu)\frac{\rho_0\omega^2}{8}\left\{b^2 - r^2\right\} \qquad \sigma_{\theta\theta} = \frac{\rho_0\omega^2}{8}\left\{(3+\nu)b^2 - (3\nu+1)r^2\right\}$$

The stress field in a thin disk with radius *b* with a hole of radius *a* in its center, subjected to pressure $p_a$ on the interior of the hole, is given by




$$\sigma_{rr} = -p_a \frac{\left(b^2 - r^2\right)a^2}{\left(b^2 - a^2\right)r^2} \qquad \sigma_{\theta\theta} = p_a \frac{\left(b^2 + r^2\right)a^2}{\left(b^2 - a^2\right)r^2}$$

Use the principle of superposition to determine the stress state in a spinning disk with a (traction free) hole with radius *a* at its center.

We need to add the two solutions to satisfy $\sigma_{rr} = 0 \quad r = a$

$$\sigma_{rr} = (3+\nu)\frac{\rho_0\omega^2}{8}\left\{b^2 - a^2\right\} - p_a \frac{\left(b^2 - a^2\right)a^2}{\left(b^2 - a^2\right)a^2} = 0$$
$$\Rightarrow p_a = (3+\nu)\frac{\rho_0\omega^2}{8}\left\{b^2 - a^2\right\}$$

Hence

$$\sigma_{rr} = (3+\nu)\frac{\rho_0\omega^2}{8}\left(b^2 - r^2\right)\left\{1 - \frac{a^2}{r^2}\right\}$$
$$\sigma_{\theta\theta} = \frac{\rho_0\omega^2}{8}\left\{(3+\nu)\left[b^2 + a^2 + \frac{a^2b^2}{r^2}\right] - (3\nu+1)r^2\right\}$$