



EN1750: Advanced Mechanics of Solids

Homework 2: Math review; tensors; index notation.

Due Friday Sept 28, 2018

1. The figure shows a triangular finite element. The displacement vector (as Cartesian components in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis, in arbitrary units) at each of its corners is $\mathbf{u}^{(1)} = [0, 0]$; $\mathbf{u}^{(2)} = [0.1, 0.1]$; $\mathbf{u}^{(3)} = [0, 0.2]$. The displacement at an arbitrary point inside the element is computed using a linear interpolation between values at the three corners (this only works for the triangle shown, it is not a general formula for all triangles)

$$\mathbf{u} = \frac{1}{2}(2 - x_2)\mathbf{u}^{(1)} + x_1\mathbf{u}^{(2)} + \frac{1}{2}(x_2 - 2x_1)\mathbf{u}^{(3)}$$

where \mathbf{x} is the position of a point in the element before deformation.

1.1 Find the components of $\nabla \mathbf{u}$ (give the answer as a 2×2 matrix)

We can use the formula

$$\nabla \mathbf{v} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

Here we just have a 2D problem so

$$\nabla \mathbf{u} = \begin{bmatrix} u_1^{(2)} - u_1^{(3)} & -u_1^{(1)}/2 + u_1^{(3)}/2 \\ u_2^{(2)} - u_2^{(3)} & -u_2^{(1)}/2 + u_2^{(3)}/2 \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ -0.1 & 0.1 \end{bmatrix}$$

[2 POINTS]

1.2 Use the gradient to calculate $\mathbf{u}^a - \mathbf{u}^b$, where the two points a and b are shown in the figure.

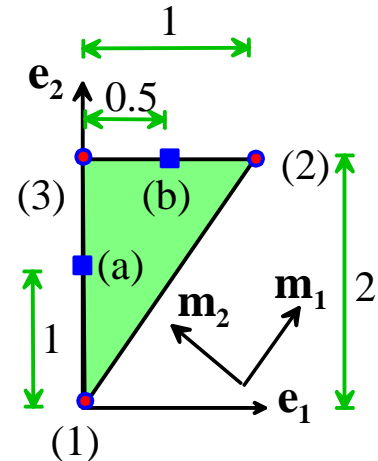
The gradient is constant, so we can write

$$\mathbf{u}^a - \mathbf{u}^b = \nabla \mathbf{u} \cdot (\mathbf{x}^a - \mathbf{x}^b) = \begin{bmatrix} 0.1 & 0 \\ -0.1 & 0.1 \end{bmatrix} \begin{bmatrix} -0.5 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.05 \\ -0.05 \end{bmatrix}$$

[1 POINT]

1.3 Calculate the divergence of \mathbf{u} (i.e. $\nabla \cdot \mathbf{u}$)

Use the formula again



$$\nabla \cdot \mathbf{u} = \text{trace}(\nabla \mathbf{u}) = \text{trace} \begin{bmatrix} 0.1 & 0 \\ -0.1 & 0.1 \end{bmatrix} = 0.2$$

[1 POINT]

1.4 Find

$$\int_A \nabla \cdot \mathbf{u} dA$$

where A denotes the area of the triangle. Check your answer using the divergence theorem; i.e. calculate

$$\int_C \mathbf{n} \cdot \mathbf{u} ds$$

around the perimeter of the triangle

The divergence is constant, and the area of the triangle is 1, so

$$\int_A \nabla \cdot \mathbf{u} dA = 0.2$$

To do this by integrating around the triangle, note that (taking arc-length counterclockwise)

$$\text{On side (1)-(2)} \quad \mathbf{u} \cdot \mathbf{n} = \mathbf{u}^{(2)} \cdot \mathbf{n}(s/\sqrt{5}) = [0.1, 0.1] \begin{bmatrix} 2 \\ -1 \end{bmatrix} \frac{s}{5} = 0.1s/5$$

On side (2)-(3)

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{u}^{(2)} \cdot \mathbf{n}(1-s) + \mathbf{u}^{(3)} \cdot \mathbf{n}s = [0.1, 0.1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1-s) + [0, 0.2] \begin{bmatrix} 0 \\ 1 \end{bmatrix} s = 0.1(1-s) + 0.2s = 0.1(1+s)$$

On side (1)-(3) $\mathbf{u} \cdot \mathbf{n} = 0$

Hence

$$\int_C \mathbf{n} \cdot \mathbf{u} ds = \int_0^{\sqrt{5}} 0.1s/5 ds + \int_0^1 0.1(1+s) ds = 0.2$$

[3 POINTS]

1.5 Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be unit vectors parallel and perpendicular to the side (1-2) of the triangle. Calculate the components of $\nabla \mathbf{u}$ in this basis. Check your answer by calculating $\nabla \cdot \mathbf{u}$ using the components in $\{\mathbf{m}_1, \mathbf{m}_2\}$.

$$\text{From the figure } \mathbf{m}_1 = (\mathbf{e}_1 + 2\mathbf{e}_2)/\sqrt{5} \quad \mathbf{m}_2 = (-2\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{5}$$

Use the formula from class

$$\begin{aligned} [\nabla \mathbf{v}]^{(\mathbf{m})} &= \begin{bmatrix} \mathbf{m}_1 \cdot \mathbf{e}_1 & \mathbf{m}_1 \cdot \mathbf{e}_2 \\ \mathbf{m}_2 \cdot \mathbf{e}_1 & \mathbf{m}_2 \cdot \mathbf{e}_2 \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ -0.1 & 0.1 \end{bmatrix} \begin{bmatrix} \mathbf{m}_1 \cdot \mathbf{e}_1 & \mathbf{m}_1 \cdot \mathbf{e}_2 \\ \mathbf{m}_2 \cdot \mathbf{e}_1 & \mathbf{m}_2 \cdot \mathbf{e}_2 \end{bmatrix}^T \\ &= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ -0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0.06 & 0.08 \\ -0.02 & 0.14 \end{bmatrix} \end{aligned}$$

The divergence is 0.2 again, as it should be (the divergence has to be independent of the basis)

[2 POINTS]

2. Suppose that \mathbf{a} and \mathbf{b} are two mutually perpendicular vectors (i.e. $\mathbf{a} \cdot \mathbf{b} = 0$). Create a tensor \mathbf{S} by forming the dyadic product

$$\mathbf{S} = \mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}$$

2.1 Is \mathbf{S} symmetric?

Yes, both terms are symmetric

[1 POINT]

2.2 Let $\alpha = |\mathbf{a}|, \beta = |\mathbf{b}|$. What are the three eigenvalues of \mathbf{S} in terms of α, β , and what are the corresponding eigenvectors? (Answer this by inspection, don't try to use the formulas)

Recall that eigenvalues satisfy $\mathbf{S} \cdot \mathbf{m} = \lambda \mathbf{m}$. Note \mathbf{a} and \mathbf{b} are mutually perpendicular, and using the definition of dyadic product

$$\mathbf{S}\mathbf{a} = \mathbf{a}(\mathbf{a} \cdot \mathbf{a}) + \mathbf{b}(\mathbf{b} \cdot \mathbf{a}) = \alpha^2 \mathbf{a}$$

$$\mathbf{S}\mathbf{b} = \mathbf{a}(\mathbf{a} \cdot \mathbf{b}) + \mathbf{b}(\mathbf{b} \cdot \mathbf{b}) = \beta^2 \mathbf{b}$$

so the eigenvectors are \mathbf{a} / α , \mathbf{b} / β , and the eigenvalues are α^2, β^2 .

[3 POINTS]

3. Which of the following equations are valid expressions using index notation? If you decide an expression is invalid, state which rule is violated.

(a) $\sigma_{ij} = C_{ijkl} \varepsilon_{ij}$

(b) $\varepsilon_{kkk} = 0$

(c) $\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho \frac{\partial^2 u_i}{\partial t^2}$

(a) is incorrect; the free indices on the left and right hand side do not agree

(b) is incorrect, an index is repeated three times

(c) is fine.

[3 POINTS]

4. What is the value of $\varepsilon_{ijk} \varepsilon_{ijk}$?

$$\text{Recall } \varepsilon_{ijk} = \begin{cases} 1 & i, j, k = 1, 2, 3; \quad 2, 3, 1 \text{ or } 3, 1, 2 \\ -1 & i, j, k = 3, 2, 1; \quad 2, 1, 3 \text{ or } 1, 3, 2 \\ 0 & \text{otherwise} \end{cases}$$

$\varepsilon_{ijk} \varepsilon_{ijk}$ is the sum of the squares of all the terms in ε_{ijk} ; i.e. 6

[2 POINTS]

5. Match the meaning of each index notation expression shown below with an option from the list


- (a) $\lambda = T_{ij}S_{ij}$
- (b) $E_{ij} = T_{ik}S_{kj}$
- (c) $E_{ij} = S_{ki}T_{kj}$
- (d) $a_i = \epsilon_{kij} b_j c_k$
- (e) $\lambda = a_i b_i$
- (f) δ_{ij}
- (g) $T_{ij}m_j = \lambda m_i$
- (h) $a_i = S_{ij}b_j$
- (i) $A_{ki}A_{kj} = \delta_{ij}$
- (j) $A_{ij} = A_{ji}$

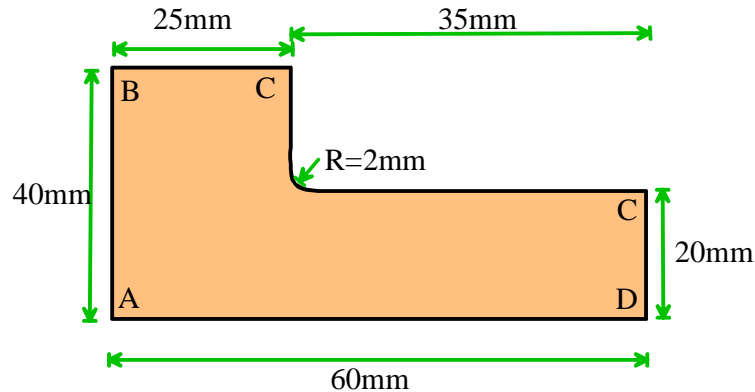
- (1) Product of two tensors
- (2) Product of the transpose of a tensor with another tensor
- (3) Cross product of two vectors
- (4) Product of a vector and a tensor
- (5) Components of the identity tensor
- (6) Equation for the eigenvalues and eigenvectors of a tensor
- (7) Contraction of a tensor
- (8) Dot product of two vectors
- (9) The definition of an orthogonal tensor
- (10) Definition of a symmetric tensor

- (a) $\lambda = T_{ij}S_{ij}$ Inner product of two tensors
- (b) $E_{ij} = T_{ik}S_{kj}$ Product of two tensors
- (c) $E_{ij} = S_{ki}T_{kj}$ Product of the transpose of **S** and **T**
- (d) $a_i = \epsilon_{kij} b_j c_k$ Cross product
- (e) $\lambda = a_i b_i$ Dot product
- (f) δ_{ij} Identity
- (g) $T_{ij}m_j = \lambda m_i$ Definition of eigenvector/value
- (h) $a_i = S_{ij}b_j$ Product of a tensor and a vector
- (i) $A_{ki}A_{kj} = \delta_{ij}$ Definition of an orthogonal tensor
- (j) $A_{ij} = A_{ji}$ Definition of a symmetric tensor.

[5 POINTS]

6. FEA Mesh Design: The objective of this problem is to investigate the influence of element size on the FEA predictions of stresses near a stress concentration. Follow these steps to set up the problem in ABAQUS/CAE

1. Create the 2D part shown below (select the 2D button when creating the part, and make the little rounded radius by creating a fillet radius tool  after you have drawn the profile with straight lines everywhere. Enter a radius of 2mm for the fillet radius)



2. Use a linear elastic constitutive equation with $E = 210\text{GNm}^{-2}$, $\nu = 0.3$ (use units of N for force and mm for length; this means that any quantities with stress units should be entered as N/mm^2); create a 2D homogeneous section and assign it to the part.
3. Create an instance of the part in the Assembly module
4. Create a General Static step with time period 1 and NLGEOM off (the defaults)
5. Skip the Interaction module
6. For boundary conditions, use zero horizontal displacement on AB, zero vertical displacement on AD, and apply a uniform horizontal displacement of 0.01mm on CD.

Run computations with the following meshes:

1. Seed the part using the default spacing; then mesh the instance with default mesh. Note that by default, ABAQUS will generate *plane stress* elements in the part (so it is assumed to be a thin sheet). The element type will be quadrilaterals, and ABAQUS will calculate the displacement field using a linear interpolation within each element. If you want plane strain elements, or a quadratic interpolation scheme, you have to use Mesh > Element Type to change the type of the element. If you want to change the element shape, you have to use Mesh > Controls and then select the shape you want. We will explore the different element types in more detail in next weeks HW (I use the term 'we' loosely – I will not be doing very much of the work). Once you have generated the mesh, create a job and then run it, and look at the contours of von Mises stress. Find the maximum value of Von Mises stress and make a note of it. (You can use Options > Contour, select the 'Limits' tab and check the 'Show location' box to have ABAQUS tell you what the max von Mises stress is, and where it occurs). You will change the mesh in the steps below, run the simulation again, and make a
2. Seed the part using a mesh size of 1mm; and mesh with the default mesh
3. Seed the part using a mesh size of 0.5mm; and mesh with the default mesh
4. Seed using a mesh size of 0.25mm; and mesh with the default mesh
5. Seed using a mesh size of 0.125mm and mesh with the default mesh. (this will have around 100000 elements and may take some time to run)

6. Seed the instance using a mesh size of 0.5mm, but mesh the part with quadratic elements. To select quadratic elements, go to the Mesh>Element type and check the Quadratic button
7. Seed the instance with a 0.25mm mesh size and mesh with quadratic elements.

For each mesh, calculate the maximum von Mises stress in the solid. Display your results in a table showing the max. stress, element type and mesh size.

Clearly, proper mesh design is critical to get accurate numbers out of FEA computations. As a rough rule of thumb the element size near a geometric feature should be about 1/5 of the characteristic dimension associated with the feature – in this case the radius of the fillet. If there were a sharp corner instead of a fillet radius, you would find that the stresses go on increasing indefinitely as the mesh size is reduced (the stresses are theoretically infinite at a sharp corner in an elastic solid).

ABAQUS has an option to automatically refine a mesh in regions of high stress concentration. Automatic mesh generation is not too difficult in 2D, and this process will work with both quadrilateral and triangular elements. It's much harder to generate meshes automatically in 3D, and the method only works with tetrahedral elements (it's important to use quadratic tetrahedral, as linear tets don't generally give very accurate solutions).

To try automatic remeshing, follow these steps:

1. In the Mesh module, seed the part with a 1mm mesh size, and assign a linear element type to the part (Mesh > Element type, etc). Then mesh the part.
2. Still in the Mesh module select Adaptivity> Remeshing Rule> Create, then select the part in the viewport, and select Done. Then check the Mises equivalent stress box in the remeshing rule window (you can leave the element energy box checked as well). In the 'sizing method' select 0.01 for the minimum element size
3. In the Job module select Adaptivity > Create, and accept all the defaults (you can change the number of times ABAQUS attempts to generate new meshes in the menu, which is sometimes useful).
4. Select Adaptivity > submit. There is no 'monitor' option for an adaptivity analysis but ABAQUS will print messages in below the viewport, and you'll see the new meshes being generated.
5. To see the results, go to the Visualization module and select File > Open – you will see 3 different output databases containing the results with the three successive meshes. The one with the highest number is the last one.

As a solution to this problem, please hand in a table showing your calculated V-M stress for each of 1-7, and with the automatic mesh.

[10 POINTS]