EN175: Advanced Mechanics of Solids

Midterm Examination
Oct 31 2019

School of Engineering
Brown University

NAME: ____

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may use two pages of reference notes
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

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1. (5 points) _____________

2. (7 points) _____________

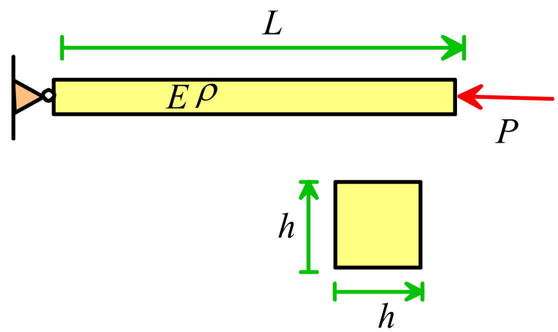
3. (5 points) _____________

4. (8 points) _____________

5. (10 points) _____________

TOTAL (35 points)_____________

1. The figure shows a beam with square cross-section hxh and length L, made from a material with Young's modulus E and mass density ρ. The beam is subjected to an axial force P. Its natural frequency of vibration can be written as a function



$$\omega = f(Eh^4, \rho, h, L, P)$$

1.1 Re-write the equation $\omega = f(Eh^4, \rho, h, L, P)$
in dimensionless form (there is more than one possible solution – any solution is fine.

The units of the variables are

$$\omega = T^{-1} \qquad Eh^4 = (mLT^{-2})L^{-2}L^4 = mL^3T^{-2} \qquad \rho = mL^{-3}$$
$$h = L \qquad L = L \qquad P = mLT^{-2}$$

So variables can be combined as

$$\omega L^3 \sqrt{\frac{\rho}{Eh^4}} = f(\frac{L}{h}, \frac{PL^2}{Eh^4})$$

Other possibilities exist.

[3 POINTS]

1.2 A beam with $h=1cm$, $L=1m$, $E=210GPa$, $\rho = 1000\ kg.m^{-3}$ subjected to an axial load P=200N has a natural frequency of 100 Hz. If h and L are both doubled, and P is increased by a factor of 4, what is the new natural frequency?

Let h,L,E,P denote the original values of the variables. Then

$$\omega_1 = \frac{1}{L^3}\sqrt{\frac{Eh^4}{\rho}} f(\frac{L}{h}, \frac{PL^2}{Eh^4})$$
$$\omega_2 = \frac{1}{(2L)^3}\sqrt{\frac{E(2h)^4}{\rho}} f(\frac{2L}{2h}, \frac{4P(2L)^2}{E(2h)^4}) = \frac{4}{2^3}\frac{1}{L^3}\sqrt{\frac{Eh^4}{\rho}} f(\frac{L}{h}, \frac{PL^2}{Eh^4}) = 0.5\omega_1 = 50Hz$$

[2 POINTS]

2. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors, let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be tensors, and let $\mathbf{I}$ denote the identity tensor. Express the following equations in index notation

(a) $\eta = \mathbf{a} \cdot \mathbf{b}$

$\eta = a_i b_i$

(b) $\mathbf{c} = \mathbf{A}\mathbf{b}$

$c_i = A_{ik} b_k$

(c) $\mathbf{C} = \mathbf{B}^T \mathbf{A}$

$C_{ij} = B_{ki} A_{kj}$

(d) $\mathbf{a} = \mathbf{I}\mathbf{a}$

$a_i = \delta_{ij} a_j$

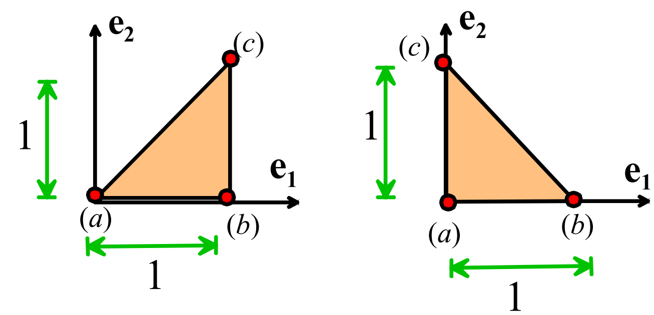
4 POINTS]

Hence, use index notation to show that $\mathbf{b} \cdot \left(\mathbf{A}^T \mathbf{A} - \mathbf{I} \right) \mathbf{b} = \left(\mathbf{A}\mathbf{b} \right) \cdot (\mathbf{A}\mathbf{b}) - \mathbf{b} \cdot \mathbf{b}$.

$$\mathbf{b} \cdot \left(\mathbf{A}^T \mathbf{A} - \mathbf{I} \right) \mathbf{b} = b_i (A_{ki} A_{kj} - \delta_{ij}) b_j = \left(A_{ki} b_i \right) \left(A_{kj} b_j \right) - b_i b_i = \left(\mathbf{A}\mathbf{b} \right) \cdot (\mathbf{A}\mathbf{b}) - \mathbf{b} \cdot \mathbf{b}$$

[3 POINTS]

3. The figure shows a plane 2D triangular constant strain finite element before and after deformation. Calculate the Lagrange strain in the element.



We can use the lengths of the sides of the element and the relation between Lagrange strain and the lengths of material fibers before and after deformation

$$\frac{l^2 - l_0^2}{2l_0^2} = \mathbf{m} \cdot (\mathbf{Em})$$

For side (ab)

$$0 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = E_{11} \Rightarrow E_{11} = 0$$

For side (bc)

$$\frac{2-1}{2} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = E_{22} \Rightarrow E_{22} = 1/2$$

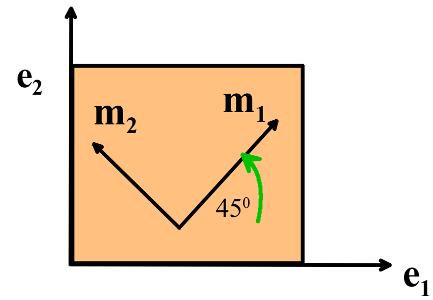
For side (ac)

$$\frac{1-2}{4} = \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{E_{11} + E_{22} + 2E_{12}}{2} \Rightarrow \frac{2E_{12} + 1/2}{2} = -1/4 \Rightarrow E_{12} = -\frac{1}{2}$$

[5 POINTS]

4. A sheet of material that lies in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ plane is subjected to a state of stress with the following properties:

- The stress is in a state of *plane stress*
- The principal stress directions are parallel to the $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ directions shown in the figure
- The hydrostatic stress is zero
- The Von-Mises stress has magnitude 250 MPa



4.1 For the plane stress state, which components of stress are zero?

$\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$

[1 POINT]

4.2 Write down the formulas for hydrostatic stress and von-Mises stress in terms of the principal stresses. Hence, find the components of stress in the $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ basis (i.e. the principal basis). If you find more than one possible solution give them all…

In this basis the first two conditions show that the stress state is $\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The hydrostatic stress is zero so $\sigma_{11} + \sigma_{22} = 0 \Rightarrow \sigma_{11} = \sigma \quad \sigma_{22} = -\sigma$

The von-Mises stress is $\sigma_e = \sqrt{\frac{1}{2}\left\{(\sigma_{11} - \sigma_{22})^2 + \sigma_{11}^2 + \sigma_{22}^2\right\}} = \sqrt{3}|\sigma|$

$\Rightarrow \sigma = \pm 250/\sqrt{3}$

[3 POINTS]

4.3 Hence, find the stress components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis

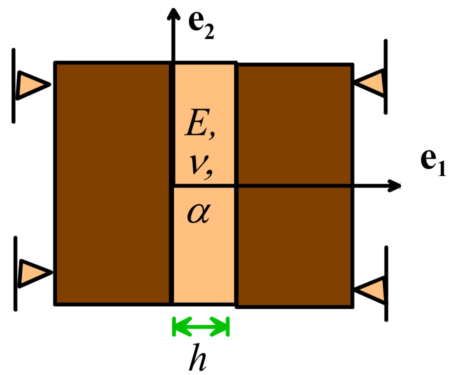
We can use the basis change formulas

$$\begin{bmatrix} \sigma_{11}^{\mathbf{e}} & \sigma_{12}^{\mathbf{e}} \\ \sigma_{21}^{\mathbf{e}} & \sigma_{22}^{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{m}_1 & \mathbf{e}_1 \cdot \mathbf{m}_2 \\ \mathbf{e}_2 \cdot \mathbf{m}_1 & \mathbf{e}_2 \cdot \mathbf{m}_2 \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ 0 & -\sigma \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{m}_1 & \mathbf{e}_2 \cdot \mathbf{m}_1 \\ \mathbf{e}_1 \cdot \mathbf{m}_2 & \mathbf{e}_2 \cdot \mathbf{m}_2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ 0 & -\sigma \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma & \sigma \\ \sigma & -\sigma \end{bmatrix} = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}$$

[4 POINTS]

5. The figure shows a thin elastic film that is bonded between two large rigid solids, which are held fixed. The film has Young's modulus E, Poisson's ratio ν and thermal expansion coefficient α. The rigid solids have zero thermal expansion coefficient.



The film is stress free at some reference temperature. The solid on the right is heated, which causes the temperature of the film to increase by

$$\Delta T = \lambda x_1$$

where λ is a constant.

The goal of this problem is to calculate the displacement and stress fields in the film. You can assume that the displacement in the film is only in the $\mathbf{e}_1$ direction, and is only a function of x_1 i.e.

$$\mathbf{u} = u(x_1)\mathbf{e}_1$$

5.1 Find a formula the 3x3 infinitesimal strain tensor (matrix) in the film, in terms of derivatives of the unknown displacement u

The strain tensor is $\begin{bmatrix} \partial u / \partial x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

[1 POINT]

5.2 Hence, find a formula for the stress components in the film in terms of u (and its derivatives) and an appropriate sub-set of $(E, \nu, \alpha, \lambda, x_1)$.

We can use the elastic stress-strain relations

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} - \frac{E\alpha\Delta T}{1-2\nu} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}(1-\nu)\partial u/\partial x_1\\ \nu\partial u/\partial x_1\\ \nu\partial u/\partial x_1\\ 0\\ 0\\ 0\end{bmatrix}-\frac{E\alpha\lambda x_1}{1-2\nu}\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}=\frac{E}{(1-2\nu)}\begin{bmatrix}(1-\nu)/(1+\nu)\partial u/\partial x_1-\alpha\lambda x_1\\ \nu/(1+\nu)\partial u/\partial x_1-\alpha\lambda x_1\\ \nu/(1+\nu)\partial u/\partial x_1-\alpha\lambda x_1\\ 0\\ 0\\ 0\end{bmatrix}$$

[2 POINTS]

5.3 Write down the three equations of static equilibrium in the film. Show that two of them are satisfied trivially, and express the third one in terms of *u* (and its derivatives) and $(E,\nu,\alpha,\lambda,x_1)$.

Only one component of the equilibrium equation is nonzero

$$\frac{\partial\sigma_{11}}{\partial x_1}=0\Rightarrow\frac{1-\nu}{1+\nu}\frac{\partial^2 u}{\partial x_1^2}-\alpha\lambda=0$$

[1 POINT]

5.4 Write down the boundary conditions for *u* at $x_1=0 \quad x_1=h$

The rigid platens prevent the film from displacing, so *u=0* at $x_1=0 \quad x_1=h$

[1 POINT]

5.5 Hence, solve (5.3) and (5.4) to calculate $u(x_1)$

The solution to the ODE is

$$u=\frac{1+\nu}{1-\nu}\alpha\lambda\frac{1}{2}x_1^2+Ax_1+B$$

And the boundary conditions give *B=0*, $A=-\frac{1+\nu}{1-\nu}\alpha\lambda\frac{1}{2}h$

So we have that

$$u=\frac{1+\nu}{1-\nu}\frac{\alpha\lambda}{2}(x_1^2-hx_1)$$

[3 POINTS]

5.6 Hence, find a formula for the stresses in the film, in terms of E, ν, α, λ

Substituting back into 5.5 gives

$$\sigma_{11} = \frac{E}{1-2\nu}\left[\frac{(1-\nu)}{(1+\nu)}\frac{\partial u}{\partial x_1} - \alpha\lambda x_1\right] = -\frac{E\alpha\lambda h}{2(1-2\nu)}$$

$$\sigma_{22} = \sigma_{33} = \frac{E}{1-2\nu}\left[\frac{\nu}{(1+\nu)}\frac{\partial u}{\partial x_1} - \alpha\lambda x_1\right] = \frac{\alpha\lambda E}{1-2\nu}\left[\frac{\nu}{(1-\nu)}\frac{1}{2}(2x_1 - h) - x_1\right]$$

[2 POINTS]